hp–adaptive two–level methods for boundary integral equations on curves

N. Heuer, Bremen and Concepción, **M. E. Mellado,** Bremen, and **E. P. Stephan,** Hannover [1]

Abstract

We consider the p– and hp–versions of the Galerkin boundary element method for hypersingular and weakly singular integral equations of the first kind on curves. We derive a–posteriori error estimates that are based on stable two–level decompositions of enriched ansatz spaces. The Galerkin errors are estimated by inverting local projection operators that are defined on small subspaces of the second level. A p–adaptive and two hp–adaptive algorithms are defined and numerical experiments confirm their efficiency.

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1 Introduction

In this paper we present a–posteriori error estimators based on two–level space decompositions for the p– and hp–versions of the Galerkin boundary element method applied to both hypersingular and weakly singular integral equations in two dimensions. Our strategies use the ideas of Bank and Smith [3] who deal with the adaptive h–version of the finite element method. As demonstrated, e.g., in [12] the additive Schwarz framework of stable decompositions of ansatz spaces can be used to generalize the idea by Bank and Smith. In principle, we enlarge the ansatz space H_N of the Galerkin boundary element method (BEM) and define error indicators by projecting the difference $\tilde{u}_N - u_N$ onto local subspaces of the new ansatz space. Here, $u_N \in H_N$ is the old Galerkin approximation to the solution u of the underlying problem and $\tilde{u}_N$ is an improved approximation of the enlarged space $\tilde{H}_N$. The function $\tilde{u}_N$ does not need to be computed. For $\tilde{H}_N$ we assume a saturation property, i.e., $\tilde{u}_N$ is a better approximation to u than u_N. Combining the local error indicators we end up with an a–posteriori error estimator. Efficiency and reliability of this estimator are obtained

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by proving stability of the underlying decomposition, provided the above mentioned saturation property is fulfilled.

In this paper we propose to use local subspaces that are purely defined by so–called bubble functions on individual elements. Then, in two dimensions, the local problems are only one–dimensional and easily invertible. As we will demonstrate, this strategy leads to adaptive *hp*–methods that converge exponentially for our model problem.

For previous results that are related with adaptive methods based on subspace decompositions for the BEM we refer to [10, 12, 11, 9]. As far as we know this is the first paper presenting theoretical and numerical results for the adaptive *hp*–version of the BEM, based on subspace decompositions.

Our model hypersingular and weakly singular integral equations are, respectively,

$$Du(x) := -\frac{1}{\pi}\frac{\partial}{\partial \nu_x}\int_\Gamma \frac{\partial}{\partial \nu_y}\log|x-y|\,u(y)\,ds_y = f(x) \quad \forall\, x\in\Gamma\,, \tag{1.1}$$

where ν_z denotes the unit outward normal to Γ at z, and

$$Vu(x) := -\frac{1}{\pi}\int_\Gamma \log|x-y|\,u(y)\,ds_y = f(x) \quad \forall\, x\in\Gamma\,. \tag{1.2}$$

In the hypersingular integral equation (1.1), the right hand side f is prescribed in $H^{-1/2}(\Gamma)$ while in (1.2) it is given in $H^{1/2}(\Gamma)$.

Now, we introduce the required Sobolev spaces for our analysis which includes open and closed curves. Let $\Gamma\subset\mathbf{R}^2$ be an open Lipschitz curve and $\tilde\Gamma$ be a closed curve containing Γ. We define as in [5],

$$H^s(\tilde\Gamma) := \begin{cases} \{\phi|_{\tilde\Gamma}\,;\ \phi\in H^{s+1/2}_{loc}(\mathbf{R}^2)\}\,, & s>0\,,\\ L^2(\tilde\Gamma)\,, & s=0\,,\\ (H^{-s}(\tilde\Gamma))'\,, & s<0\,. \end{cases}$$

Also, for $s>0$, let

$$H^s(\Gamma) = \{\phi|_\Gamma\,;\ \phi\in H^s(\tilde\Gamma)\}\,,$$
$$\tilde H^s(\Gamma) = \{\phi\in H^s(\tilde\Gamma)\,;\ \mathrm{supp}(\phi)\subset\bar\Gamma\}\,,$$

and for $s<0$,

$$H^s(\Gamma) = (\tilde H^{-s}(\Gamma))'\,,\quad \tilde H^s(\Gamma) = (H^{-s}(\Gamma))'\,.$$

It is important to note that, if Γ is closed, the Sobolev spaces $\tilde H^s(\Gamma)$ and $H^s(\Gamma)$ coincide. We recall from [4] that the hypersingular integral operator D is a continuous and positive definite mapping from $\tilde H^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$ for open curves Γ and also, from $H^{1/2}(\tilde\Gamma)/\mathbf{R}$ onto $H^{-1/2}(\tilde\Gamma)$ for closed curves $\tilde\Gamma$. Here, $H^{1/2}(\tilde\Gamma)/\mathbf{R}$ denotes the quotient space of $H^{1/2}(\tilde\Gamma)$ where all functions are identified which differ only by a constant function. Moreover, the weakly singular integral operator $V:\tilde H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ is a continuous and positive definite mapping if the conformal radius of Γ is different from 1. This can always be achieved by an appropriate scaling, see [4, 16]. (This also holds for closed curves $\tilde{\Gamma}$.)

In order to deal with the integral equations (1.1) and (1.2), we introduce the following abstract problem: *given* $f \in H'$, *find* $u \in H$ *such that*

$$< Au, v >_{L^2(\Gamma)} = < f, v >_{L^2(\Gamma)} \quad \forall\, v \in H\,, \tag{1.3}$$

where H denotes the adequate Sobolev space for the weak formulation, H' is its dual space, A is either the hypersingular or weakly singular operator and $< \cdot, \cdot >_{L^2(\Gamma)}$ stands for the $L^2(\Gamma)$–inner product.

In order to obtain an approximation to the unique solution $u \in H$ of (1.3), we consider a finite dimensional subspace $H_N \subset H$ and state the Galerkin scheme: *find* $u_N \in H_N$ *such that*

$$< Au_N, v >_{L^2(\Gamma)} = < f, v >_{L^2(\Gamma)} \quad \forall\, v \in H_N\,. \tag{1.4}$$

Here, N refers to the dimension of H_N. Since we are dealing with positive definite operators A, the system (1.4) is uniquely solvable and the discrete solution u_N converges quasi–optimally to u in the energy norm. For convergence results we refer to [15] for the hp–version on quasi–uniform meshes and to [2, 8] for the hp–version on geometrically refined meshes.

An outline of the remainder of this paper is as follows. In Section 2 we describe the abstract setting of our a–posteriori error estimator based on a two–level decomposition of an enriched ansatz space. In Section 3 we deal with hypersingular integral operators on open and closed curves. A special enrichment of the ansatz space by bubble functions is defined and stability of an associated two–level decomposition is proved. This leads to an efficient and reliable error estimator as long as a saturation assumption is fulfilled (Theorems 3.1 and 3.2). Based on the analysis in Section 3 we prove reliability and efficiency of an a–posteriori error estimator for weakly singular operators in Section 4 (Theorem 4.1). The error estimators of Sections 3 and 4 make use of local error indicators for which one needs to solve only one–dimensional problems. In Section 5 these error indicators are taken to define p– and hp–adaptive algorithms. Section 6 reports on several numerical experiments that underline efficiency and reliability of our methods.

From now on, C denotes a positive constant which may take different values at different places.

2 Abstract two–level space decomposition and a–posteriori error estimator

In what follows we describe an abstract framework which allows to define an a–posteriori error estimator for the discrete problem (1.4). Let $\tilde{H}_N$ denote a locally enriched space

$$\tilde{H}_N := H_N + Z_1 + \cdots + Z_J \,, \tag{2.1}$$

where $Z_j \cap H_N = \{0\}$. Below we will use projections onto the subspaces Z_j to define error indicators. Therefore, these spaces will be taken to be low–dimensional (e.g. one–dimensional) and to consist of functions with small support (e.g. one element of the boundary element mesh). The so–called second level space

$$Z := Z_1 + \cdots + Z_J \,,$$

is assumed to be directly decomposed. Then, also (2.1) is a direct decomposition of $\tilde{H}_N$.

To define our a–posteriori error estimator, we introduce the projection operators $P_0 : \tilde{H}_N \to H_N$ and $P_j : \tilde{H}_N \to Z_j$, $j = 1, \ldots, J$, defined respectively by

$$a(P_0\phi, \psi) = a(\phi, \psi) \quad \forall\, \psi \in H_N$$

and

$$a(P_j\phi, \psi) = a(\phi, \psi) \quad \forall\, \psi \in Z_j \,.$$

Here, $a(\cdot,\cdot)$ denotes the bilinear form induced by the boundary integral operator A, i.e.,

$$a(u, v) :=< Au, v >_{L^2(\Gamma)} \quad \forall\, u, v \in H \,.$$

Let $\tilde{u}_N \in \tilde{H}_N$ be the Galerkin solution of the enriched space $\tilde{H}_N$. Then, our a–posteriori error estimator is $\left\{\sum_{j=1}^J \theta_j^2\right\}^{1/2}$, where

$$\theta_j := \|P_j(\tilde{u}_N - u_N)\|_A \,, \quad j = 1, \ldots, J$$

and $\|\cdot\|_A$ denotes the norm induced by $a(\cdot,\cdot)$. Note that to compute θ_j, we only need to solve a low–dimensional problem within the subspace Z_j and the norm is given by the local stiffness matrix.

Let us make the following two assumptions.

(A1) **Stability of the two–level decomposition.** There exist C_1, $C_2 > 0$ such that

$$C_1 \sum_{j=0}^J a(\phi_j, \phi_j) \le a(\phi, \phi) \le C_2 \sum_{j=0}^J a(\phi_j, \phi_j)$$

for all $\phi = \sum_{j=0}^J \phi_j \in \tilde{H}_N$ with $\phi_0 \in H_N$ and $\phi_j \in Z_j$, $j = 1, \ldots, J$.

(A2) **Saturation assumption.** There exists $\sigma \in [0, 1)$ such that

$$\|u - \tilde{u}_N\|_A \le \sigma \, \|u - u_N\|_A \,.$$

We recall that the assumption (A1) is equivalent to the fact that the minimum and maximum eigenvalues of the additive Schwarz operator, $P := \sum_{j=0}^{J} P_j$, are bounded respectively by $\lambda_{min}(P) \geq C_1$ and $\lambda_{max}(P) \leq C_2$. The corresponding additive Schwarz method amounts to a two–level preconditioner for the Galerkin system defined in $\tilde{H}_N$. The first level is given by the ansatz space H_N and the second level is formed by the space Z. However, in this paper, the decomposition of the temporarily enriched ansatz space $\tilde{H}_N$ is only performed to define the error indicators and estimator.

The next lemma shows that it is necessary to study the behavior of the parameters c_1, c_2 and σ to obtain bounds for the efficiency and reliability of the presented error estimator. A proof of this lemma can be found, e.g., in [12].

Lemma 2.1 *Let the assumptions (A1) and (A2) be satisfied. Then, there holds*

$$\frac{1}{C_2} \sum_{j=1}^{J} \theta_j^2 \leq \|u - u_N\|_A^2 \leq \frac{1}{C_1(1-\sigma^2)} \sum_{j=1}^{J} \theta_j^2 \,.$$

3 A–posteriori error estimator for the hypersingular integral equation

In this section we consider the hypersingular integral equation, i.e., $A = D$ in (1.3), on open and closed curves Γ. We start with the model problem on an open curve and then we extend the analysis to closed curves.

3.1 Hypersingular integral equation on an open curve Γ

The problem under consideration reads: *given $f \in H^{-1/2}(\Gamma)$, find $u \in \tilde{H}^{1/2}(\Gamma)$ such that*

$$< Du, v >_{L^2(\Gamma)} = < f, v >_{L^2(\Gamma)} \quad \forall\, v \in \tilde{H}^{1/2}(\Gamma)\,. \tag{3.1}$$

Now, we define the finite dimensional spaces H_N and $Z := Z_1 + \cdots + Z_J$ for the *hp*–version BEM applied to (3.1). Let $\mathcal{T}_h := \{\Gamma_1, \ldots, \Gamma_J\}$ be a set of non–intersecting straight line pieces of Γ such that $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$. This mesh is not required to be quasi–uniform. We also note that we consider straight elements only for ease of presentation. Since our analysis is based on affine mappings of the elements onto a reference element this theory can be extended to meshes with curved elements by using smooth mappings. Therefore, only for presentation we assume that Γ is a polygon.

We consider the piecewise polynomial space

$$H_N := S_{h,p}(\Gamma) = \{\phi \in C^0(\Gamma); \phi|_{\partial\Gamma} = 0, \phi|_{\Gamma_j} \in \mathcal{P}_{p_j}(\Gamma_j), j = 1, \ldots, J\}\,,$$

where $p := (p_1, \ldots, p_J)$ is the vector which specifies the degrees of the polynomials on each element Γ_j and $\mathcal{P}_{p_j}(\Gamma_j)$ denotes the space of polynomials of degree p_j on Γ_j. The locally enriched space is defined by

$$\tilde{H}_N := S_{h,p}(\Gamma) + Z_1 + \cdots + Z_J\,, \tag{3.2}$$

where $Z_j = \mathcal{P}_{p_j+1}(\Gamma_j)\backslash\mathcal{P}_{p_j}(\Gamma_j)$, $j = 1, \ldots, J$, consists of bubble functions of degree $p_j + 1$: $\psi \in Z_j$ vanishes at the endpoints of Γ_j. The functions of Z_j are assumed to be extended by zero to the whole curve Γ. The space $\mathcal{P}_{p_j}(\Gamma_j)$ is considered to be spanned by affine images, from the reference interval $I := [-1, 1]$ onto Γ_j, of the shape functions $\psi_0, \psi_1, \psi_2, \ldots, \psi_{p_j}$, where $\psi_0(x) = \frac{1-x}{2}$, $\psi_1(x) = \frac{1+x}{2}$ and

$$\psi_j(x) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{x} L_{j-1}(t)\,dt\,, \quad 2 \le j \le p_j\,,$$

with $L_{j-1}(t)$ being the Legendre polynomial of degree $j-1$. Hereafter, by using the affine transformation $T_j : I \to \Gamma_j$, we write $\tilde{\phi}_j = \phi_j \circ T_j$ for functions ϕ_j defined on Γ_j. Also, $\|\cdot\|_D$ denotes the norm induced by the hypersingular operator D.

Theorem 3.1 *There exist positive constants c_1 and c_2 such that for the error estimator based upon the decomposition (3.2), there holds*

$$c_1 \sum_{j=1}^{J} \theta_j^2 \le \|u - u_N\|_D^2 \le \frac{c_2}{1-\sigma^2}(1 + \log p_{max})^2 \sum_{j=1}^{J} \theta_j^2\,.$$

Here, $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the solution u of (3.1), σ is the saturation parameter and $p_{max} = max\{p_1, \ldots, p_J\}$.

In order to prove Theorem 3.1, it suffices to prove the stability of the decomposition (3.2), see Lemma 2.1. To do this, we need the following technical lemma which proves the stability of the decomposition of the second level Z.

Lemma 3.1 *There exist positive constants c_1 and c_2, independent of h and p, such that*

$$c_1(1 + \log p_{max})^{-2} \sum_{j=1}^{J} \|\phi_j\|_D^2 \le \|\phi_1^*\|_D^2 \le c_2 \sum_{j=1}^{J} \|\phi_j\|_D^2$$

for all $\phi_1^* = \sum_{j=1}^{J} \phi_j \in Z$, where $\phi_j \in Z_j$, $j = 1, \ldots, J$. Here $p_{max} = max\{p_1, \ldots, p_j\}$.

Proof: First we deduce that

$$\|\phi_1^*\|_D^2 \le C\,\|\phi_1^*\|_{\tilde{H}^{1/2}(\Gamma)}^2 \le C \sum_{j=1}^{J} \|\phi_j\|_{\tilde{H}^{1/2}(\Gamma_j)}^2\,,$$

which is the upper bound in the assertion. Here, for the second inequality, we used Lemma 3.2 in [19] (see also Lemma 1 in [6]).

On the other hand there holds

$$\sum_{j=1}^{J} \|\phi_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le C \sum_{j=1}^{J} \|\tilde{\phi}_j\|^2_{\tilde{H}^{1/2}(I)}$$

$$\le C \sum_{j=1}^{J} (1+\log(p_j+1))^2 \|\tilde{\phi}_j\|^2_{H^{1/2}(I)} \le C(1+\log p_{max})^2 \sum_{j=1}^{J} \|\tilde{\phi}_j\|^2_{H^{1/2}(I)}\,, \tag{3.3}$$

where $\tilde{\phi}_j = \phi_j \circ T_j$. For the first inequality we used that the $\tilde{H}^{1/2}$–norm scales with a constant that is independent of the element size under affine transformations, see Lemma 3.1 in [15] and Lemma 2 in [6]. The second inequality is Theorem 6.5 in [1]. Note that $\tilde{\phi}_j(-1) = \tilde{\phi}_j(+1) = 0$. Therefore, we can apply the Poincaré–Friedrichs inequality and obtain

$$\sum_{j=1}^{J} \|\tilde{\phi}_j\|^2_{H^{1/2}(I)} \le C \sum_{j=1}^{J} |\tilde{\phi}_j|^2_{H^{1/2}(I)} \le C \sum_{j=1}^{J} |\phi_j|^2_{H^{1/2}(\Gamma_j)} \le C\,|\phi_1^*|^2_{H^{1/2}(\Gamma)}\,. \tag{3.4}$$

The last inequality is a direct application of the definition of the Sobolev–Slobodeckij seminorm $|\cdot|_{H^{1/2}(\Gamma)}$.

Finally, by combining (3.3) and (3.4) and by using the fact that D is positive definite on $\tilde{H}^{1/2}(\Gamma)$, the proof is finished. □

Using the previous result we can prove the stability of the two–level decomposition (3.2).

Lemma 3.2 *There exist positive constants c_3 and c_4, independent of h and p, such that*

$$c_3(1+\log p_{max})^{-2} \sum_{j=0}^{J} a(\phi_j,\phi_j) \le a(\phi,\phi) \le c_4 \sum_{j=0}^{J} a(\phi_j,\phi_j)$$

for all $\phi = \sum_{j=0}^{J} \phi_j \in \tilde{H}_N$, where $\phi_0 \in H_N$ and $\phi_j \in Z_j$, $j = 1,\dots,J$.

Proof: First, in virtue of Lemma 3.1, we note that there holds

$$a(\phi,\phi) = \|\phi_0+\phi_1^*\|^2_D \le 2\,\{\|\phi_0\|^2_D + \|\phi_1^*\|^2_D\} \le C \sum_{j=0}^{J} a(\phi_j,\phi_j)\,,$$

which is the right inequality of the assertion. To prove the left inequality, we see that, by using Lemma 3.1 again, it is only necessary to prove that there exists $C > 0$ such that

$$C\,\{\|\phi\|^2_D + \|\phi_1^*\|^2_D\} \le \|\phi\|^2_D\,.$$

It suffices to show that $\|\phi_0\|_D \le C\,\|\phi\|_D$ since this implies with $\phi_1^* = \phi - \phi_0$ that $\|\phi_1^*\|_D \le C\,\|\phi\|_D$ by the triangle inequality. We prove that $|\phi_0|_{H^1(\Gamma)} \le C\,|\phi|_{H^1(\Gamma)}$ and

$\|\phi_0\|_{L^2(\Gamma)} \le C\,\|\phi\|_{L^2(\Gamma)}$. The desired inequality then follows by interpolating between $L^2(\Gamma)$ and $H_0^1(\Gamma)$, where $H_0^1(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ within $H^1(\Gamma)$. To prove $|\phi_0|_{H^1(\Gamma)} \le C\,|\phi|_{H^1(\Gamma)}$ we show that $|\phi_0|^2_{H^1(\Gamma_j)} \le C\,|\phi|^2_{H^1(\Gamma_j)}$ and then sum over j.

On the reference interval I, we represent

$$\tilde{\phi}(x) = \left(\phi|_{\Gamma_j} \circ T_j\right)(x) = \sum_{i=0}^{p+1} c_i \psi_i(x)\,.$$

Here, the functions ψ_j, $j = 0, \ldots, p+1$, are the shape functions on I as defined at the beginning of this section. Then,

$$\tilde{\phi}_0(x) := \left(\phi_0|_{\Gamma_j} \circ T_j\right)(x) = c_0\psi_0(x) + c_1\psi_1(x) + \sum_{i=2}^{p} c_i\psi_i(x)$$

and

$$\tilde{\phi}_1^*(x) := \left(\phi_1^*|_{\Gamma_j} \circ T_j\right)(x) = c_{p+1}\psi_{p+1}(x)\,.$$

Since

$$\frac{d\tilde{\phi}_0}{dx} = -\frac{1}{2}c_0 + \frac{1}{2}c_1 + \sum_{i=2}^{p} c_i \sqrt{\frac{2i-1}{2}}\,L_{i-1}(x)$$

and

$$\frac{d\tilde{\phi}_1^*}{dx} = c_{p+1}\sqrt{\frac{2p+1}{2}}\,L_p(x)$$

we find that $\frac{d\tilde{\phi}_0}{dx}$ and $\frac{d\tilde{\phi}_1^*}{dx}$ are orthogonal in $L^2(I)$. Therefore,

$$|\phi|^2_{H^1(\Gamma_j)} = |\phi_0|^2_{H^1(\Gamma_j)} + |\phi_1^*|^2_{H^1(\Gamma_j)}\,,$$

which shows that $|\phi_0|^2_{H^1(\Gamma_j)} \le C\,|\phi|^2_{H^1(\Gamma_j)}$ and thus, $|\phi_0|_{H^1(\Gamma)} \le C\,|\phi|_{H^1(\Gamma)}$.

In order to prove that $\|\phi_0\|_{L^2(\Gamma)} \le C\,\|\phi\|_{L^2(\Gamma)}$, we again consider an element Γ_j and transform onto the reference element I. We consider the normalized shape functions $\psi_i^* := \psi_i/\|\psi_i\|_{L^2(I)}$, $i = 0, \ldots, p+1$. Then, we can represent

$$\tilde{\phi}_0(x) := \left(\phi_0|_{\Gamma_j} \circ T_j\right)(x) = v_0\psi_0^*(x) + v_1\psi_1^*(x) + \sum_{i=2}^{p} v_i\psi_i^*(x)$$

and

$$\tilde{\phi}(x) := \left(\phi|_{\Gamma_j} \circ T_j\right)(x) = v_0\psi_0^*(x) + v_1\psi_1^*(x) + \sum_{i=2}^{p+1} v_i\psi_i^*(x)\,.$$

There holds $\|\tilde{\phi}_0\|^2_{L^2(I)} = v^T A v$ where A is a matrix of dimension $(p+1)\times(p+1)$ with entries given by

$$a_{ij} = \int_{-1}^{1} \psi_i^*(x)\psi_j^*(x)\,dx \quad i,j = 0, \ldots, p$$

and $v^T = (v_0, \ldots, v_p)$. The matrix A can be explicitly calculated by using the orthogonality of the Legendre polynomials (see [14]) and we get

$$A = \begin{bmatrix} 1 & c_0 & b_0 & -b_1 & & & & \\ c_0 & 1 & b_0 & b_1 & & & & \\ b_0 & b_0 & 1 & 0 & b_2 & & & \\ -b_1 & b_1 & 0 & 1 & 0 & b_3 & & \\ & & b_2 & 0 & 1 & \ldots & \ldots & \\ & & & \ldots & \ldots & \ldots & \ldots & b_{p-2} \\ & & & & \ldots & \ldots & 1 & 0 \\ & & & & & b_{p-2} & 0 & 1 \end{bmatrix}$$

where $c_0 = \frac{1}{2}$, $b_0 = -\sqrt{\frac{5}{8}}$, $b_1 = -\sqrt{\frac{7}{40}}$ and

$$b_j = -\frac{1}{2}\sqrt{\frac{(2j-3)(2j+5)}{(2j-1)(2j+3)}} \quad \forall\, j \geq 2\,.$$

Moreover, it follows that

$$\|\tilde{\phi}\|^2_{L^2(I)} = \tilde{u}^T \begin{bmatrix} A & b^T \\ b & 1 \end{bmatrix} \tilde{u}\,,$$

where $\tilde{u}^T = (v_0, \ldots, v_p, v_{p+1})$ and $b = (0, 0, \ldots, b_{p-1}, 0)$.

Therefore, to show that $\|\phi_0\|_{L^2(\Gamma_j)} \leq C\,\|\phi\|_{L^2(\Gamma_j)}$, we need to bound the maximum eigenvalue of the following generalized eigenvalue problem:

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} A & b \\ b^T & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

This eigenvalue problem was already analyzed in Lemma 2 in [14]. The structure of the matrix A is slightly different from the matrix A_{p+1} in that paper, due to the presence of the entry d_0, but the proof given in [14] can be applied here without modifications. Therefore, by using the properties of the sequence b_j and following [14], we deduce that λ is bounded which shows that $\|\phi_0\|^2_{L^2(\Gamma_j)} \leq C\,\|\phi\|^2_{L^2(\Gamma_j)}$. This gives $\|\phi_0\|^2_{L^2(\Gamma)} \leq C\,\|\phi\|^2_{L^2(\Gamma)}$ and the proof of the theorem is finished by interpolation.

□

Remark 3.1. By using the procedure described in the proof of the above lemma, we note that there also holds $|\phi_1^*|_{H^{1/2}(\Gamma)} \leq C\,|\phi|_{H^{1/2}(\Gamma)}$. This fact will be used later to extend the results presented in this section to closed curves.

To end this section, we note again that Theorem 3.1 follows by using Lemmata 2.1 and 3.2.

3.2 Hypersingular integral equation on a closed curve Γ

In what follows, we extend the results of the last section to the hypersingular equation on a closed curve Γ. Given $f \in H^{-1/2}(\Gamma)$, we consider the problem: *find* $u \in H^{1/2}(\Gamma)/\mathbf{R}$ *such that*

$$< Du, v >_{L^2(\Gamma)} = < f, v >_{L^2(\Gamma)} \quad \forall\, v \in H^{1/2}(\Gamma)/\mathbf{R}\,. \tag{3.5}$$

On closed curves, the kernel of the hypersingular operator D consists of constant functions. Therefore, D is no longer positive definite on $\tilde{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$ in this case, there only holds

$$< Dv, v >_{L^2(\Gamma)} \geq C\, |v|^2_{H^{1/2}(\Gamma)} \quad \forall\, v \in H^{1/2}(\Gamma)\,. \tag{3.6}$$

In fact, the hypersingular integral operator defines a norm in $H^{1/2}(\Gamma)/\mathbf{R}$ that is equivalent to the seminorm $|\cdot|_{H^{1/2}(\Gamma)}$ on $H^{1/2}(\Gamma)$. In order to have unique solvability of our Galerkin system, we need to fix the constant function in the ansatz space. We choose H_N to be a subspace of $H_0^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) :< v, 1 >_{L^2(\Gamma)} = 0\}$. However, since we want to use standard bubble functions ψ_j to construct the local subspaces Z_j, the enriched ansatz space $\tilde{H}_N$ will not always be a subspace of $H_0^{1/2}(\Gamma)$. Within $\tilde{H}_N$ the constant is fixed by combining the condition of integral mean zero within H_N and requiring for functions within Z to be zero at the nodes. Therefore we choose the ansatz space

$$H_N := S_{h,p}(\Gamma) \cap H_0^{1/2}(\Gamma)$$

and the two–level decomposed enriched space

$$\tilde{H}_N := H_N + Z_1 + \cdots + Z_J\,. \tag{3.7}$$

Here $Z_j := \mathcal{P}_{p_j+1}(\Gamma_j) \backslash \mathcal{P}_{p_j}(\Gamma_j)$, $j = 1, \ldots, J$. Note that $Z_j \subset H_0^{1/2}(\Gamma)$ only if $p_j + 1 > 2$, since ψ_{p_j+1} is a linear combination of the Legendre polynomials of degrees $p_j + 1$ and $p_j - 1$.

Analogously to the case of an open curve Γ, we obtain the following result.

Theorem 3.2 *There exist positive constants c_1 and c_2 such that for the error estimator based upon the decomposition (3.7), there holds*

$$c_1 \sum_{j=1}^{J} \theta_j^2 \leq \|u - u_N\|_D^2 \leq \frac{c_2}{1 - \sigma^2} (1 + \log p_{max})^2 \sum_{j=1}^{J} \theta_j^2\,.$$

Here, $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the solution u of (3.5), σ is the saturation parameter and $p_{max} = max\{p_1, \ldots, p_J\}$.

Proof: It suffices to prove the stability of the decomposition (3.7), i.e., there exist positive constants c_1 and c_2, independent of h and p, such that

$$c_1(1+\log p_{max})^{-2}\sum_{j=0}^{J} a(\phi_j,\phi_j) \le a(\phi,\phi) \le c_2\sum_{j=0}^{J} a(\phi_j,\phi_j) \tag{3.8}$$

for all $\phi = \sum_{j=0}^{J}\phi_j \in \tilde{H}_N$, where $\phi_0 \in H_N$ and $\phi_j \in Z_j$, $j=1,\ldots,J$.

First, we note that

$$c_1(1+\log p_{max})^{-2}\sum_{j=1}^{J} \|\phi_j\|_D^2 \le \|\phi_1^*\|_D^2 \le c_2\sum_{j=1}^{J} \|\phi_j\|_D^2 \tag{3.9}$$

holds with positive constants c_1 and c_2, independent of h and p, for all $\phi_1^* = \sum_{j=1}^{J}\phi_j \in Z$, where $\phi_j \in Z_j$, $j=1,\ldots,J$. The right inequality follows by the same arguments used in the proof of Lemma 3.1. Also, by using (3.3), (3.4) and (3.6) the left inequality holds.

On the other hand, we note that the right inequality in (3.8) follows from (3.9) and the triangle inequality. Finally, taking into account Remark 3.1, we have

$$a(\phi_1^*,\phi_1^*) \le C\,|\phi_1^*|^2_{H^{1/2}(\Gamma)} \le C\,|\phi|^2_{H^{1/2}(\Gamma)} \le C\,a(\phi,\phi)$$

and then, the left inequality in (3.8) follows by using the triangle inequality. □

4 A–posteriori error estimator for the weakly singular integral equation

In this section we extend the results given in Section 3 to the weakly singular equation, i.e., $A=V$ in (1.3), both on open curves and closed curves Γ. Let $f \in H^{1/2}(\Gamma)$ be given. The weakly singular problem reads as follows: *find* $u \in \tilde{H}^{-1/2}(\Gamma)$ *such that*

$$< Vu, v >_{L^2(\Gamma)} = < f, v >_{L^2(\Gamma)} \quad \forall\, v \in \tilde{H}^{-1/2}(\Gamma)\,. \tag{4.1}$$

Since the problem is posed in $\tilde{H}^{-1/2}(\Gamma)$, continuity of the functions of the boundary element space is not required. Therefore, instead of using antiderivatives of Legendre polynomials, we take Legendre polynomials to define our finite dimensional approximation space. In what follows, Γ can be considered either as an open or a closed curve. Note again that $\tilde{H}^{-1/2}(\Gamma)$ coincides with $H^{-1/2}(\Gamma)$ on closed curves.

Let $\mathcal{T}_h := \{\Gamma_1,\ldots,\Gamma_J\}$ be a set of non–intersecting straight line pieces such that $\bar{\Gamma} = \cup_{j=1}^{J}\bar{\Gamma}_J$. We define the piecewise polynomial space $\mathcal{P}_{p_j-1}(\Gamma_j)$ spanned by affine images of the basis functions $\varphi_1, \varphi_2, \ldots, \varphi_{p_j}$, where $\varphi_1(x) = \frac{1}{2}$ and

$$\varphi_j(x) = \sqrt{\frac{2j-1}{2}}L_{j-1}(x)\,, \quad 2 \le j \le p_j\,.$$

Then, the ansatz space H_N for the Galerkin scheme associated with (4.1) is

$$H_N := \{\phi \in L^2(\Gamma) : \phi|_{\Gamma_j} \in \mathcal{P}_{p_j-1}(\Gamma_j)\,,\ j = 1, \ldots J\}\,.$$

Consequently, we define the locally enriched space

$$\tilde{H}_N := H_N + Z_1 + \cdots + Z_J\,, \tag{4.2}$$

where $Z_j = \text{span}\,\{L_{p_j} \circ T_j^{-1}\}$, $j = 1, \ldots, J$, and the functions of Z_j are assumed to be extended by zero onto Γ. In what follows, $\|\cdot\|_V$ stands for the norm induced by the weakly singular operator V.

Theorem 4.1 *There exist positive constants c_1 and c_2 such that for the error estimator based upon the decomposition (4.2), there holds*

$$c_1 \sum_{j=1}^{J} \theta_j^2 \leq \|u - u_N\|_V^2 \leq \frac{c_2}{1-\sigma^2}(1 + \log p_{max})^2 \sum_{j=1}^{J} \theta_j^2\,.$$

Here, $u_N \in H_N$ is the Galerkin approximation of the solution u of (4.1), σ is the saturation parameter and $p_{max} = max\{p_1, \ldots, p_j\}$.

Proof: It is enough to prove the stability of the decomposition (4.2), i.e., there exist positive constants c_1 and c_2, independent of h and p, such that

$$c_1(1 + \log p_{max})^{-2} \sum_{j=0}^{J} a(\phi_j, \phi_j) \leq a(\phi, \phi) \leq c_2 \sum_{j=0}^{J} a(\phi_j, \phi_j)$$

for all $\phi = \sum_{j=0}^{J} \phi_j \in \tilde{H}_N$, where $a(\cdot,\cdot)$ denotes the bilinear form induced by the weakly singular operator V, $\phi_0 \in H_N$ and $\phi_j \in Z_j$, $j = 1, \ldots, J$. We note that the spaces H_N and Z_j, $j = 1, \ldots, J$, of (4.2) can be obtained by taking the derivatives with respect to the arc length of the functions of the corresponding spaces in (3.2) and (3.7) (for open and closed curves, respectively). Therefore, the required stability of the decomposition (4.2) is obtained by using Lemma 3.2 (for open curves) and (3.8) (for closed curves) together with the relation (see [13])

$$< Du, v >_{L^2(\Gamma)} = < Vu', v' >_{L^2(\Gamma)} \quad \forall\, u, v \in \tilde{H}^{1/2}(\Gamma)\,.$$

Here the prime indicates differentiation with respect to the arc length. □

Remark 4.1. From the proof of the previous theorem it is obvious that the appearing linear systems in the discretizations of weakly singular and hypersingular operators are strongly related, as long as one chooses related ansatz spaces as we did here. Then not only the analyzes for both operators coincide, but also the numerical experiments in both cases can be expected to show the same behavior. Therefore, in the numerical Section 6 we only report on numerical results for the weakly singular operator.

5 The adaptive algorithms

In this section we describe three adaptive algorithms for the boundary integral equations under consideration. The first algorithm is a pure p–adaptive strategy that keeps a fixed mesh for the ansatz space. The other two algorithms also consider refinements of the mesh on Γ, they are hp–adaptive algorithms. Within the so–called two fold hp–adaptive algorithm every element is either bisected or its polynomial degree is increased by 1 during every adaption step. This strategy leads to high polynomial degrees and simulates a standard hp–method with geometric mesh refinement, cf. [2, 8]. However, since the bisection yields elements of the same length, this method relates to a geometrically graded mesh with mesh grading parameter 0.5. Such a method converges exponentially but not in an optimal way, see Section 6 below. We therefore study a second hp–adaptive strategy that avoids increasing polynomial degrees in every adaption step. This method is three fold in the sense that the error indicators are classified by two parameters. Small indicators do not lead to mesh refinement or increase of polynomial degrees. Indicators of medium values lead to increasing polynomial degrees and large error indicators force to bisect elements.

The p– and hp–adaptive algorithms

Define an initial ansatz space H_N, prescribe an error tolerance ϵ and steering parameters $\delta_1 \in (0,1)$ for the p– and the two fold hp–adaptive algorithms or δ_1, δ_2 with $0 < \delta_1 < \delta_2 < 1$ for the three fold hp–adaptive algorithm. Then, the adaptive algorithms read as follows:

1. Galerkin solution. Compute the Galerkin solution $u_N \in H_N$.

2. Error estimator. Calculate the error indicators θ_j, the estimator $\delta := (\sum_{j=1}^{J} \theta_j^2)^{1/2}$ and compute $\theta_{max} = \max \theta_j$. **Stop** if $\delta \leq \epsilon$.

3. Adaption step.

(a) **p–adaptive algorithm:** If $\theta_j > \delta_1 \theta_{max}$, increase by 1 the polynomial degree on Γ_j.

(b) **two fold hp–adaptive algorithm:** If $\theta_j > \delta_1 \theta_{max}$, split the element Γ_j into two elements of the same size and inherit the polynomial degree. Otherwise, increase the polynomial degree by 1 on Γ_j.

(c) **three fold hp–adaptive algorithm:** If $\delta_1 \theta_{max} \leq \theta_j \leq \delta_2 \theta_{max}$, increase by 1 the polynomial degree on Γ_j. If $\theta_j > \delta_2 \theta_{max}$, split the element Γ_j into two elements of the same size and inherit the polynomial degree. Do nothing on Γ_j if $\theta_j < \delta_1 \theta_{max}$.

Define the corresponding adapted space H_N and **goto** *1.*

6 Numerical Results

In what follows we present numerical experiments for the p– and hp–adaptive schemes presented in Section 3. We only consider the weakly singular integral equation, cf. Remark 4.1, and note that the underlying situation is the Dirichlet boundary value problem for the L–shaped domain Ω shown in Figure 1,

$$-\Delta u = 0 \quad \text{in } \Omega\,,$$

$$u = f \quad \text{on } \Gamma = \partial\Omega\,,$$

where f is chosen such that $u(x, y) = Im(z^{2/3})$ for $z = x + i\,y$. It is well known that this problem can be reformulated as the weakly singular equation (4.1) (with different right hand side f). We recall that the finite dimensional subspaces of $\tilde{H}^{-1/2}(\Gamma)$ are constructed by affine images of scaled Legendre polynomials on each subinterval of the boundary element mesh.

We always start with an uniform mesh consisting of 8 elements of equal length. As already noted, the p–adaptive algorithm just uses this fixed mesh and temporarily increases the polynomial degree by 1 on each element to compute the error indicators. From [15] we recall that the convergence of the p– version of the boundary element method on quasi–uniform meshes is twice as fast as the h–version, provided the solution of the problem has a corner singularity as in our model problem.

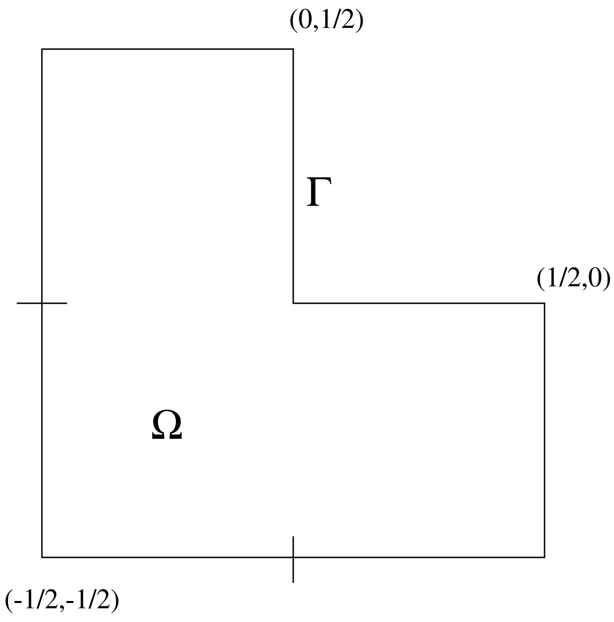

Figure 1: *The L–shaped computational domain.*

The two and three fold hp–adaptive algorithms combine mesh refining with a suitable polynomial degree distribution. In fact, these algorithms produce a sequence of geometric meshes towards the incoming corner $(0,0)$ with a grading parameter equal to 0.5. As proved in [2, 8] an exponential rate of convergence is expected.

In the numerical experiments we use a steering parameter $\delta_1 = 0.8$ for the p–adaptive and the two fold hp–adaptive algorithms. For the three fold hp–adaptive algorithm, the parameters $\delta_1 = 0.2$ and $\delta_2 = 0.8$ are chosen. Figure 2 presents the error in energy norm for different methods. The errors obtained by the adaptive algorithms are much smaller than those of the uniform h– and p–versions. Figure 3 demonstrates, for this model problem, that the hp–adaptive algorithms yield an exponential rate of convergence. Here we plot the logarithm of the errors against the square root of N. The almost straight lines confirm the convergence $\|u - u_N\|_V \le C \exp(-b\sqrt{N})$ which is in agreement with Theorem 4.3 in [8]. The polynomial degree distributions produced by the adaptive schemes, after a given number of refinement steps, are shown in Figures 10, 11 and 12. Note that, for this example, the three fold hp–adaptive algorithm almost amounts to an h–adaptive scheme, see Figure 12. The highest polynomial degree after four refinement steps is one. Nevertheless, the convergence is exponential in this range of unknowns as it is underlined by Figure 3.

Figure 4 shows the behavior of the error estimators. The estimators for the hp–adaptive schemes are very close to the errors in energy norm. However, for the p–adaptive scheme the reliability deteriorates for higher polynomial degrees. This behavior appears to result from an increase of the saturation parameter σ, cf. Figure 5. Indeed, in this situation the saturation assumption (A2) does not seem to be uniformly satisfied. On the other hand, our hp–strategies always yield low polynomial degrees close to the corner singularity. This is the region of large errors in this example, and increasing here the low polynomial degrees by 1 this leads to a small saturation parameter, cf. again Figure 5. This explains the good quality of our error estimator for the hp–adaptive strategies. The effectivity indices (error estimator divided by error in energy norm) are presented in Figure 6. Note again the good efficiency of the estimators for the hp–adaptive algorithms.

As we mentioned, the p–adaptive algorithm looses reliability when the polynomial degrees are increased. We also noted that this is due to an increase of the saturation parameter σ. To avoid this violation of a uniform saturation property we propose an alternative strategy to compute an error estimator (the indicators to steer the adaption are taken as before). The error indicators for the new estimator are computed by using an overlapping decomposition of an ansatz space that is enriched by halfing the mesh size (in contrast to increasing the polynomial degrees and keeping the mesh as before). By noting that the convergence of the quasi–uniform hp–method is like $h^{\alpha} p^{-2\alpha}$ for some positive α (see [15]) we expect a much better saturation when halfing h than when increasing p by 1. Since we do not use the new indicators to steer the adaption process we use a different decomposition that is uniformly stable (independent of h and p for quasi–uniform meshes and degrees). This decomposition is obtained by using local overlapping spaces. In any way, a decomposition of the enriched space is necessary to reduce the sizes of the problems that must be solved to compute the indicators (otherwise we had to solve a problem in the full enriched space).

The new enriched space is defined by halfing the mesh size and keeping the polynomial degrees, i.e, we consider the space

$$\tilde{H}_N := H_{h/2,0}(\Gamma) \cup H_{h/2,q_1}(\Gamma'_1) \cup \cdots \cup H_{h/2,q_J}(\Gamma'_J)\,,$$

where J denotes the number of nodes of the mesh (in the interior of Γ if Γ is an open curve) with halfed size and Γ'_j is the union of the two elements that are adjacent to the node j. Here, $H_{h/2,0}(\Gamma)$ is the space of piecewise constant functions on the mesh with size $h/2$. Moreover, $H_{h/2,q_j}(\Gamma'_j)$ denotes the local subspace spanned by one Haar basis function on Γ'_j (being 1 on one element and -1 on the other element) and transformed Legendre polynomials from degree 1 up to q_j on each piece of Γ'_j (q_j is the inherited degree from the old space H_N). For analyses of overlapping decompositions for the quasi–uniform hp–version we refer to [7, 18, 17].

The behavior of the estimator based on this new space decomposition is shown in Figures 7, 8 and 9. From Figure 7 we observe that this estimator is much more reliable for the p–adaptive version than the previous estimator. However, we note that this new strategy is more expensive than the previous one since the local spaces defining the indicators are not one–dimensional anymore. We suggest to use the new estimator as a stopping criterion for the p–adaptive algorithm. It is only necessary to compute this estimator from time to time after several refinement steps. Figures 8 and 9 show that for the hp–adaptive algorithms the new estimator is also reliable and efficient. But since the previous estimator is already almost optimal, the new one is not worth being considered for these strategies in this example.

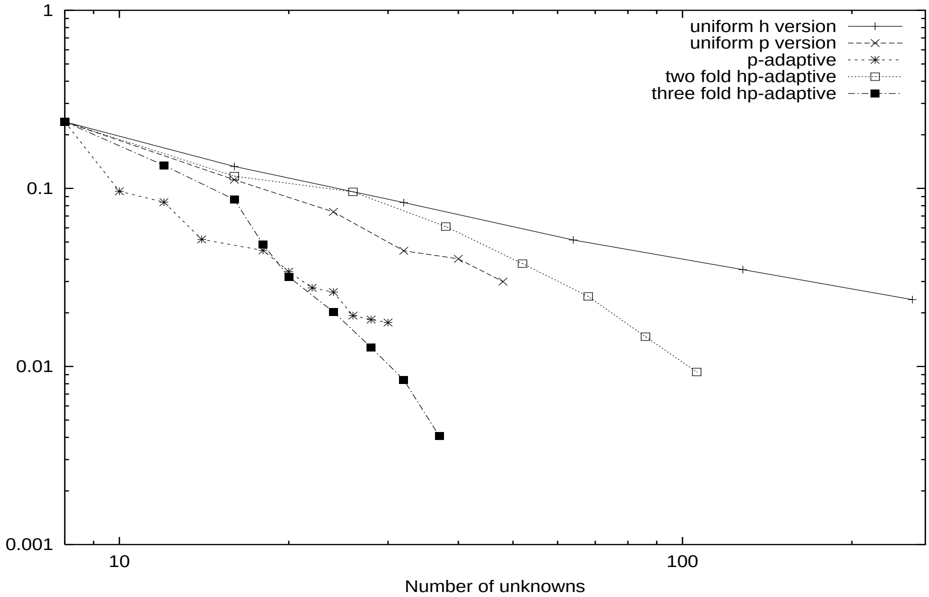

Figure 2: *Error in energy norm for different methods.*

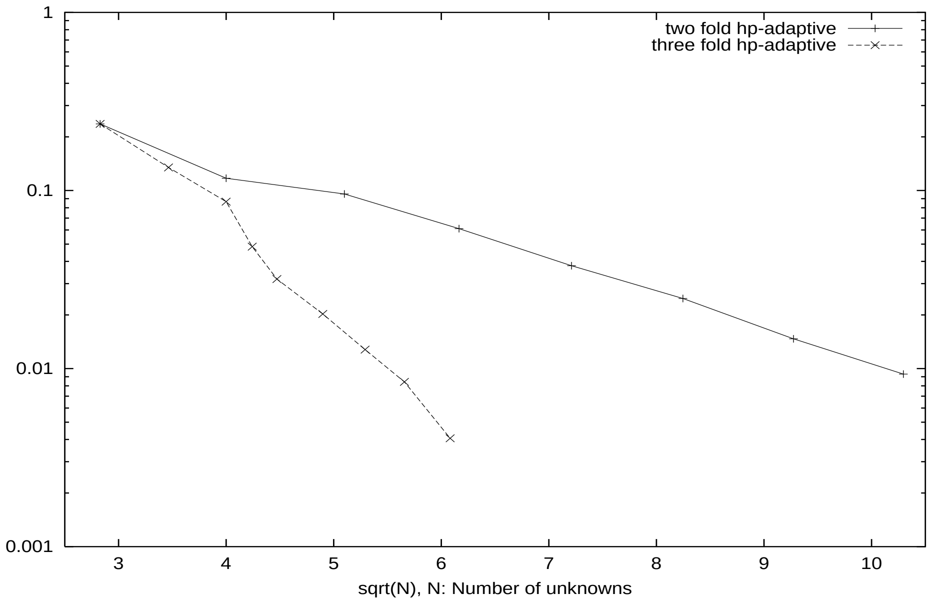

Figure 3: *Error in energy norm for the hp–adaptive methods.*

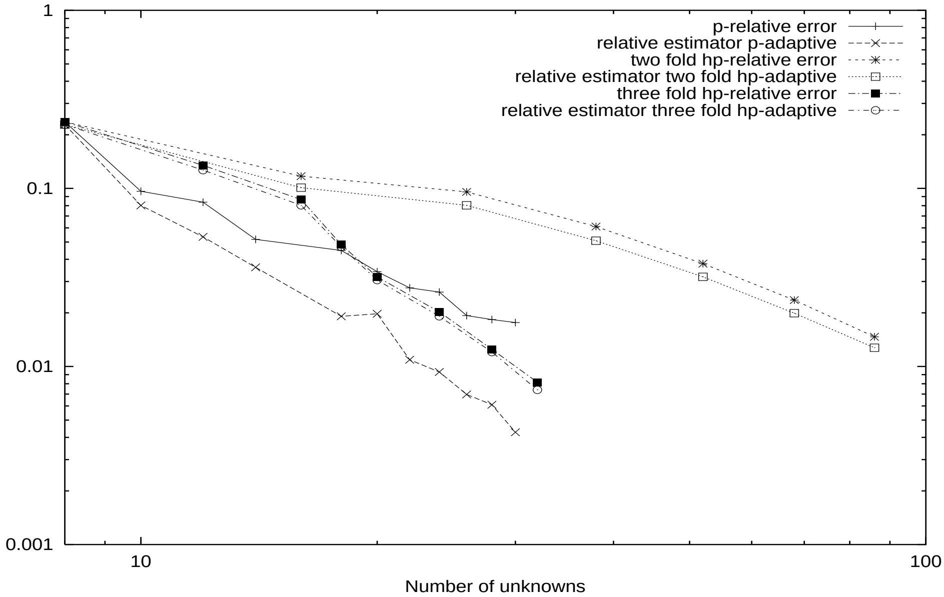

Figure 4: *Relative error in energy norm and relative estimators.*

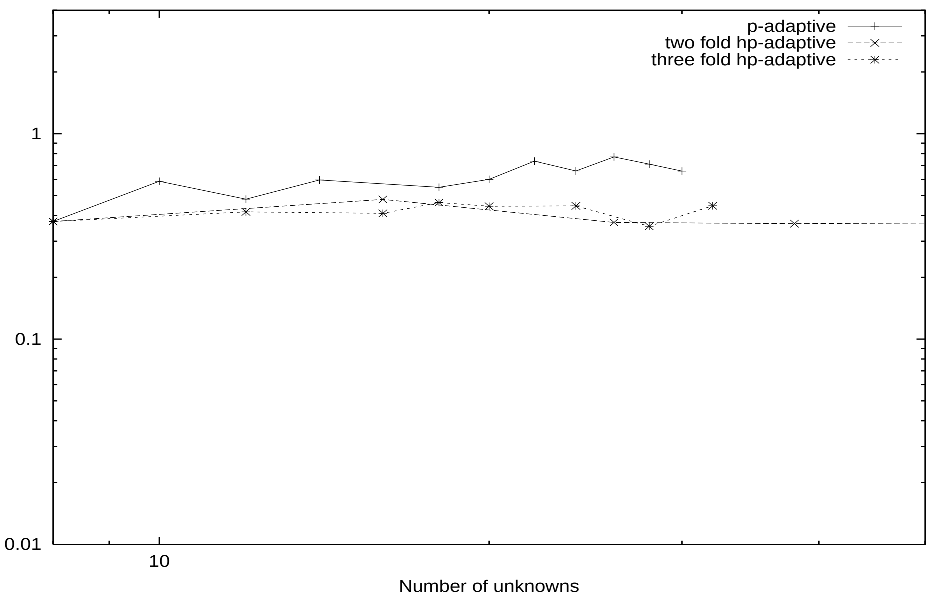

Figure 5: *Saturation parameter of the different algorithms.*

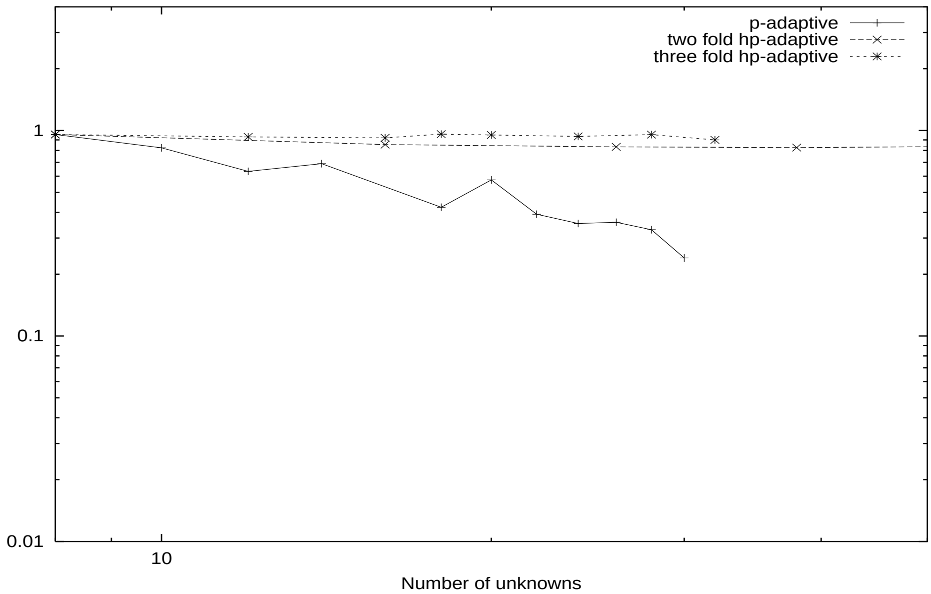

Figure 6: *Effectivity index of the different estimators.*

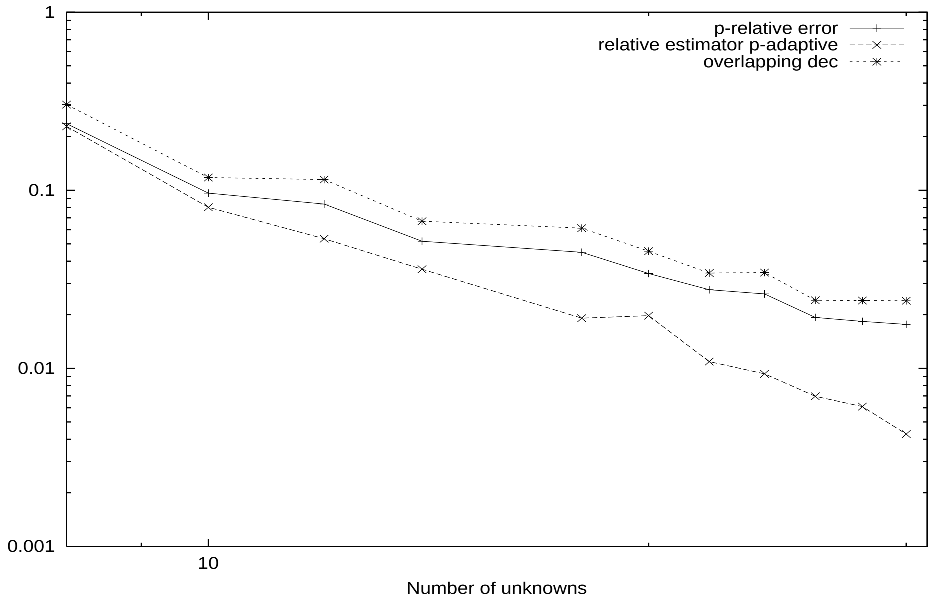

Figure 7: *Overlapping decomposition for the p–adaptive algorithm.*

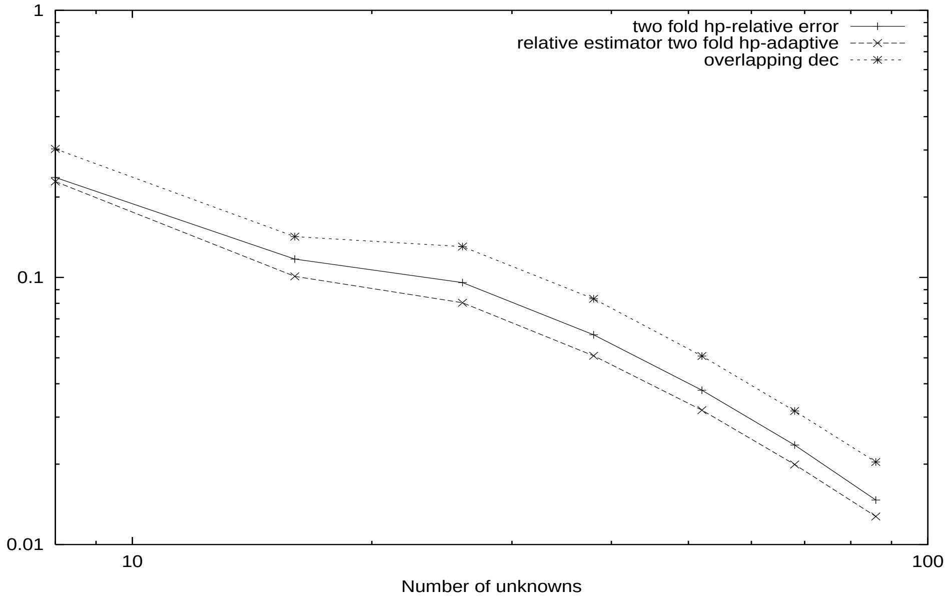

Figure 8: *Overlapping decomposition for the two fold hp–adaptive algorithm.*

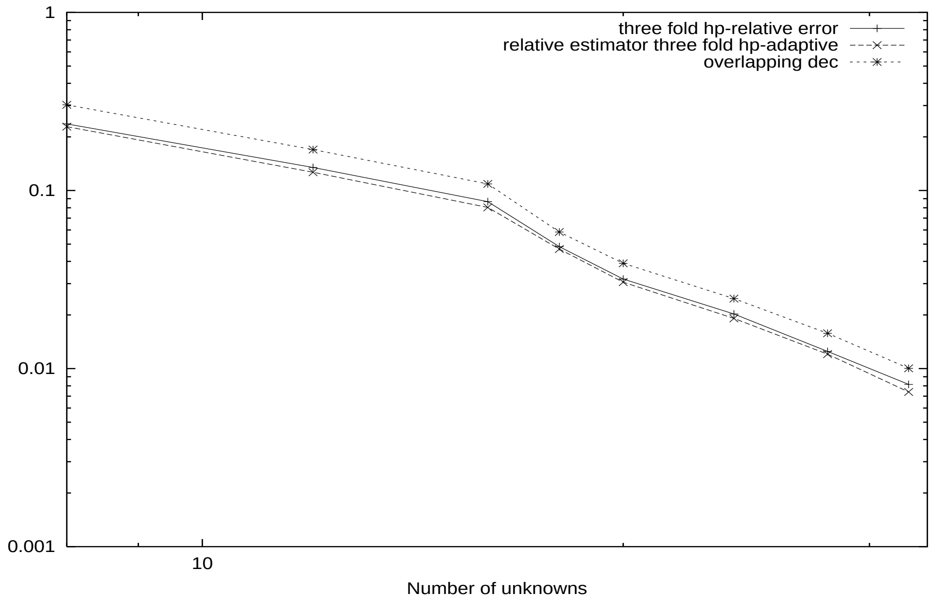

Figure 9: *Overlapping decomposition for the three fold hp–adaptive algorithm.*

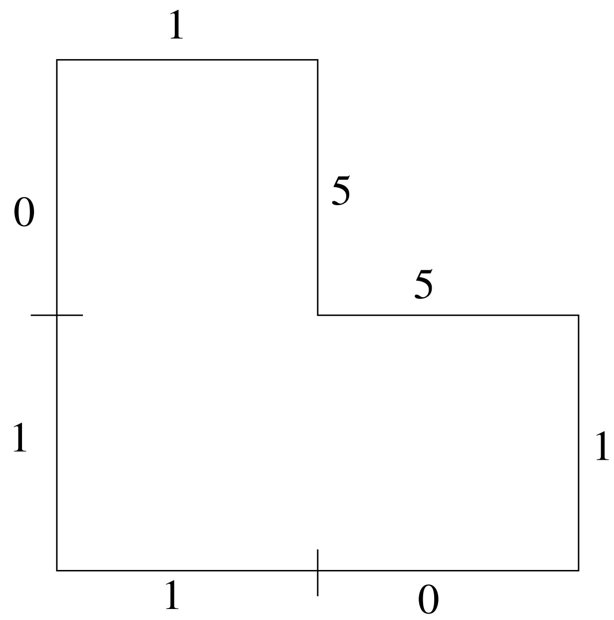

Figure 10: *The polynomial degrees after 6 p–adaptive refinements.*

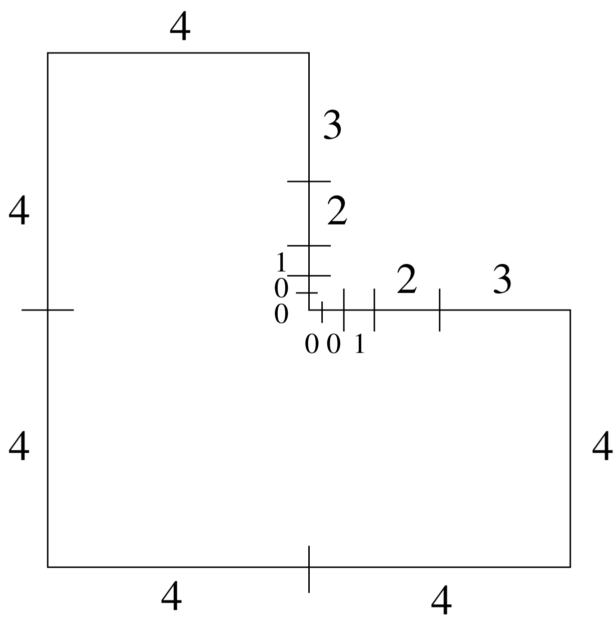

Figure 11: *The polynomial degrees after 4 two fold hp–adaptive refinements.*

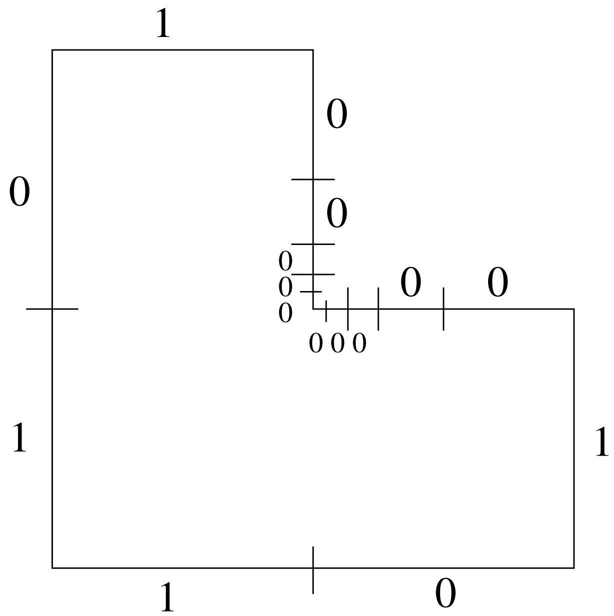

Figure 12: *The polynomial degrees after 4 three fold hp–adaptive refinements.*

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N. Heuer

Institut für Wissenschaftliche Datenverarbeitung, Universität Bremen, Postfach 330440, 28334 Bremen, Germany, e–mail: heuer@iwd.uni–bremen.de

and

Departamento de Ingeniería Matematica, Universidad de Concepción, Casilla 160-C, Concepción, Chile.

M. E. Mellado

Institut für Wissenschaftliche Datenverarbeitung, Universität Bremen, Postfach 330440, 28334 Bremen, Germany.

E. P. Stephan

Institut für Angewandte Mathematik, Universität Hannover, Welfengarten 1, 30167 Hannover, Germany. e–mail: stephan@ifam.uni–hannover.de