An extension theorem for polynomials on triangles *

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Abstract

We present an extension theorem for polynomial functions that proves a quasi-optimal bound for a lifting from L^2 on edges onto a fractional order Sobolev space on triangles. The extension is such that it can be further extended continuously by zero within the trace space of H^1. Such an extension result is critical for the analysis of non-overlapping domain decomposition techniques applied to the p- and hp-versions of the finite and boundary element methods for elliptic problems of second order in three dimensions.

Key words: p- and hp-versions, finite element method, boundary element method, polynomial extension, domain decomposition, additive Schwarz method
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1 Introduction

This paper presents an extension theorem for polynomials from edges of triangles into its interior. Quasi-boundedness (i.e. weakly depending on the polynomial degree of the data) is proved within Sobolev spaces that are inherent to elliptic problems of second order in three space dimensions (L^2 on edges and a specific fractional order Sobolev space on triangles).

Polynomial extension theorems are critical for the analysis of approximation errors and domain decomposition strategies when dealing with high order finite element (FEM) or boundary element methods (BEM), in particular the p- and hp-versions. Non-overlapping domain decomposition techniques rely on the stable splitting of different types of basis functions (nodal, edge, interior functions), see, e.g., [18]. Associated with individual subspaces are specific bilinear forms, e.g. L^2-bilinear form for functions associated with the so-called wire basket and energy bilinear form for local subspaces associated with elements and faces of elements. The analysis of the L^2-part living on the wire basket of the mesh requires an extension of nodal and edge basis functions onto elements, continuously in the energy norm. One way to do this is to first extend

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from the wire basket of elements onto faces, and then e.g. discrete harmonically onto elements. For hexahedral meshes, where a tensor product structure of basis functions can be conveniently employed, such extensions are well analysed, see [17, 11, 2, 13]. For tetrahedral meshes (or triangular meshes when dealing with the BEM) this problem has not been adequately solved so far. It is essential to find low energy extensions in order to get the best possible condition number for corresponding additive Schwarz methods. Existing results show condition number bounds of the order $C(p)(\log p)^2$ (FEM, with an unspecified function C), see [7], or $(\log p)^7$ (BEM), see [8]. Here, p denotes the polynomial degree of basis functions. (Here and in the following, we always write $\log p$ for simplicity, meaning $1+\log(p+1)$ when $p=0$ or $p=1$. The extension results are trivial for these polynomial degrees.) Both results mentioned before are not optimal and need to be improved. In fact, for moderate polynomial degrees $O(\log p)$ is comparable with $O(p)$ and removing any unnecessary power of $\log p$ greatly improves the theoretical bound. A key tool for the analysis is, as mentioned before, an appropriate extension theorem. Such an extension theorem is the subject of this paper. Cao and Guo [8] avoided the use of an explicit extension construction by assuming that basis functions are extended from the sides in a discrete harmonic fashion (with respect to the hypersingular integral operator) to the elements. In contrast, Bică followed a construction introduced by Maday [15] and analysed by Muñoz-Sola for the approximation theory of finite elements in three dimensions [16]. Muñoz-Sola analysed an extension from faces to tetrahedra and Bică considered the analogue in two dimensions, extending from sides to triangles. In this paper we follow this construction and fill several gaps which were left open in the theory.

Let us describe in some detail what the procedure is. Extensions can be defined locally on patches of elements. For the extension of basis functions associated with edges (so-called edge basis functions) the situation is as indicated in Figure 1(a). A polynomial f defined on the edge I vanishes at the endpoints of I and needs to be extended to a piecewise polynomial U on $K := T_1 \cup T_2$ such that it can be extended continuously by zero onto an enlarged patch $\tilde{K}$ which contains K.

For three-dimensional elliptic problems of second order the right norm on $\tilde{K}$ is $H^{1/2}(\tilde{K})$, the trace space of $H^1(\Omega)$ onto $\tilde{K}$, assuming that $\tilde{K}$ is part of the boundary of a domain $\Omega \subset \mathbb{R}^3$. The corresponding intrinsic space on K is denoted by $\tilde{H}^{1/2}(K)$ (often used in the BEM literature) or $H_{00}^{1/2}(K)$ (often used in the FEM literature). Any element of $\tilde{H}^{1/2}(K)$ can be extended by zero to an element of $H^{1/2}(\tilde{K})$. With this notation, our above mentioned continuity of edge function extensions renders like:

Given a polynomial f of degree p on I that vanishes at the end points of I, find a function U on $K = T_1 \cup T_2$ such that $U|_{T_1}$ and $U|_{T_2}$ are polynomials of degree p, $U|_I = f$, $U = 0$ on ∂K (the boundary of K) and

$$\|U\|_{\tilde{H}^{1/2}(K)} \le C(p)\,\|f\|_{L^2(I)}. \tag{1}$$

For functions associated with nodes (nodal functions) the situation is analogous, see Figure 1(b). In this situation let us denote $K = \cup_{i=1}^6 T_i$. The task is as follows:

For a given continuous function f on $\cup I_i$ which is a polynomial of degree p on I_i ($i = 1, \dots, 6$) and which vanishes at the endpoints of the edges I_i that lie on ∂K, find U on K such that

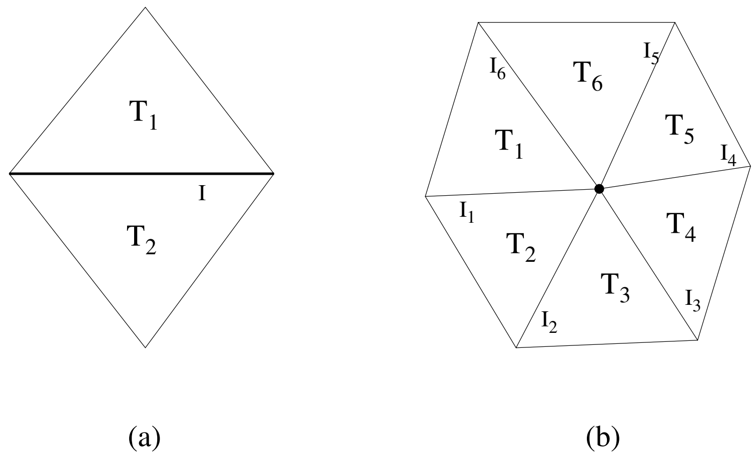

Figure 1: Constructing edge and nodal basis functions by extension.

$U|_{I_i} = f|_{I_i}$, $U|_{T_i}$ is a polynomial of degree p ($i = 1, \dots, 6$), $U = 0$ on ∂K and

$$\|U\|_{\tilde{H}^{1/2}(K)} \le C(p)\,\|f\|_{L^2(\cup_i I_i)}. \tag{2}$$

In this paper we show that both (1) and (2) can be satisfied with $C = O(\log p)^{1/2}$. This result is only quasi-optimal since the constant grows (very moderately) with p. But it is better than any existing result we know of.

In order to prove (1), (2) we estimate separately the $H^{1/2}(K)$-norm of an extended function U, and the part of the $\tilde{H}^{1/2}(K)$-norm that makes $\tilde{H}^{1/2}(K)$ a subspace of $H^{1/2}(K)$. The latter part is a weighted L^2-norm. In fact, our technical results estimating the weighted L^2-term follow the lines given [7] where, however, several gaps had to be filled. Our estimate of the term $\|U\|_{H^{1/2}(K)}$ is new.

We do not try to give a complete overview of existing extension theorems as the one presented here is very specifically designed towards p- and hp-FEM and BEM theory for three-dimensional problems. More "standard" extension theorems, i.e. dealing with the Sobolev spaces L^2, $H^{1/2}$ and H^1, are given, e.g., in [5, 16, 1]. In particular in [1], Ainsworth and Demkowicz recall existing literature on extensions and we refer the reader to that paper for a more detailed discussion.

In the next section we recall definitions of the needed Sobolev spaces and state the main extension theorem (Theorem 1). All the technical details and proofs are collected in Section 3. An overview of that section is given there (before giving proofs in subsections).

Throughout the paper, C denotes a generic constant which may take on different values at different occurrences, but which is independent of polynomial degrees p, if not otherwise stated.

2 Sobolev spaces and the main result

We define several Sobolev spaces and formulate our main extension theorem.

We use standard Sobolev spaces where the following norms are needed: For $\Omega \subset \mathbb{R}^n$ and $0 < s < 1$ we define

$$\|u\|^2_{H^s(\Omega)} := \|u\|^2_{L^2(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|^2_{H^s(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{2s+n}}\, dx\, dy.$$

For a Lipschitz domain Ω and $0 < s < 1$ the space $\tilde{H}^s(\Omega)$ is defined as the completion of $C_0^\infty(\Omega)$ under the norm

$$\|u\|^2_{\tilde{H}^s(\Omega)} := |u|^2_{H^s(\Omega)} + \int_\Omega \frac{u(x)^2}{(\operatorname{dist}(x, \partial\Omega))^{2s}}\, dx.$$

For $s \in (0, 1/2)$, $\|\cdot\|_{\tilde{H}^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$ are equivalent norms whereas for $s \in (1/2, 1)$ there holds $\tilde{H}^s(\Omega) = H_0^s(\Omega)$, the latter space being the completion of $C_0^\infty(\Omega)$ with norm in $H^s(\Omega)$. Also we note that functions from $\tilde{H}^s(\Omega)$ are continuously extendible by zero onto a larger domain. For all these results we refer to [14, 9].

For $s > 0$ the spaces $H^{-s}(\Omega)$ and $\tilde{H}^{-s}(\Omega)$ are the dual spaces of $\tilde{H}^s(\Omega)$ and $H^s(\Omega)$, respectively. For a Lipschitz domain (or bounded interval) Ω with boundary $\partial\Omega$ and subset $\Gamma \subset \partial\Omega$ we also define the space $\tilde{H}^s(\Omega, \Gamma)$ by completion of $C_0^\infty(\Omega)$ and using the norm

$$\|u\|^2_{\tilde{H}^s(\Omega,\Gamma)} := |u|^2_{H^s(\Omega)} + \int_\Omega \frac{u(x)^2}{(\operatorname{dist}(x, \Gamma))^{2s}}\, dx \quad (0 < s < 1).$$

There holds $\tilde{H}^s(\Omega) \subset \tilde{H}^s(\Omega, \Gamma) \subset H^s(\Omega)$. Fractional order Sobolev spaces can be equivalently defined by interpolation. We will use the real K-method, see [6].

Our main result is as follows.

Theorem 1. *Let $\tilde{T}$ be a triangle and let Γ be one of its sides or the union of two. Then, for a given continuous function f on $\partial\tilde{T}$ which is a polynomial of degree up to p on each of the sides and which vanishes on Γ, there exists an extension U on $\tilde{T}$ such that U is a polynomial of total degree up to p, $U = f$ on $\partial\tilde{T}$ and*

$$\|U\|_{\tilde{H}^{1/2}(\tilde{T},\Gamma)} \le C \log^{1/2} p\, \|f\|_{L^2(\partial\tilde{T})}. \tag{3}$$

Here, the constant $C > 0$ is independent of f and p.

Remark 1. *An application of this theorem provides the extension results for the construction of low-energy basis functions, as discussed in the introduction. To this end one applies it locally on elements. For instance, for the construction of an edge basis function, a polynomial f is given on I with $f(0) = f(1) = 0$. Using a piecewise affine transformation we can assume that $I_1 = [0, 1] \times \{0\}$ and that T_2 is the reflection of T_1 at I_1. Then, applying the theorem twice, one* *defines U_1 on T_1 and U_2 on T_2 yielding a continuous function U by setting $U|_{T_i} := U_i$, $i = 1, 2$. The result* (1) *then follows with $C(p) = \log^{1/2} p$ by noting that (with $K = T_1 \cup T_2$)*

$$\begin{aligned}\|U\|^2_{\tilde H^{1/2}(K)} &\simeq \sum_{i=1}^{2} \|U_i\|^2_{\tilde H^{1/2}(T_i, \partial T_i \cap \partial K)} + \int_{T_1} \frac{(U_1(x,y) - U_2(x,-y))^2}{\operatorname{dist}((x,y), I_1)}\, d(x,y)\\ &= \sum_{i=1}^{2} \|U_i\|^2_{\tilde H^{1/2}(T_i, \partial T_i \cap \partial K)},\end{aligned}$$

see [14, 9].

3 Proof of the extension theorem

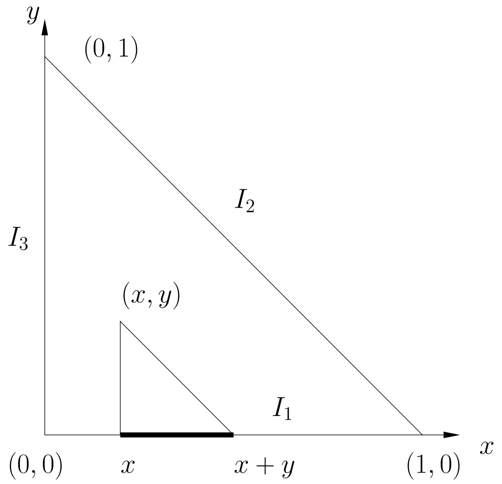

Figure 2: The reference triangle T.

Without loss of generality we assume that the triangle $\tilde T$ under consideration is the reference triangle $T := \{(x, y) : 0 \le x, y;\ x + y \le 1\}$. The edges of T are denoted by I_i, $i = 1, 2, 3$, see Figure 2. The edges I_1 and I_3 will be identified with the Interval $I := [0, 1]$, and $I = I_1$ will be used without further notice. We also need the polynomial spaces

$$P_p(I) := \operatorname{span}\{x^i,\ 0 \le i \le p\}, \qquad P_p(T) := \operatorname{span}\{x^i y^j,\ 0 \le i + j \le p\}.$$

For the proof of the main theorem we need two extension operators, the operator F frequently used in finite element analysis (see [5, 4]), and the operator E used for problems in three dimensions (see [15, 16]). The operator E is needed for the actual construction of polynomial extensions whereas F is required only for the analysis of E.

The operator F is defined by

$$F(f)(x,y) := \frac{1}{y}\int_x^{x+y} f(t)\,dt.$$

It extends polynomials of degree p on I to polynomials of total degree p on T. It cannot be used to construct the extension needed for Theorem 1 since, e.g., a root of f in 0 does not extend to a zero trace of $F(f)$ on I_3. This is precisely the property of E which is defined by

$$E(f)(x,y) := \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt \quad (f(0)=0).$$

More generally, for $f \in P_p(I)$ we define extension operators from I_1 by

$$\begin{aligned}
E_1^1(f)(x,y) :=& E(f)(x,y) = \frac{x}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt, && \text{if } f(0)=0,\\
E_2^1(f)(x,y) :=& \frac{1-x-y}{y}\int_x^{x+y} \frac{f(t)}{1-t}\,dt && \text{if } f(1)=0,\\
E^1(f)(x,y) :=& \frac{x(1-x-y)}{y}\int_x^{x+y} \frac{f(t)}{t(1-t)}\,dt && \text{if } f(0)=f(1)=0.
\end{aligned}$$

We note that there holds

$$E^1(f)(x,y) = (1-x-y)E_1^1(f)(x,y) + xE_2^1(f)(x,y).$$

Moreover, $E_2^1(f) = 0$ on I_2 and $E^1(f) = 0$ on $I_2 \cup I_3$.

Extension operators E_1^3 (for $f \in P_p(I_3)$ with $f(1)=0$), E_1^3 (if $f(0)=0$) and E^3 (if $f(0) = f(1) = 0$) from I_3 onto T are defined analogously.

For a polynomial $f \in P_p(I_2)$ we define

$$\begin{aligned}
E_2^2(f)(x,y) :=& \frac{y}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{(1-t)}\,dt, && \text{if } f(1,0)=0,\\
E_3^2(f)(x,y) :=& \frac{x}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{t}\,dt, && \text{if } f(0,1)=0,\\
E^2 f(x,y) :=& \frac{xy}{1-x-y}\int_x^{1-y} \frac{f(t,1-t)}{t(1-t)}\,dt, && \text{if } f(1,0)=f(0,1)=0.
\end{aligned}$$

There holds

$$E^2 f(x,y) = xE_2^2(f) + yE_3^2(f)$$

and $E_2^2(f) = 0$ on I_1, $E_3^2(f) = 0$ on I_3, $E^2(f) = 0$ on $I_1 \cup I_3$.

It is easy to see that all the extensions are polynomials of degree p on T. Furthermore, all the operators which deal with polynomials that vanish in only one vertex are linear transformations of the operator $E = E_1^1$. Therefore we only have to analyse the operator E. The main results concerning this operator are given in the next theorem.

Theorem 2. *For $f \in P_p(I)$ with $f(0) = 0$ there holds*

$$\|E(f)\|_{\tilde{H}^{1/2}(T,I_3)} \le C \log^{1/2} p \, \|f\|_{L^2(I)}. \tag{4}$$

For $f \in P_p(I)$ with $f(1) = 0$ there holds

$$\|E_2^1(f)\|_{\tilde{H}^{1/2}(T,I_2)} \le C \log^{1/2} p \, \|f\|_{L^2(I)}. \tag{5}$$

For $f \in P_p(I)$ with $f(0) = f(1) = 0$ there holds

$$\|E^1(f)\|_{\tilde{H}^{1/2}(T,I_2 \cup I_3)} \le C \log^{1/2} p \, \|f\|_{L^2(I)}. \tag{6}$$

The proof of this theorem is divided into three parts. In Section 3.1 we analyse E as a mapping $\tilde{H}^{1/2}(I,0) \to H^1(T)$ and in Section 3.2 from $H^{-1/2}(I)$ onto $L^2(T)$. These results are then used in Section 3.3 to prove the theorem. The proof of the main theorem (Theorem 1) is given in Section 3.4.

3.1 Boundedness of $E: \ P_p(I) \cap \tilde{H}^{1/2}(I,0) \to H^1(T)$

Lemma 1. *Let $0 \le x \le 1$ and $f \in L^2(x,1)$. Then there holds*

$$\int_0^{1-x} \frac{1}{y^2} \left(\int_x^{x+y} f(t)\, dt \right)^2 dy \le 4 \int_x^1 f^2(t)\, dt. \tag{7}$$

Proof. Recall Hardy's inequality ($p > 1$, $r \ne 0$):

$$\int_0^\infty y^{-r} (F(y))^p \, dy \le \left(\frac{p}{|r-1|} \right)^p \int_0^\infty y^{-r} (y f(y))^p \, dy,$$

where $F(y) = \int_y^\infty f(t)\, dt$ for $r < 1$, and $F(y) = \int_0^y f(t)\, dt$ for $r > 1$, see [10, Theorem 330].

We use this inequality for $r = 2$, $p = 2$ and with

$$F(y) := \int_0^y f(x+t)\, dt = \int_x^{x+y} f(t)\, dt.$$

Extending f by zero from $(x,1)$ onto (x,∞) we then obtain

$$\begin{aligned} \int_0^{1-x} \frac{1}{y^2} \left(\int_x^{x+y} f(t)\, dt \right)^2 dy &= \int_0^{1-x} \frac{1}{y^2} \left(\int_0^y f(x+t)\, dt \right)^2 dy \\ \le \ & \int_0^\infty \frac{1}{y^2} \left(\int_0^y f(x+t)\, dt \right)^2 dy \le 4 \int_0^{1-x} f^2(x+y)\, dy = 4 \int_x^1 f^2(y)\, dy. \end{aligned}$$

$\square$

Lemma 2. *There exists a constant $C > 0$ such that, for any $f \in H^{1/2}(I)$, there holds*

$$\|F(f)\|_{H^1(T)} \le C \, \|f\|_{H^{1/2}(I)}, \tag{8}$$

$$\|F(f)\|_{L^2(T)} \le C \, \|x^{1/2} f(x)\|_{L^2(I)}. \tag{9}$$

Proof. The bound (8) is [4, Lemma 7.1]. To prove (9) we use Lemma 1 to conclude that there holds

$$\begin{aligned}
\|F(f)\|^2_{L^2(T)} &= \int_0^1 \int_0^{1-x} \frac{1}{y^2} \left(\int_x^{x+y} f(t)\, dt \right)^2 dy\, dx \\
&\le 4 \int_0^1 \int_x^1 f(t)^2 \, dt\, dx = 4 \int_0^1 f(t)^2 \int_0^t dx\, dt = 4 \|t^{1/2} f\|^2_{L^2(I)}.
\end{aligned}$$

This proves the lemma. □

The following lemma states the main result of this subsection.

Lemma 3. *There exists a constant $C > 0$ such that*

$$\|E(f)\|_{H^1(T)} \le C \, \|f\|_{\tilde{H}^{1/2}(I,0)} \quad \forall f \in P_p(I), \ f(0) = 0.$$

Proof. The proof follows the techniques from [16, Lemma 6] where the three-dimensional case is considered. Using (9) we estimate

$$\|E(f)\|^2_{L^2(T)} \le \|F(|f|)\|^2_{L^2(T)} \le C \, \|x^{1/2} \, f(x)\|^2_{L^2(I)} \le C \, \|f\|^2_{L^2(I)}. \tag{10}$$

To estimate the $H^1(T)$-semi-norm we calculate the first order derivatives of E and F.

$$\begin{aligned}
\frac{\partial E(f)}{\partial x} &= \frac{1}{y} \int_x^{x+y} \frac{f(t)}{t} \, dt + \frac{x}{y} \left(\frac{f(x+y)}{x+y} - \frac{f(x)}{x} \right), \\
\frac{\partial E(f)}{\partial y} &= -\frac{x}{y^2} \int_x^{x+y} \frac{f(t)}{t} \, dt + \frac{x}{y} \frac{f(x+y)}{x+y}, \\
\frac{\partial F(f)}{\partial x} &= \frac{1}{y} f(x+y) - \frac{1}{y} f(x), \\
\frac{\partial F(f)}{\partial y} &= -\frac{1}{y^2} \int_x^{x+y} f(t)\, dt + \frac{1}{y} f(x+y).
\end{aligned}$$

By (8) there holds $\|F(f)\|_{H^1(T)} \le C \, \|f\|_{H^{1/2}(I)}$ such that it is enough to bound the differences of E and F in the $H^1(T)$-semi-norm.

Calculating

$$\frac{\partial E(f)}{\partial x} - \frac{\partial F(f)}{\partial x} = \frac{1}{y} \int_x^{x+y} \frac{f(t)}{t} \, dt + \frac{f(x+y)}{y} \left(\frac{x}{x+y} - 1 \right), \tag{11}$$

let us denote the two terms on the right-hand side by

$$R_1 := \frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt, \qquad R_2 := \frac{f(x+y)}{y}\left(\frac{x}{x+y}-1\right).$$

It is clear that

$$|R_1| = \left|\frac{1}{y}\int_x^{x+y} \frac{f(t)}{t}\,dt\right| \le \frac{1}{y}\int_x^{x+y} \frac{|f(t)|}{t}\,dt = F\left(\frac{|f(t)|}{t}\right).$$

Using (9) we find that

$$\|R_1\|_{L^2(T)} \le \left\|F\left(\frac{|f(t)|}{t}\right)\right\|_{L^2(T)} \le C\left\|x^{1/2}\frac{f(x)}{x}\right\|_{L^2(I)} = C\,\|x^{-1/2}f(x)\|_{L^2(I)}. \tag{12}$$

Next we estimate R_2. Since $0 \le 1 - \frac{x}{x+y} \le 1$ we obtain

$$\begin{aligned}|R_2| &= \left|\frac{f(x+y)}{y}\left(\frac{x}{x+y}-1\right)\right| = \frac{1}{y}\left|f(x+y)\frac{y}{x+y}\right| \\ &\le \frac{1}{y}|f(x+y)-f(x)|\frac{y}{x+y} + \frac{|f(x)|}{x+y} \le \frac{1}{y}|f(x+y)-f(x)| + \frac{|f(x)|}{x+y}.\end{aligned}$$

The first term on the right-hand side can be bounded by

$$\begin{aligned}&\left\|\frac{1}{y}\big(f(x+y)-f(x)\big)\right\|^2_{L^2(T)} = \int_0^1\int_0^{1-x}\frac{\big(f(x+y)-f(x)\big)^2}{y^2}\,dy\,dx \\ &= \int_0^1\int_x^1 \frac{\big(f(z)-f(x)\big)^2}{(z-x)^2}\,dz\,dx \le \int_0^1\int_0^1 \frac{\big(f(z)-f(x)\big)^2}{(z-x)^2}\,dz\,dx = |f|^2_{H^{1/2}(I)}.\end{aligned}$$

For the second term we obtain

$$\int_0^1\int_0^{1-x}\frac{f(x)^2}{(x+y)^2}\,dy\,dx \le \int_0^1 \frac{f(x)^2}{x}\,dx = \|x^{-1/2}f(x)\|^2_{L^2(I)}.$$

Combination of the last three estimates yields

$$\|R_2\|^2_{L^2(T)} \le C\left(|f|^2_{H^{1/2}(I)} + \|x^{-1/2}f(x)\|^2_{L^2(I)}\right). \tag{13}$$

Now we examine the derivatives with respect to y. There holds

$$\frac{\partial E(f)}{\partial y} - \frac{\partial F(f)}{\partial y} = \frac{1}{y^2}\int_x^{x+y} f(t)\left(1-\frac{x}{t}\right)dt + \frac{f(x+y)}{y}\left(\frac{x}{x+y}-1\right) = R_3 + R_2$$

with

$$R_3 := \frac{1}{y^2}\int_x^{x+y} f(t)\left(1-\frac{x}{t}\right)dt.$$

The term R_2 has already been estimated. It remains to bound R_3. Since $0 \le 1 - \frac{x}{t} \le \frac{y}{t}$ for $t \in [x, x+y]$, we get

$$|R_3| \le \frac{1}{y} \int_x^{x+y} \frac{|f(t)|}{t}\, dt = F\left(\frac{|f(t)|}{t}\right).$$

This can be estimated like R_1 and it follows that

$$\|R_3\|_{L^2(T)} \le C\, \|x^{-1/2} f(x)\|_{L^2(I)}. \tag{14}$$

Eventually, using (8), the triangle inequality and estimates (10) and (12), (13), (14), the lemma is proved. ☐

3.2 Quasi-boundedness of $E:\ P_p(I) \cap H^{-1/2}(I) \to L^2(T)$

Lemma 4. *There exists a constant $C > 0$, independent of p, such that for any $f \in P_p(I)$ and $\varepsilon \in (0, 1/2)$ there holds*

$$\left\| y^{\varepsilon-1}\, \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \right\|_{L^2(T)} \le \frac{C}{\sqrt{\varepsilon}} \|f\|_{H^{-1/2+\varepsilon}(I)}.$$

Proof. For $y \in [0,1]$ the function $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ is continuous with respect to $x \in [0, 1-y]$. This can be seen by estimating

$$\begin{aligned}
&\Big| \|f\|_{H^{-1/2+\varepsilon}(x_1,x_1+y)} - \|f\|_{H^{-1/2+\varepsilon}(x_2,x_2+y)} \Big| \\
&= \Big| \|f(x_1+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)} - \|f(x_2+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)} \Big| \\
&\le \|f(x_1+\cdot) - f(x_2+\cdot)\|_{H^{-1/2+\varepsilon}(0,y)} \le \|f(x_1+\cdot) - f(x_2+\cdot)\|_{L^2(0,y)}
\end{aligned}$$

and using the uniform continuity of f.

Therefore, $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ is Riemann integrable in x and we can calculate the integral as the limit of Riemann sums. To this end we define a partition of $[0, 1-y]$ into N_h intervals:

$$x_i := i\frac{1-y}{N_h}, \quad i = 0, \dots, N_h, \quad h := \frac{1-y}{N_h}.$$

We obtain

$$\int_0^{1-y} \|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)}\, dx = \lim_{h\to 0} \sum_{i=0}^{N_h-1} h \|f\|^2_{H^{-1/2+\varepsilon}(x_i,x_i+y)}.$$

Every interval $(x_i, x_i + y) = (ih, ih + y)$, $i = 0, \dots, N_h - 1$, overlaps with at most $O(\frac{y}{h})$ intervals. Therefore, we can use a colouring argument together with the estimate

$$\sum_i \|f\|^2_{H^s(\gamma_i)} \le C\, \|f\|^2_{H^s(0,1)}, \quad \cup_i \bar{\gamma}_i \subset (0,1),\ \gamma_i \cap \gamma_i = \emptyset\ (i \ne j),$$

(see, e.g., [3]) to obtain

$$\lim_{h\to 0}\sum_{i=0}^{N_h-1} h\|f\|^2_{H^{-1/2+\varepsilon}(x_i,x_i+y)} \le C\,\lim_{h\to 0} h\,\frac{y}{h}\|f\|^2_{H^{-1/2+\varepsilon}(0,1)} = C\,y\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}.$$

We finish the proof by calculating

$$\begin{aligned}\big\|y^{\varepsilon-1}\,\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}\big\|^2_{L^2(T)} &= \int_0^1 y^{2\varepsilon-2}\int_0^{1-y}\|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)}\,dx\,dy\\ &\le C\int_0^1 y^{2\varepsilon-2}\,y\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}\,dy = C\,\|f\|^2_{H^{-1/2+\varepsilon}(0,1)}\int_0^1 y^{2\varepsilon-1}\,dy = C\,\frac{1}{2\varepsilon}\|f\|^2_{H^{-1/2+\varepsilon}(I)}.\end{aligned}$$

□

The next lemma is the main result of this subsection. It represents the key ingredient for the proof of our extension theorems (Theorems 1 and 2).

Lemma 5. *There exists a constant $C > 0$, independent of p, such that*

$$\|E(f)\|_{L^2(T)} \le C\,\log p\,\|f\|_{H^{-1/2}(I)}\quad \forall f\in P_p(I),\ f(0)=0.$$

Proof. Let $\varepsilon\in(0,1/4)$ and $(x,y)\in T$, $x>0$. Making use of the duality between the spaces $H^{-1/2+\varepsilon}(x,x+y)$ and $\tilde H^{1/2-\varepsilon}(x,x+y)$ we obtain

$$\int_x^{x+y}\frac{f(t)}{t}\,dt \le \|t^{-1}\|_{\tilde H^{1/2-\varepsilon}(x,x+y)}\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}. \tag{15}$$

We bound the first term on the right-hand side:

$$\begin{aligned}\|t^{-1}\|^2_{\tilde H^{1/2-\varepsilon}(x,x+y)} &= |t^{-1}|^2_{H^{1/2-\varepsilon}(x,x+y)} + \int_x^{x+y}\frac{t^{-2}}{\operatorname{dist}(t;x,x+y)^{2(1/2-\varepsilon)}}\,dt\\ &\le |t^{-1}|^2_{H^{1/2-\varepsilon}(x,x+y)} + C\left(\int_x^{x+y}\frac{t^{-2}}{|t-x|^{1-2\varepsilon}}\,dt + \int_x^{x+y}\frac{t^{-2}}{|t-x-y|^{1-2\varepsilon}}\,dt\right).\end{aligned} \tag{16}$$

To estimate the $H^{1/2-\varepsilon}(x,x+y)$-semi-norm we calculate

$$\|t^{-1}\|^2_{L^2(x,x+y)} = \frac{y}{x(x+y)}$$

and

$$|t^{-1}|^2_{H^1(x,x+y)} = \frac{x^2y+xy^2+\frac13 y^3}{x^3(x+y)^3} \le \frac{y}{x^3(x+y)}.$$

Then interpolation yields (see, e.g., [6])

$$
\begin{aligned}
|t^{-1}|_{H^{1/2-\varepsilon}(x,x+y)} &\lesssim \|t^{-1}\|^{1/2+\varepsilon}_{L^2(x,x+y)} \|t^{-1}\|^{1/2-\varepsilon}_{H^1(x,x+y)} \\
\leq \left(\frac{y^{1/2+\varepsilon}}{x^{1/2+\varepsilon}(x+y)^{1/2+\varepsilon}} \frac{y^{1/2-\varepsilon}}{x^{3/2-3\varepsilon}(x+y)^{1/2-\varepsilon}} \right)^{1/2} &= \frac{y^{1/2}x^{\varepsilon}}{x(x+y)^{1/2}} \leq \frac{y^{1/2}x^{\varepsilon}}{x^{3/2}}.
\end{aligned}
\tag{17}
$$

The second term on the right-hand side of (16) can be estimated by

$$
\int_x^{x+y} \frac{t^{-2}}{|t-x|^{1-2\varepsilon}}\, dt \leq \frac{1}{x^2} \int_x^{x+y} \frac{1}{(t-x)^{1-2\varepsilon}}\, dt = \frac{1}{x^2} \frac{y^{2\varepsilon}}{2\varepsilon}, \tag{18}
$$

and the second term by

$$
\int_x^{x+y} \frac{t^{-2}}{|t-x-y|^{1-2\varepsilon}}\, dt \leq \frac{1}{x^2} \int_x^{x+y} \frac{1}{(x+y-t)^{1-2\varepsilon}}\, dt = \frac{1}{x^2} \frac{y^{2\varepsilon}}{2\varepsilon}. \tag{19}
$$

Thus we get from (15), together with (16), (17), (18) and (19), the bound

$$
\begin{aligned}
\int_x^{x+y} \frac{f(t)}{t}\, dt &\leq \|t^{-1}\|_{\tilde{H}^{1/2-\varepsilon}(x,x+y)} \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \\
&\leq C \left(\frac{y^{1/2}x^{\varepsilon}}{x^{3/2}} + \frac{1}{\sqrt{\varepsilon}} \frac{y^{\varepsilon}}{x} \right) \|f\|_{H^{-1/2+\varepsilon}(x,x+y)}.
\end{aligned}
$$

Using this bound we start estimating $\|E(f)\|_{L^2(T)}$:

$$
\begin{aligned}
\|E(f)\|_{L^2(T)} &\leq C \left\| \left(x^{\varepsilon-1/2}y^{-1/2} + \frac{1}{\sqrt{\varepsilon}} y^{\varepsilon-1} \right) \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \right\|_{L^2(T)} \\
&\leq C \left\| x^{\varepsilon-1/2}y^{-1/2} \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \right\|_{L^2(T)} + \frac{C}{\sqrt{\varepsilon}} \left\| y^{\varepsilon-1} \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \right\|_{L^2(T)}.
\end{aligned}
\tag{20}
$$

To bound the factor $\|f\|_{H^{-1/2+\varepsilon}(x,x+y)}$ in the first term on the right-hand side of (20) we use scaling properties of the norms involved together with a linear transformation forward and backward to the interval $(x, x+1)$, see, e.g., [12, Lemma 2]. Denoting by $\tilde{f}$ the correspondingly linearly transformed polynomial f, this gives

$$
\begin{aligned}
\|f\|^2_{H^{-1/2+\varepsilon}(x,x+y)} &\leq C\, y^{1-2(-1/2+\varepsilon)} \|\tilde{f}\|^2_{H^{-1/2+\varepsilon}(x,x+1)} = C\, y^{2\varepsilon}(y^{2-4\varepsilon}) \|\tilde{f}\|^2_{H^{-1/2+\varepsilon}(x,x+1)} \\
&\leq C\, y^{2\varepsilon}(y^{2-4\varepsilon}) \|\tilde{f}\|^2_{H^{-1/2+2\varepsilon}(x,x+1)} \\
&\leq C\, y^{2\varepsilon}(y^{2-4\varepsilon}) \frac{1}{y^{1-2(-1/2+2\varepsilon)}} \|f\|^2_{H^{-1/2+2\varepsilon}(x,x+y)} \\
&= C\, y^{2\varepsilon} \|f\|^2_{H^{-1/2+2\varepsilon}(x,x+y)} \leq C\, y^{2\varepsilon} \|f\|^2_{H^{-1/2+2\varepsilon}(I)}.
\end{aligned}
$$

Hence

$$\begin{aligned}
\left\| x^{\varepsilon-1/2} y^{-1/2} \|f\|_{H^{-1/2+\varepsilon}(x,x+y)} \right\|_{L^2(T)} &\le C\, \|x^{\varepsilon-1/2} y^{\varepsilon-1/2}\|_{L^2(T)} \cdot \|f\|_{H^{-1/2+2\varepsilon}(I)} \\
&\le C\, \|x^{\varepsilon-1/2}\|_{L^2(I)} \cdot \|y^{\varepsilon-1/2}\|_{L^2(I)} \cdot \|f\|_{H^{-1/2+2\varepsilon}(I)} \\
&\le \frac{C}{2\varepsilon} \|f\|_{H^{-1/2+2\varepsilon}(I)}.
\end{aligned}$$

For the second term on the right-hand side of (20) we apply Lemma 4. We then conclude, using the inverse property of polynomials (see [12, Lemma 4]) that for $f \in P_p(I)$ there holds

$$\begin{aligned}
\|E(f)\|_{L^2(T)} &\le C\, \left(\frac{1}{\varepsilon} \|f\|_{H^{-1/2+2\varepsilon}(I)} \;+\; \frac{1}{\varepsilon} \|f\|_{H^{-1/2+\varepsilon}(I)} \right) \\
&\le C\, \frac{1}{\varepsilon} \left(p^{4\varepsilon} + p^{2\varepsilon} \right) \|f\|_{H^{-1/2}(I)} \le C\, \log p\, \|f\|_{H^{-1/2}(I)}.
\end{aligned}$$

Here we have chosen $\varepsilon := \log^{-1} p$. This proves the lemma. □

3.3 Proof of Theorem 2

Proof of (4)**.** Lemmas 3 and 5 yield two bounds for E:

$$\|E(f)\|_{H^1(T)} \le C\, \|f\|_{\tilde{H}^{1/2}(I,0)}$$

and

$$\|E(f)\|_{L^2(T)} \le C\, \log p\, \|f\|_{H^{-1/2}(I)}$$

for all polynomials $f \in P_p(I)$ with $f(0) = 0$. Interpolation thus proves the boundedness

$$\|E(f)\|_{H^{1/2}(T)} \le C\, \log^{1/2} p\, \|f\|_{L^2(I)}. \tag{21}$$

In order to finish the proof of (4) it is therefore left to bound the weighted L^2-norm that contributes to $\|E(f)\|_{\tilde{H}^{1/2}(T,I_3)}$. Here we use Lemma 1 and obtain

$$\begin{aligned}
\|x^{-1/2} E(f)\|^2_{L^2(T)} &= \int_0^1 \int_0^{1-x} \frac{x}{y^2} \left(\int_x^{x+y} \frac{f(t)}{t}\, dt \right)^2 dy\, dx \\
&\le 4 \int_0^1 x \left(\int_x^1 \frac{f(t)^2}{t^2}\, dt \right) dx = 4 \int_0^1 \frac{f(t)^2}{t^2} \int_0^t x\, dx\, dt = 2\|f\|^2_{L^2(I)}.
\end{aligned} \tag{22}$$

This finishes the proof of (4).

Proof of (5)**.** This can be obtained by a linear transformation to the previous case.

Proof of (6)**.** For this estimate we use techniques that had been proposed in [7]. Recall that there holds

$$E^1(f)(x,y) = (1-x-y)E(f)(x,y) + xE_2^1(f)(x,y).$$

We consider only the first term. The second can be bounded analogously. In order to bound the $H^{1/2}$-semi-norm we need the following estimate: For $(x,y)\in T$ and $(x',y')\in T$ there holds

$$\begin{aligned}
&|(1-x-y)E(f)(x,y) - (1-x'-y')E(f)(x',y')|^2\\
&\quad = |(1-x-y)E(f)(x,y) - (1-x-y)E(f)(x',y')\\
&\qquad\qquad\qquad\qquad + (1-x-y)E(f)(x',y') - (1-x'-y')E(f)(x',y')|^2\\
&\quad \le 2(1-x-y)^2|E(f)(x,y) - E(x',y')|^2 + 2(x'-x+y'-y)^2|E(f)(x',y')|^2\\
&\quad \le |E(f)(x,y) - E(f)(x',y')|^2 + 2(x'-x+y'-y)^2|E(f)(x',y')|^2.
\end{aligned}$$

Then, using the definition of the $H^{1/2}(T)$-norm and the estimate (21), we obtain

$$\begin{aligned}
&|(1-x-y)E(f)|^2_{H^{1/2}(T)}\\
&\le C\left(|E(f)|^2_{H^{1/2}} + \int_0^1\int_0^{1-y}\int_0^1\int_0^{1-y'} \frac{(x'-x+y'-y)^2\big(E(f)(x',y')\big)^2}{\big((x'-x)^2+(y'-y)^2\big)^{3/2}}\,dx'\,dy'\,dx\,dy\right)\\
&\le C\left(|E(f)|^2_{H^{1/2}} + \|E(f)\|^2_{L^2}\right) = C\,\|E(f)\|^2_{H^{1/2}(T)} \le C\,\log p\,\|f\|^2_{L^2(I)}.
\end{aligned}$$

Eventually we estimate the weighted L^2-norm corresponding to the edge I_3 (the one for I_2 is straightforward):

$$\begin{aligned}
\|x^{-1/2}(1-x-y)E(f)\|^2_{L^2(T)} &= \int_0^1\int_0^{1-y} \frac{x(1-x-y)^2}{y^2}\left(\int_x^{x+y}\frac{f(t)}{t}\,dt\right)^2 dx\,dy\\
&\le \int_0^1\int_0^{1-y} \frac{x}{y^2}\left(\int_x^{x+y}\frac{f(t)}{t}\,dt\right)^2 dx\,dy \le C\,\|f\|^2_{L^2(I)}.
\end{aligned}$$

The last step is (22). This finishes the proof of Theorem 2.

3.4 Proof of Theorem 1

As mentioned at the beginning of this section we consider, instead of $\tilde T$, the reference triangle T. We define $f_i := f|_{I_i} \in P_p(I_i)$, $i = 1, 2, 3$. It is enough to consider the two cases $\Gamma = I_2$ (i.e. $f_2 = 0$) and $\Gamma = I_2 \cup I_3$ (i.e. $f_2 = 0$, $f_3 = 0$).

Case $\Gamma = I_2 \cup I_3$**.** This case is covered by Theorem 2.

Case $\Gamma = I_2$**.** We have to show that there exists $U \in P_p(T)$ such that $U = f$ on ∂T and

$$\|U\|_{\tilde H^{1/2}(T,I_2)} \le C\,\log^{1/2} p\,\|f\|_{L^2(\partial T)}.$$

To this end define $U_1 := E_2^1(f_1)$. By Theorem 2 there holds

$$\|U_1\|_{\tilde{H}^{1/2}(T,I_2)} \le C\, \log^{1/2} p\, \|f\|_{L^2(I)}. \tag{23}$$

Further let g_3 be the trace of U_1 on I_3. Making use of Lemma 1 we can bound

$$\begin{aligned}\|g_3\|^2_{L^2(I_3)} = \|E_2^1(f_1)\|^2_{L^2(I_3)} &= \int_0^1 \frac{(1-y)^2}{y^2} \left(\int_0^y \frac{f_1(t)}{1-t}\, dt\right)^2 dy \\ &\le \int_0^1 \frac{1}{y^2} \left(\int_0^y |f_1(t)|\, dt\right)^2 dy \le 4\|f_1\|^2_{L^2(I)}.\end{aligned}$$

Due to the continuity of f there holds $(g_3 - f_3)(1,0) = (g_3 - f_3)(0,1) = 0$. Using E^3 we extend $g_3 - f_3$ to a polynomial $U_3 \in P_p(T)$ with $U_3 = 0$ on I_1 and I_2, and applying the case before (i.e. Theorem 2) and the previous estimate we obtain

$$\|U_3\|_{\tilde{H}^{1/2}(T,I_2)} \le \|U_3\|_{\tilde{H}^{1/2}(T,I_1\cup I_2)} \le C\, \log^{1/2} p\, \|g_3 - f_3\|_{L^2(I_3)} \le C\, \log^{1/2} p\, \|f\|_{L^2(\partial T)}. \tag{24}$$

Finally, setting $U := U_1 - U_3$ and combining the estimates (23) and (24) we finish the proof.

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