A p-adaptive algorithm for the BEM with hypersingular operators on surfaces

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Abstract. We propose a p-adaptive algorithm for the Galerkin method solving hypersingular integral operators on surfaces. The error indicators/estimators are based on projections of the actual error onto local subspaces. These subspaces are defined by decompositions of specially designed enriched ansatz spaces. Our algorithm uses different strategies for the refinement and the stopping criterion. The error estimator that stops the algorithm is based on an overlapping decomposition of an ansatz space that is defined by mesh refinement. The error indicators that steer the p-refinement are computed via an almost direct decomposition of an enriched ansatz space that uses the same mesh but higher polynomial degrees. Numerical results support the efficiency of our algorithm.

1 Introduction

In this paper we study a p-adaptive algorithm for the boundary element method on surfaces. As a model problem we consider the weak form of the hypersingular integral equation

$$\langle Du, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on a plane rectangular surface piece $\Gamma \subset \mathbf{R}^3$ where $f \in H^{-1/2}(\Gamma)$ is a given function. Here D is the hypersingular integral operator

$$Du(x) = \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma$$

which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [7]. For standard Sobolev spaces $L^2(\Gamma)$ and $H^1_0(\Gamma)$ the space $\tilde{H}^{1/2}(\Gamma)$ is an interpolation space between $L^2(\Gamma)$ and $H^1_0(\Gamma)$, and $H^{-1/2}(\Gamma)$ is the dual space of $\tilde{H}^{1/2}(\Gamma)$. The solution u of (1) is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathbf{R}^3 \setminus \bar{\Gamma}$, cf. [7].

To define our standard hp-ansatz space for the Galerkin method let $\mathcal{T}_h = \{\Gamma_1, \ldots, \Gamma_J\}$ be a regular mesh of quadrilaterals with $\cup_j \bar{\Gamma}_j = \bar{\Gamma}$. We consider the piecewise polynomial space

$$S_{h,p}(\Gamma) = \{\phi \in C^0(\Gamma);\ \phi|_{\partial\Gamma} = 0,\ \phi|_{\Gamma_j} \in \mathcal{P}_{p_j,q_j}(\Gamma_j), j = 1, \ldots, J\}.$$

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Here, p stands for the vector $((p_1, q_1), \ldots, (p_J, q_J))$ specifying the polynomial degrees on the elements in the two coordinate directions, and

$$\mathcal{P}_{p_j,q_j}(\Gamma_j) := \left\{\phi|_{\Gamma_j};\ \phi \in \operatorname{span}\{x_1^k x_2^l;\ k = 0, \ldots, p_j,\ l = 0, \ldots, q_j\}\right\}.$$

Then the Galerkin scheme for (1) reads as follows. *Find $u_N \in S_{h,p}(\Gamma)$ such that*

$$\langle Du_N, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \textit{for all} \quad v \in S_{h,p}(\Gamma). \tag{2}$$

Here, N refers to the dimension of $S_{h,p}(\Gamma)$.

The solution u of (1) behaves singularly at the edges and corners of Γ, cf. [8, 9]. Due to these singularities an adaptive strategy that refines the ansatz space only where the approximation is poor is more appropriate than uniform mesh refinements or uniform increase of the polynomial degrees. When appropriately combining mesh refinements and polynomial degree distributions in a non-uniform fashion, even an exponential rate of convergence is achievable, cf. [5].

In this paper we consider p-adaptive strategies that keep a fixed mesh for the ansatz space. By defining local error indicators that are based on two-level subspace decompositions we steer local p-refinements. Ausfuehren und Referenzen angeben.

An outline of the paper is as follows. In Section 2 we introduce our strategy to define error indicators and error estimators. Section 2.1 is devoted to an adaptive version that is based on local incease of the polynomial degrees. In Section 2.2 we consider a mesh refinement strategy that leads to better error estimators, whereas the previous idea is used to steer our adaptive algorithm. This algorithm is prescribed in Section 3. In Section 4 we collect various numerical results that underline the efficiency of the adaptive algorithm.

2 Error indicators and estimators for the hp-version

To introduce error indicators in an abstract way let us consider an ansatz space H_N of dimension N and an enriched space $\tilde{H}_N$ with $H_N \subset \tilde{H}_N$ that is decomposed like

$$\tilde{H}_N = Z_0 + Z_1 + \cdots + Z_J \tag{3}$$

where $Z_j \not\subset H_N$ for $j > 0$. Below we consider two different strategies to define enriched ansatz spaces and decompositions thereof. In the first case, which yields local error indicators that are used to steer the refinement, there holds $Z_0 = S_{h,p}(\Gamma)$. In the second case, Z_0 is only a low-dimensional subspace of $S_{h,p}(\Gamma)$.

Let $u_N \in H_N$ be the Galerkin approximation to the true solution u of (1), and let $\tilde{u}_N \in \tilde{H}_N$ be the improved approximation (obtained by projecting u with respect to $a(\cdot, \cdot)$ onto $\tilde{H}_N$). Then our *a-posteriori error estimator* is defined by $\left(\sum_{j=0}^{J} \theta_j^2\right)^{1/2}$, where

$$\theta_j := \|P_j(\tilde{u}_N - u_N)\|_D, \quad j = 0, \ldots, J,$$

are the *error indicators* and

$$P_j : \tilde{H}_N \to Z_j \quad \langle DP_j\phi, \psi\rangle_{L^2(\Gamma)} = \langle D\phi, \psi\rangle_{L^2(\Gamma)} \quad \text{for any } \psi \in Z_j$$

are the corresponding projection operators. Efficiency and reliability of this error estimator depend on the stability of the decomposition and on a saturation property, which are assumed as follows:

(A1) **Stability of the decomposition**: there exist two numbers c_1, $c_2 > 0$ such that

(i) for any $\phi \in \tilde{H}_N$ there exist $\phi_j \in Z_j$ $(j = 1, \ldots, J)$ with $\phi = \sum_{j=0}^{J} \phi_j$ such that

$$c_1 \sum_{j=0}^{J} \langle D\phi_j, \phi_j \rangle_{L^2(\Gamma)} \leq \langle D\phi, \phi \rangle_{L^2(\Gamma)},$$

(ii) for any $\phi \in \tilde{H}_N$ and any $\phi_j \in Z_j$ $(j = 1, \ldots, J)$ with $\phi = \sum_{j=0}^{J} \phi_j$ there holds

$$\langle D\phi, \phi \rangle_{L^2(\Gamma)} \leq c_2 \sum_{j=0}^{J} \langle D\phi_j, \phi_j \rangle_{L^2(\Gamma)}.$$

(A2) **Saturation assumption**: there exists $\sigma \in [0, 1)$ such that

$$\|u - \tilde{u}_N\|_D \leq \sigma \|u - u_N\|_D.$$

Remark. *Assumption* (A1) *is equivalent to bound the extremum eigenvalues of the so-called additive Schwarz operator* $P := \sum_{j=0}^{J} P_j$. *Condition* (i) *means that the minimum eigenvalue of* P *is bounded from below by* c_1 *and* (ii) *leads to a bound* c_2 *for the maximum eigenvalue of* P.

It is well known that the efficiency and reliability of the error estimator can be bounded by making use of (A1) and (A2), see, e.g., [2, 6]:

Lemma 1 *Let the assumptions* (A1) *and* (A2) *be fulfilled. Then there holds*

$$\frac{1}{c_2} \sum_{j=0}^{J} \theta_j^2 \leq \|u - u_N\|_D^2 \leq \frac{1}{c_1(1 - \sigma^2)} \sum_{j=0}^{J} \theta_j^2.$$

2.1 Local enrichment with direction control

We start considering an enriched ansatz space that is decomposed in such a way that one can measure the error on individual elements in different directions. In the notation of (3) we choose $Z_0 = S_{h,p}(\Gamma)$ the actual ansatz space and temporarily add bubble functions of higher degrees on the elements. Note that the additional bubble functions are only used to define the local error indicators. The bubble functions of higher degrees in x_1-direction are collected to form the subspaces Z_{j1}, $j = 1, \ldots, J$. They are used to measure the errors on the elements in x_1-direction. Accordingly, the polynomials of higher degrees in x_2-direction form the subspaces Z_{j2} and are used to estimate the errors in x_2-direction. The decomposition is as follows:

$$\tilde{S}_{h,p}(\Gamma) = S_{h,p}(\Gamma) + Z_{11} + Z_{12} + \cdots + Z_{J1} + Z_{J2}, \tag{4}$$

where

$$Z_{j1} := \begin{cases} \Big(\mathcal{P}_{p_j+1,q_j}(\Gamma_j) \setminus \mathcal{P}_{p_j,q_j}(\Gamma_j)\Big) \cap H_0^1(\Gamma_j) & \text{if } p_j > 1, q_j > 1 \\ \mathcal{P}_{p_j+1,2}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j > 1, q_j = 1 \\ \mathcal{P}_{2,q_j}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j = 1, q_j > 1 \\ \mathcal{P}_{3,2}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j = 1, q_j = 1 \end{cases}$$

is the local enrichment on Γ_j in x_1-direction and

$$Z_{j2} := \begin{cases} \left(\mathcal{P}_{p_j,q_j+1}(\Gamma_j) \setminus \mathcal{P}_{p_j,q_j}(\Gamma_j)\right) \cap H_0^1(\Gamma_j) & \text{if } p_j > 1, q_j > 1 \\ \mathcal{P}_{p_j,2}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j > 1, q_j = 1 \\ \mathcal{P}_{2,q_j+1}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j = 1, q_j > 1 \\ \mathcal{P}_{2,3}(\Gamma_j) \cap H_0^1(\Gamma_j) & \text{if } p_j = 1, q_j = 1 \end{cases}$$

is the local enrichment on Γ_j in x_2-direction. As basis functions for the subspaces Z_{j1}, Z_{j2} we take affine mappings of the tensor products

$$\psi_p(x_1)\psi_q(x_2) \quad \text{with} \quad \psi_p(x) := \int_{-1}^{x} L_{p-1}(t)\,dt \quad (p, q \geq 2)$$

defined on $[-1,1]^2$ where L_{p-1} is the Legendre polynomials of degree $p-1$.

The three typical cases that appear in the temporary increase of the polynomial degrees on a certain element are shown in Figure 1. Case (a) is the general case if the local polynomial degrees (denoted by p and q) are bigger than 1. We increase the degrees by 1 in each direction and take the local subspaces Z_{j1} and Z_{j2} as indicated. Here, we use the symbol (p,q) to indicate the full space of polynomials and p, q stands for a certain basis function of the corresponding degrees in x_1- and x_2-direction. Cases (b) and (c) deal with the situation that p or q is just 1. These situations are not covered by (a) and we need a special definition of the p-enrichment to obtain subspaces that measure the error in different directions (increasing $(p,q) = (1,1)$ just to $(2,2)$ would give only one additional bubble function that could not be used to get error indicators for different directions).

We note that one cannot expect (4) to be an efficient decomposition of the enriched space $\tilde{S}_{h,p}(\Gamma)$. It is known that the coupling of standard basis functions of high degrees is rather strong, for the FEM see, e.g., [1]. Therefore we consider improved decompositions by decoupling the spaces $S_{h,p}(\Gamma)$ and $Z := Z_{11} + Z_{12} + \cdots + Z_{J2}$. A full orthogonalization is equivalent to a Schur complement step and requires inverting the matrix A_2 that corresponds to the space Z. This orthogonalization induces a basis transformation that changes the basis of $S_{h,p}(\Gamma)$. We call the new space $T_{A_2}(S_{h,p}(\Gamma))$ and a projection of the error $\tilde{u}_N - u_N$ onto the new space does not vanish in general (whereas $P_0(\tilde{u}_N - u_N) = 0$ if $Z_0 = S_{h,p}(\Gamma)$). The corresponding decomposition is

$$\tilde{S}_{h,p}(\Gamma) = T_{A_2}(S_{h,p}(\Gamma)) + Z_{11} + Z_{12} + \cdots + Z_{J1} + Z_{J2}, \tag{5}$$

and the error estimator comprises a global contribution θ_0. From [3] we cite the following result.

Theorem 1 *There exist constants c_1, c_2 such that for the error estimator based upon the decomposition* (5) *there holds*

$$c_1 \left(\theta_0^2 + \sum_{j=1}^{J}\sum_{i=1}^{2} \theta_{ji}^2\right) \leq \|u - u_N\|_D^2 \leq \frac{c_2}{1-\sigma^2}(1 + \log p_{\max})^2 \left(\theta_0^2 + \sum_{j=1}^{J}\sum_{i=1}^{2} \theta_{ji}^2\right).$$

Here, $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the true solution u of (1), *$p_{\max} = \max\{p_1, q_1, \ldots, p_J, q_J\}$ and σ is the saturation parameter.*

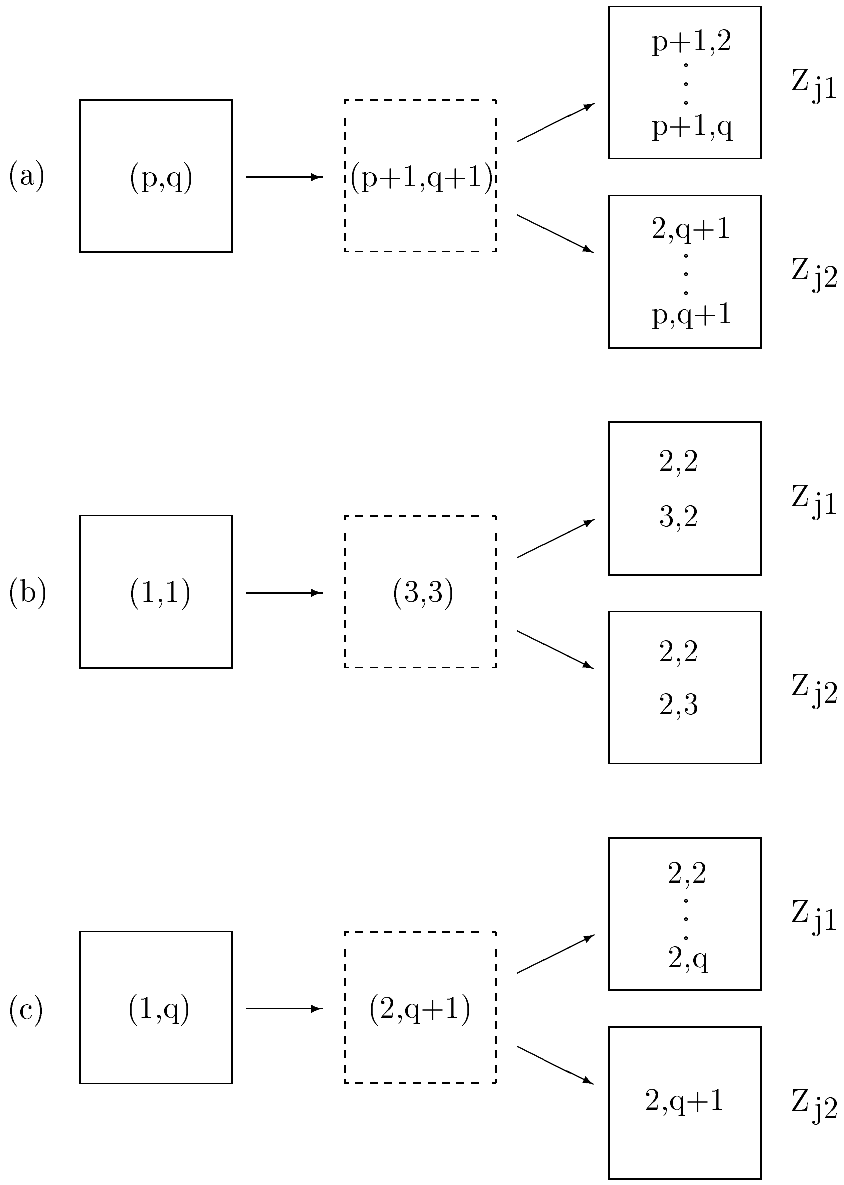

Figure 1: Strategies for local p-increase with direction control.

We note that, in general, the matrix A_2 that needs to be inverted for the Schur complement step is dense. Therefore, the transformation that is necessary to implement the decomposition (5) is not cheap. For that reason we study another transformation that is based only on a partial Schur complement step. Instead of inverting the full matrix A_2 we only take the block diagonal part C_2 of A_2 that corresponds to the decomposition

$$Z = Z_1 + \ldots + Z_J$$

with subspaces $Z_j = Z_{j1} \cup Z_{j2}$, $j = 1, \ldots, J$. In [3] it has been proved that this partial Schur complement also leads to a stable decomposition. Essential part of that analysis is the positive definiteness of the Schur complement and we now give some details.

The standard Schur complement step transforms the original stiffness matrix

$$\begin{pmatrix} A_1 & B^T \\ B & A_2 \end{pmatrix}$$

into the new matrix

$$\begin{pmatrix} S & 0 \\ 0 & A_2 \end{pmatrix} := \begin{pmatrix} A_1 - B^T A_2^{-1} B & 0 \\ 0 & A_2 \end{pmatrix}.$$

Here, A_1 belongs to $S_{h,p}(\Gamma)$, the matrix B represents the coupling between $S_{h,p}(\Gamma)$ and Z, and S is the Schur complement. When performing the partial Schur complement we obtain the transformed matrix

$$\begin{pmatrix} \tilde{S} & \tilde{B}^T \\ \tilde{B} & A_2 \end{pmatrix} := \begin{pmatrix} A_1 - B^T(2C_2^{-1} - C_2^{-1}A_2C_2^{-1})B & B^T(I - C_2^{-1}A_2) \\ (I - A_2C_2^{-1})B & A_2 \end{pmatrix}$$

where $\tilde{B}$ is non-zero in general. The transformation defined by the partial Schur complement also induces a transformation of the basis in $S_{h,p}(\Gamma)$. The new space is called $T_{C_2}(S_{h,p}(\Gamma))$ and the corresponding decomposition is

$$\tilde{S}_{h,p}(\Gamma) = T_{C_2}(S_{h,p}(\Gamma)) + Z_{11} + Z_{12} + \cdots + Z_{J1} + Z_{J2}. \tag{6}$$

The influence of the coupling matrix $\tilde{B}$ to the stability of this decomposition can be estimated by making use of the positive definiteness of the Schur complement S. Since $S = A_1 - B^T A_2^{-1} B$ is positive definite we may assume the following:

(A3) **Positive definiteness of Schur complement:** there exists a positive number $\kappa < 1$ such that

$$x^T B^T A_2^{-1} B x \le \kappa x^T A_1 x$$

for all $x \in \mathbf{R}^N$ (N is the dimension of $S_{h,p}(\Gamma)$).

From [3] we obtain the following result.

Theorem 2 *There exist constants c_1, c_2 such that for the error estimator based upon the decomposition* (6) *there holds*

$$c_1 \left(\theta_0^2 + \sum_{j=1}^{J} \sum_{i=1}^{2} \theta_{ji}^2 \right) \le \|u - u_N\|_D^2 \le \frac{c_2}{(1-\kappa)(1-\sigma^2)} (1 + \log p_{\max})^2 \left(\theta_0^2 + \sum_{j=1}^{J} \sum_{i=1}^{2} \theta_{ji}^2 \right).$$

Here, $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the true solution u of (1), *σ is the saturation parameter in* (A2) *and κ is the parameter in* (A3).

2.2 Mesh refinement and overlapping decomposition

In the previous section we constructed the enriched ansatz space $\tilde{S}_{h,p}(\Gamma)$ by locally increasing the polynomial degrees by 1 or 2. Since, for problems with singularities, the p-version converges algebraically in p, i.e., like $O(p^{-\alpha})$ for some $\alpha > 0$, the error of the new approximation $\tilde{u}_N$ can roughly be expected to be $(p/p+1)^{\alpha}$-times that of the old approximation u_N. That means one can expect that the saturation parameter σ in (A2) approaches 1 if p tends to be large. This behavior reduces the reliability of the error estimators defined previously, cf. Theorems 1 and 2. On the other hand, the uniform h-version converges like $O(h^{\alpha/2})$. Therefore, defining the enriched space $\tilde{S}_{h,p}(\Gamma)$ by halving the mesh size and keeping the polynomial degrees in a certain way, one can expect a much better saturation parameter σ. This strategy is proposed in this section. However, we note that the numerical experiments below show that the error indicators of the previous section lead to efficient refinements. Just the reliability of the estimators is reduced. The error indicator defined now is only used as an estimator to serve as a stopping criterion.

We define an enriched ansatz space by halving the mesh size. The error estimator is based on an overlapping decomposition that, for uniform degree distributions, leads to uniformly stable decompositions [4]:

$$\tilde{S}_{h,p}(\Gamma) := S_{h/2,q}(\Gamma) = S_{h/2,1}(\Gamma) \cup S_{h/2,q_1}(\Gamma'_1) \cup \cdots \cup S_{h/2,q_J}(\Gamma'_J). \tag{7}$$

Here, $S_{h/2,1}(\Gamma)$ is the space of piecewise bilinear functions and J denotes the number of nodes of the mesh with halved size that are not on the boundary of Γ. The space $S_{h/2,q_j}(\Gamma'_j)$ ($j = 1, \ldots, J$) is a local subspace of piecewise polynomials on the union Γ'_j of elements that are adjacent to the node with number j. For the numerical studies below we consider two possibilities to define the new polynomial degree vector q for the space $S_{h/2,q}(\Gamma)$. Either we use the maximum degree $p_{\max}$ from the original space everywhere ("$q = p_{\max}$") or we inherit the polynomial degrees ("$q = p$"). For the latter strategy we just inherit the degrees for the interior functions from the corresponding father element. The degrees for the side functions then are defined as the minimum degrees of the degrees in side directions of the neighboring elements. This is also the standard procedure to define the degrees for the space $S_{h,p}(\Gamma)$.

Then we have the following result, see [3].

Theorem 3 *For the error estimator based upon the decomposition* (7) *with* $q = p_{\max}$ *there holds*

$$c_1 \sum_{j=0}^{J} \theta_j^2 \le \|u - u_N\|_D^2 \le \frac{c_2}{1-\sigma^2} \sum_{j=0}^{J} \theta_j^2$$

for constants c_1 *and* c_2 *which are independent of* h *and* p*. Here,* σ *is the saturation parameter and* $u_N \in S_{h,p}(\Gamma)$ *is the Galerkin approximation of the true solution* u *of* (1).

Comparing this result with Theorem 2 it is obvious that the reliability of the error estimator defined by (7) is theoretically superior to that of the estimator based on (6). The only parameter that may reduce the reliability now is the saturation parameter. However, we note that the enriched space $S_{h/2,q}(\Gamma)$ in (7) is in general much larger than $\tilde{S}_{h,p}(\Gamma)$ in (6). Moreover, (7) is an overlapping decomposition whereas (6) is almost direct and cheaper to perform (only if $(p_j, q_j) = (1, 1)$ then Z_{j1} and Z_{j2} overlap, cf. Figure 1).

In conclusion, we propose the indicators based on (6) as a cheap method to define an adaptive refinement strategy. The error estimator based on (7) is more expensive but also more reliable. We suggest to use it as a stopping criterion which needs only be computed from time to time (not after each refinement step).

3 The p-adaptive algorithm

We define an initial ansatz space $S_{h,p}(\Gamma)$ for the Galerkin scheme and prescribe an error tolerance ε and steering parameters $\delta_1,\ \delta_2 \in (0,1)$. Then the adaptive algorithm reads as follows:

1. Galerkin solution. Compute the Galerkin approximation $u_N \in S_{h,p}(\Gamma)$.

2. Error estimator. If wanted, consider the enriched space and corresponding decomposition (7) to calculate the associated estimator $\delta := (\sum_{j=0}^{J} \theta_j^2)^{1/2}$. **Stop** if $\delta \leq \varepsilon$.

3. Error indicator. Consider the enriched space and corresponding decomposition (6) to calculate the error indicators $\theta_{j1},\ \theta_{j2}$ $(j = 1, \ldots, J)$. Compute $\theta_{\max} := \max_j (\theta_{j1}^2 + \theta_{j2}^2)^{1/2}$.

4. Adaption step. Increase the polynomial degree on Γ_j if $(\theta_{j1}^2 + \theta_{j2}^2)^{1/2} > \delta_1 \theta_{\max}$ in the following way:

$$\begin{aligned} q_j &\mapsto q_j + 1 && \text{if } \theta_{j1} < \delta_2 \theta_{j2}, \\ p_j &\mapsto p_j + 1 && \text{if } \theta_{j2} < \delta_2 \theta_{j1}, \\ (p_j, q_j) &\mapsto (p_j + 1, q_j + 1) && \text{else.} \end{aligned}$$

Take the new degrees to define the adapted space $S_{h,p}(\Gamma)$ and **goto** 1.

We note that for the third step one does not need to calculate the global error indicator θ_0 which, however, is non-zero in general. This term cannot be used to localize an element of large error. Therefore, this strategy also does not require to calculate any Schur complement. If, on the other hand, one wants to avoid the calculation of the error estimator based on the overlapping decomposition then this term needs to be computed.

4 Numerical results

For our model problem we choose the domain Γ to be the square plate $(-1/2, 1/2)^2 \times \{0\}$ and we use a uniform mesh of 16 squares or a geometrically refined mesh of 16 elements as indicated in Figures 2 and 3. Figure 2 shows the polynomial degrees of the ansatz space that is obtained after 8 steps of the adaptive algorithm of Section 3. The steering parameters δ_1 and δ_2 both are taken to be 0.8. We always start with uniform degrees $(p_j, q_j) = (1, 1)$, $j = 1, \ldots, 16$. Accordingly, Figure 3 shows the polynomial degrees on the geometrically graded mesh after 5 steps. In both cases it appears that the polynomial degrees resolve the singularities of the true solution in an appropriate way. This property is underlined by Figure 4 that presents the relative error in energy norm for different methods. The two curves for the uniform mesh and the geometrically graded mesh show that the errors obtained by the adaptive algorithm are much smaller than those of the uniform p-version. We note that a pure p-adaption in general does not improve the convergence rate of the p-version, and this is also confirmed by Figure 4.

(8,8)	(1,8)	(1,8)	(8,8)
(8,1)	(2,2)	(2,2)	(8,1)
(8,1)	(2,2)	(2,2)	(8,1)
(8,8)	(1,8)	(1,8)	(8,8)

Figure 2: The polynomial degrees on the uniform mesh after 8 p-refinements.

(3,3)	(1,5)	(1,5)	(3,3)
(5,1)	(2,2)	(2,2)	(5,1)
(5,1)	(2,2)	(2,2)	(5,1)
(3,3)	(1,5)	(1,5)	(3,3)

Figure 3: The polynomial degrees on a geometric mesh after 5 p-refinements.

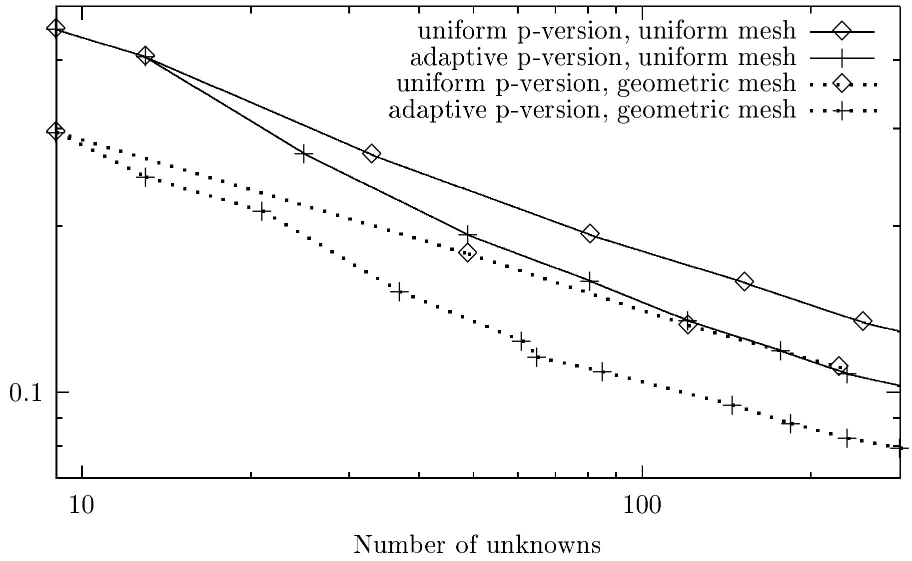

Figure 4: The relative error in energy norm for different methods (p-version).

Now let us check the error estimators. Figure 5 compares the relative error in energy norm with the three relative estimators based on the decompositions (6) and (7) with "$q = p$" (inherited degrees) and "$q = p_{\max}$" (uniform degree within $\tilde{S}_{h,p}(\Gamma)$). Obviously the overlapping decomposition (7) leads to much better estimators than decomposition (6) which is based on local p-refinements. This behavior is justified by two facts. First, the saturation parameter σ is smaller when defining $\tilde{S}_{h,p}(\Gamma)$ by mesh refinement rather than by increasing p, see Table 1. This parameter enters the estimates of Theorem 2 (error estimator based on an increas of p) and Theorem 3 (estimator based on mesh refinement). Second, the estimate given by Theorem 2 also comprises a $\log p_{\max}$ term that further reduces the reliability of the estimator based on decomposition (6). Moreover, the parameter κ appears in Theorem 2. This parameter reflects the positive definiteness of the Schur complement, cf. Assumption (A3). However, as Figure 6 indicates, this parameter appears to be bounded away from 1 in our situations. Figure 6 gives this parameter for different versions (uniform h-version with constant polynomial degree $p = 2$, non-uniform h-version defined by geometrically graded meshes with grading parameter 0.5 and constant polynomial degree $p = 2$, uniform p-version with different meshes $1/h = 3, 4$, and our adaptive version on the unform mesh with 16 elements). In any case κ is small or seems to be bounded.

Let us consider the effictivity indices (error estimator divided by error in energy norm). Figure 7 gives the results on the uniform mesh (16 elements) for all the cases. The solid lines belong to the estimators based upon increasing the polynomial degrees. They all behave similar but the effectivity of decompositions (6) (partial Schur complement) and (5) (full Schur complement) is better than when not performing a Schur complement (decomposition (4)). The results also show that it makes not a big difference whether performing the full or the partial Schur complement. This confirms the assertions of Theorems 1 and 2 which only differ by the parameter κ. But this parameter was already realized to be small or bounded,

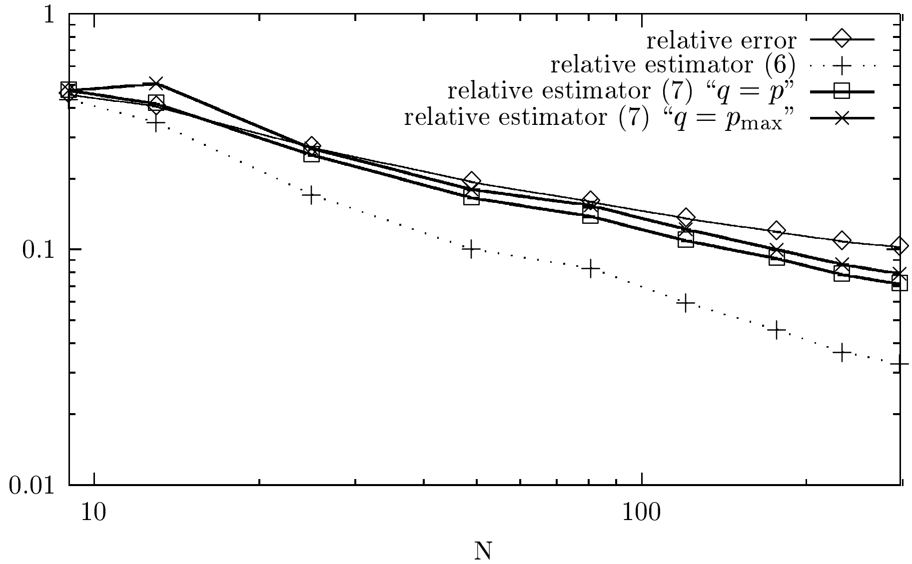

Figure 5: The relative error in energy norm and different relative estimators for the adaptive p-version (uniform mesh $1/h = 4$).

N	$p_{\max}$	1	2	3
9	1	0.654	0.695	0.695
13	2	0.734	0.696	0.471
25	2	0.790	0.707	0.704
49	3	0.839	0.745	0.741
81	4	0.859	0.754	0.749
121	5	0.889	0.790	0.779
177	6	0.916	0.754	0.801
233	7	0.934	0.834	0.837
297	8	0.943	0.849	0.845

Table 1: The saturation parameter σ of Assumption (A2) for the adaptive p-version on the uniform mesh of Figure 2. 1: $\tilde{S}_{h,p}$ defined by (6), 2: $\tilde{S}_{h,p} = S_{h/2,q}$ with “$q = p$” in (7), 3: $\tilde{S}_{h,p} = S_{h/2,q}$ with “$q = p_{\max}$” in (7).

cf. Figure 6. Now, on the other hand, the effectivity of the estimators based on mesh refinement and overlapping decomposition (7) is much better than that of the previous estimators. The estimator for the case $\tilde{S}_{h,p}(\Gamma) = S_{h/2,p_{\max}}(\Gamma)$ (uniform degree) is slightly better than for the case $\tilde{S}_{h,p}(\Gamma) = S_{h/2,p}(\Gamma)$ (inherited degrees). This is due to the larger dimension of $S_{h/2,p_{\max}}(\Gamma) \supset S_{h/2,p}(\Gamma)$ which reduces the saturation parameter σ. The effectivity index of both estimators decays onbly slowly with increasing p (and N) and is relatively close to 1.

One main assumption to ensure the effectivity of the error estimator is the stability of the corresponding decomposition of the enriched ansatz space, cf. Assumption (A1) and Lemma 1. Figures 8 and 9 present the parameters c_1 (minimum eigenvalue of the corre-

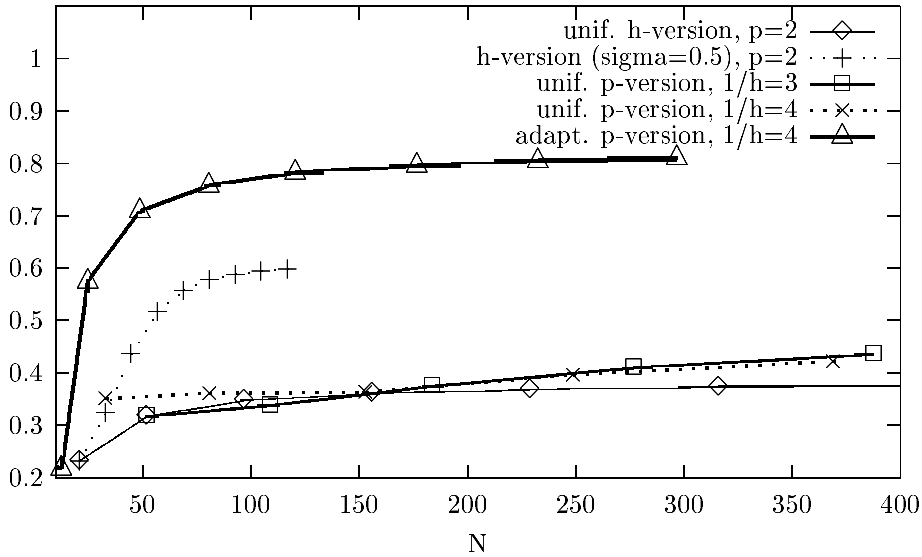

Figure 6: The parameter κ of Assumption (A3) for different versions.

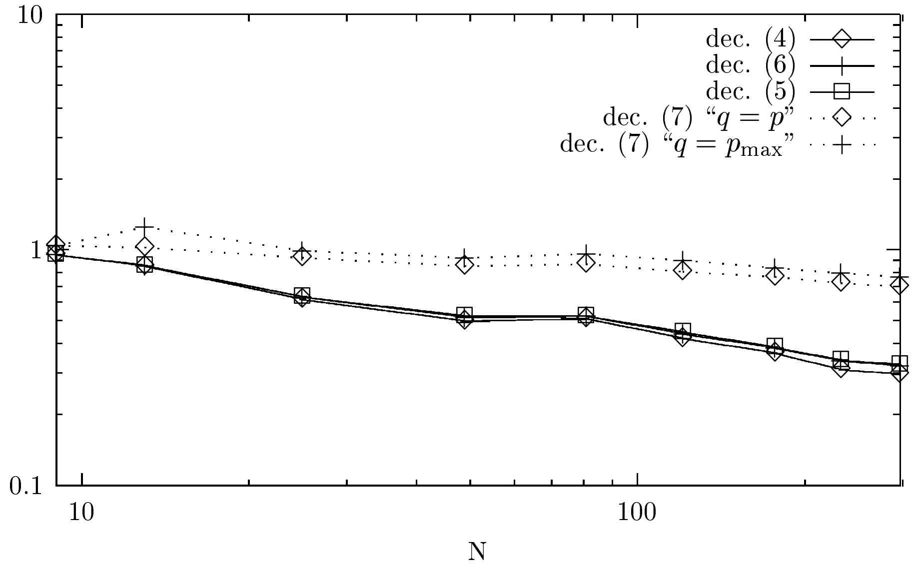

Figure 7: The effectivity index of different estimators for the adaptive p-version (uniform mesh $1/h = 4$).

sponding additive Schwarz operator P) and c_2 (maximum eigenvalue of P) appearing in Assumption (A1). Figure 8 shows the extremum eigenvalues for the decompositions (4), (5) and (6) for the adaptive p-version on the uniform mesh. Obviously the stability improves when a Schur complement if performed (decompositions (5) and (6)) since then the minimum eigenvalues are increasing and the maximum eigenvalues are decreasing. Finally, Figure 9 underlines the stability of the overlapping decomposition (7) in both cases where the extremum eigenvalues appear to be bounded.

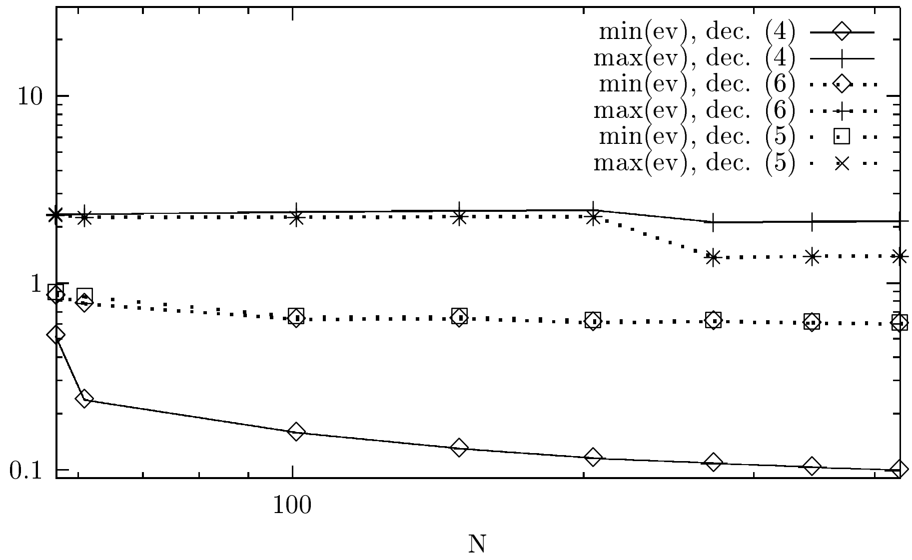

Figure 8: The stability of the decompositions (4), (5) and (6) for the adaptive p-version ($1/h = 4$).

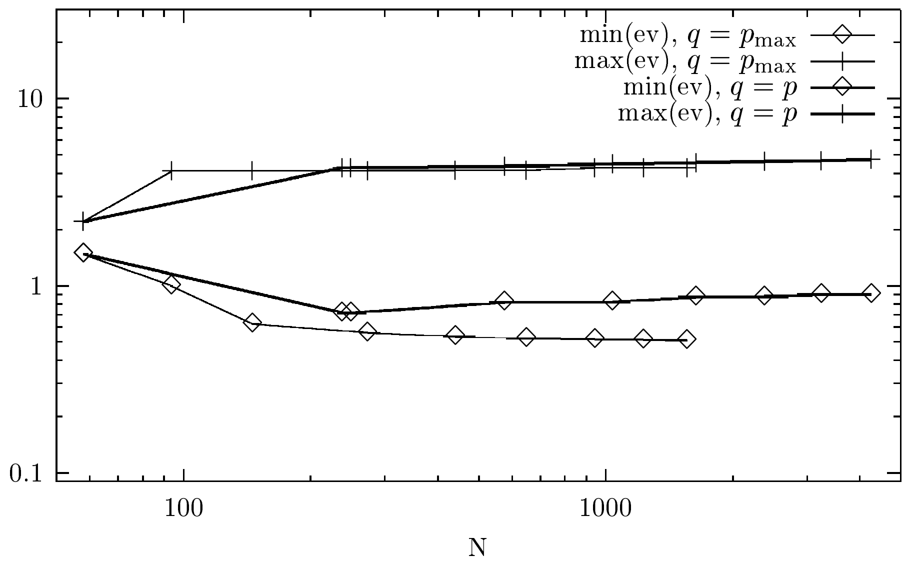

Figure 9: The stability of the overlapping decomposition (7) for the adaptive p-version ($1/h = 4$).

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