An Additive Schwarz Method for the h-p Version of the Boundary Element Method for Hypersingular Integral Equations in R^3

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Dedicated to Prof. Dr. Ian H. Sloan on the occasion of his 60th birthday.

Abstract

We study a preconditioner for the h-p version of the boundary element method for hypersingular integral equations in three dimensions. The preconditioner is based on a three-level decomposition of the underlying ansatz space, the levels being piecewise bilinear functions on a coarse grid, piecewise bilinear functions on a fine grid, and piecewise polynomials of high degree on the fine grid. We prove that the condition number of the preconditioned linear system is bounded by $\max_j(1+\log\frac{H_j p_j}{h_j})^2$ where H_j is the diameter of an element Γ_j of the coarse grid, h_j is the size of the elements of the fine grid on Γ_j, and p_j is the maximum of the polynomial degrees used in Γ_j. Numerical results supporting our theory are reported.

Key words. h-p version of the boundary element method, Schwarz methods, preconditioning

AMS subject classifications. 65N55, 65N38

1 Introduction.

In this paper we are concerned with the preconditioning of linear systems which stem from the Galerkin boundary element method for hypersingular integral equations of the first kind on surfaces in $\mathbf{R}^3$.

As a model problem we consider the weak form of the hypersingular integral equation

$$\langle Du, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all} \quad v\in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on a plane rectangular surface piece $\Gamma\subset\mathbf{R}^3$ where $f\in H^{-1/2}(\Gamma)$ is a given function. Here D is the hypersingular integral operator

$$Du(x) = \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x\in\Gamma$$

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which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [14]. Hence, there holds the equivalence of norms

$$\langle Dv, v\rangle_{L^2(\Gamma)} \simeq \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma).$$

The Sobolev spaces $\tilde{H}^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are defined in the next section. The solution u of (1) is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathbf{R}^3 \setminus \bar{\Gamma}$, cf. [14]. The extension of our results to more practical problems like exterior traction problems in linear elasticity is essentially straight forward and for ease of presentation we concentrate on the generic model problem for the Laplacian.

The Galerkin scheme for (1) reads as follows. Given a finite dimensional subspace $\Psi \subset \tilde{H}^{1/2}(\Gamma)$ with $\dim \Psi = N$ find $u_N \in \Psi$ such that

$$\langle Du_N, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all} \quad v \in \Psi. \tag{2}$$

The solution u of (1) behaves singularly at the edges and corners of Γ, cf. [15, 18]. Due to these singularities the standard h and p versions of the Galerkin method converge at a rather low rate. On the other hand, when appropriately combining mesh refinements and polynomial degree distributions in a non-uniform fashion, even an exponential rate of convergence is achievable, cf. [7]. This paper is a first step towards preconditioning methods for the general h-p version of the boundary element method in three dimensions and, to the authors' knowledge, this is the first paper investigating a preconditioner for this method. Moreover, we present numerical results underlying the almost optimal results depending only polylogarithmically on discretization parameters like the mesh size or the polynomial degrees. In this paper we consider locally quasi-uniform meshes and we assume that the polynomial degrees vary not too much within elements, i.e. the ratio of maximum and minimum polynomial degrees is bounded on individual elements. Since the polynomial degrees on neighboring elements are coupled by the continuity of the basis functions this boundedness of the ratio then holds also on patches of adjacent elements. Therefore, we call this non-uniform p version locally uniform. To the authors' knowledge it is not clear whether an exponential rate of convergence can be obtained under these restrictions.

In any case, the stiffness matrices for the h-p version in (2) are ill-conditioned and a preconditioner is necessary for an efficient solution. The method of choice for solving positive definite linear systems is the conjugate gradient method. Let A denote the stiffness matrix of the linear system with spectral condition number κ. Then a bound on the decrease of the energy norm of the error, after k steps, is given by

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \quad \text{where} \quad \kappa = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

The goal of this paper is to investigate a preconditioner for A which yields good bounds for κ. Provided the stiffness matrix is given in a suitable basis with appropriate numbering of basis functions, our preconditioner amounts to a block-Jacobi step where some of the blocks may overlap with others. Each block of this method corresponds to a discretization of the integral operator for a given subspace of the full approximation space Ψ. Therefore, the preconditioner is related to a decomposition of Ψ into subspaces. The main structure of this decomposition is given by a three-level method. There are two levels for the piecewise polynomials of lowest degree corresponding to a coarse and a fine mesh, both being locally quasi-unifom. The fine level subspace then is further decomposed into a couple of subspaces associated with the wire basket and individual elements of the coarse mesh. Finally, the third level is given by the full space of piecewise polynomials of high degrees and is also further decomposed. The subspaces of the latter decomposition belong to the wire basket and individual elements of the fine mesh.

The three-level decomposition of our method is analogous to that proposed by Guo and Cao in [4] for the finite element method in three dimensions. However, we have to deal with an integral operator on surfaces that means we have to consider trace spaces of H^1. In [4] interpolation polynomials for the Legendre-Gauss-Lobatto points and related discrete bilinear forms are used on the wire basket (the space of functions associated with the nodes and sides of the elements) to define the part of the preconditioner belonging to the wire basket. Here we take polynomials with minimal L^2-norm as given in [13] and use the L^2-bilinear form on the wire basket. Many results by Pavarino and Widlund in [13] for the uniform p version are derived by techniques described in their paper [12] where sophisticated basis functions have been used. In this way, for the decomposition of piecewise polynomials of high order, we make use of basic estimates given in [13] and [12]. The basis functions taken in [12] are also used in [5] to define preconditioners for the uniform p version of the boundary element method. However, as mentioned above, here we use the simpler basis functions from [13].

Let us also note that this paper extends the results in [8] where the pure h version of the boundary element method on quasi-uniform meshes is considered. Indeed, the first two levels of our preconditioner are identical with the subspace decomposition of the h version in [8] where, however, in each case the original bilinear form is used as preconditioner. In our previous paper [9] we also deal with preconditioners for the h-p version of the boundary element method where, in particular, geometrically graded meshes and non-uniform polynomial degrees are considered. However, that paper only deals with two-dimensional problems on polygonal domains, i.e. with integral operators on curves, and the techniques described there are not applicable to three-dimensional problems. Let us also mention the work by Tran and Stephan [16] where the additive Schwarz method for the h-version of the boundary element method with quasi-uniform meshes is under investigation. Here again, integral operators on polygonal curves are considered.

An outline of the paper is as follows. In §2 we introduce the h-p approximation space of the Galerkin scheme (2). The decomposition of the approximation space and associated bilinear forms which define the preconditioner are described in §3. The main theorem is given which proves a bound like $\max_j(1+\log\frac{H_j}{h_j}p_j)^2$ for the condition number of the preconditioned system. Here, H_j is the diameter of the jth element (which is called subdomain) of the coarse mesh and h_j is the mesh size of the fine mesh in the jth subdomain. Within the subdomains the fine mesh is assumed to be quasi-uniform. p_j indicates the maximum polynomial degree in the jth subdomain. The technical details which are needed for the proof of the main theorem are given in §4. The proof of the main theorem is then given in §5. In §6 we present some numerical experiments which underline the theoretical results.

Throughout the paper c denotes a generic constant which may take on different values at different occurrences.

2 Construction of the h-p approximation space.

In this section we define the ansatz space Ψ that will be used in (2). It consists of piecewise polynomials of varying degree on a locally quasi-uniform mesh. This mesh is defined by two levels.

One level is the **coarse mesh** which is given by a regular family of triangles or quadrilaterals, $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$. The elements Γ_j of this level are called *subdomains* and its nodal points are referred to as *vertices* whereas the lines between the vertices are the *edges*. The subdomains must be shape regular, i.e. they are not too distorted and the coarse mesh is locally quasi-uniform. The diameter of Γ_j is denoted by H_j.

The second level of the mesh is the **fine mesh**. It is given by partitioning each subdomain into a number of quasi-uniform quadrilaterals Γ_{ji} (the *elements*) which are shape regular and of diameter h_j on Γ_j. The nodal points of the fine mesh are called *nodes* and the lines between the nodes are the *sides*.

Having defined the mesh on Γ the h-p approximation space is completely determined by defining basis functions locally on the reference element $\Gamma_{\text{ref}} := (-1,1)^2$ and by specifying polynomial degrees on the elements of the mesh.

On the reference element we use the vector of polynomial degrees

$$\mathcal{P} = (p_{\gamma_1}, \dots, p_{\gamma_4}, p_{I_1}, p_{I_2})$$

where p_{γ_j} and p_{I_j} are the degrees associated with the sides and the interior of Γ_{ref} (x_1- and x_2-direction), respectively, in a certain order. The maximum polynomial degree on Γ_{ref} is denoted by $p_{\max}$. The elements Γ_{ji} are associated with degree vectors $\mathcal{P}_{ji}$, and we assume that the ratio of the maximum and minimum polynomial degrees on individual elements is bounded uniformly on the fine mesh. Note that this condition implies that also the ratio of maximum and minimum polynomial degrees on neighboring elements is bounded. As mentioned in the introduction we call this non-uniform p-distribution locally uniform.

Now let us define the basis functions. As in the standard p version we make a distinction between nodal, side, and internal shape functions. Let φ_0^p denote the pth degree polynomial on $I := (-1,1)$ with $\varphi_0^p(-1) = 0$ and $\varphi_0^p(1) = 1$ which minimizes the $L^2(I)$-norm over the space of all pth degree polynomials (subject to the same boundary conditions).

- One of the four nodal shape functions, the one for node $V_1 = (-1,-1)$, is given by

$$\varphi_0^{p_{\gamma_1}}(-x_1)\varphi_0^{p_{\gamma_4}}(-x_2).$$

- One of the sets of side shape functions, for the edge $E_1 = \{(x_1,x_2);\ x_2 = -1\}$, is given by

$$\psi^{p_{\gamma_1}}(x_1)\varphi_0^{p_{I_2}}(-x_2)$$

  where $\psi^{p_{\gamma_1}}$ is a polynomial of degree p_{γ_1} such that $\psi^{p_{\gamma_1}}(-1) = \psi^{p_{\gamma_1}}(1) = 0$.

- The interior shape functions are polynomials of degree p_{I_1} in x_1 and of degree p_{I_2} in x_2 which vanish at the sides of Γ_{ref}.

Using the shape functions defined above we introduce a polynomial space on Γ_{ref} by

$$\Psi_{\mathcal{P}}(\Gamma_{\text{ref}}) = \Psi_{\mathcal{P}}^{[N]}(\Gamma_{\text{ref}}) + \cup_{l=1}^{4}\Psi_{\mathcal{P}}^{[\gamma_l]}(\Gamma_{\text{ref}}) + \Psi_{\mathcal{P}}^{[I]}(\Gamma_{\text{ref}}).$$

Here, $\Psi_{\mathcal{P}}^{[N]}(\Gamma_{\text{ref}})$ is the space of nodal shape functions on the reference element Γ_{ref}, $\Psi_{\mathcal{P}}^{[\gamma_l]}(\Gamma_{\text{ref}})$ is the space of side shape functions on the side γ_l of Γ_{ref}, and $\Psi_{\mathcal{P}}^{[I]}(\Gamma_{\text{ref}})$ denotes the space of interior shape functions on Γ_{ref}.

The full h-p approximation space on Γ is now defined by taking affine transformations onto the elements Γ_{ji} of the polynomial space $\Psi_{\mathcal{P}}(\Gamma_{\text{ref}})$. Using the notation previously introduced we have on each element the representation

$$\Psi_{\mathcal{P}_{ji}}(\Gamma_{ji}) = \Psi_{\mathcal{P}_{ji}}^{[N]}(\Gamma_{ji}) + \cup_{l=1}^{4}\Psi_{\mathcal{P}_{ji}}^{[\gamma_l]}(\Gamma_{ji}) + \Psi_{\mathcal{P}_{ji}}^{[I]}(\Gamma_{ji})$$

and the full space

$$\Psi(\Gamma) = \{\psi;\ \psi|_{\Gamma_{ji}} \in \Psi_{\mathcal{P}_{ji}}(\Gamma_{ji})\} \cap \tilde{H}^{1/2}(\Gamma).$$

3 Additive Schwarz preconditioner.

Decomposition of the h-p approximation space. We decompose the approximation space by

$$\Psi(\Gamma) = \Psi_H(\Gamma) + \Psi_{\mathcal{W}}(\Gamma) + \cup_j \Psi_{\Gamma_j} + \Psi_W(\Gamma) + \cup_{j,i}\Psi_{\mathcal{P}_{ji}}^{[I]}(\Gamma_{ji}) \tag{3}$$

Here, $\Psi_H(\Gamma) = \Pi_h \Psi_H^*(\Gamma)$ where $\Psi_H^*(\Gamma)$ is the space of piecewise linear/bilinear functions on the coarse mesh and $\Pi_h : \Psi(\Gamma) \to \Psi_h(\Gamma)$ is the interpolation operator onto the space of piecewise bilinear functions $\Psi_h(\Gamma)$ on the fine mesh. Further, $\Psi_{\mathcal{W}}(\Gamma)$ is the space of piecewise bilinear functions on the fine mesh which are zero at the nodes which are not on the wire basket $\mathcal{W}$. The piecewise bilinear functions on the fine mesh which are non-zero only on the subdomain Γ_j span the space Ψ_{Γ_j}. The remaining spaces, $\Psi_W(\Gamma)$ and $\Psi_{\mathcal{P}_{ji}}^{[I]}(\Gamma_{ji})$, represent a decomposition of the space of high degree polynomials. $\Psi_W(\Gamma)$ is spanned by all the side and nodal functions and $\Psi_{\mathcal{P}_j i}^{[I]}(\Gamma_{ji})$ comprises all polynomials of the specified degrees on Γ_{ji} that vanish on $\Gamma \setminus \Gamma_{ji}$.

Note that, as explained in [4] for the finite element method, the above decomposition amounts to a three-level method. The first two levels, $\Psi_H(\Gamma)$ and $\Psi_{\mathcal{W}}(\Gamma) + \cup_j \Psi_{\Gamma_j}$, represent a two-level decomposition of the piecewise polynomials of lowest degree whereas the third level, $\Psi_W(\Gamma) + \cup_{j,i}\Psi_{\mathcal{P}_{ji}}^{[I]}(\Gamma_{ji})$, contains all piecewise polynomials of higher degrees.

Bilinear forms and preconditioner. For ease of presentation we use, instead of (3), also the notation

$$\Psi(\Gamma) = H_1 + \ldots + H_k$$

for the three-level decomposition of $\Psi(\Gamma)$ where the number of subspaces k equals to three plus the number of subdomains Γ_j plus the number of elements Γ_{ji} (if the polynomial degrees are large enough such that all the subspaces are non-empty).

The additive Schwarz method consists in solving, by an iterative method, the equation

$$Pu_N := (P_1 + P_2 + \cdots + P_k)u_N = f_N, \tag{4}$$

where the projections $P_j : \Psi(\Gamma) \to H_j$, $j = 1, \dots, k$, are defined for any $v \in \Psi(\Gamma)$ by

$$b_j(P_j v, \varphi) = \langle Dv, \varphi \rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_j.$$

Here, b_j, $j = 1, \dots, k$, are given bilinear forms. On all but the wire basket spaces $\Psi_{\mathcal{W}}(\Gamma)$ and $\Psi_W(\Gamma)$ we use the original bilinear form given by the integral operator, i.e.

$$b_j(v, w) := \langle Dv, w \rangle_{L^2(\Gamma)} \quad \text{for} \quad v, w \text{ both in } \Psi_H(\Gamma), \Psi_{\Gamma_j}, \text{ or } \Psi^{[I]}_{\mathcal{P}_{ji}}(\Gamma_{ji}).$$

On the wire basket spaces $\Psi_{\mathcal{W}}(\Gamma)$ and $\Psi_W(\Gamma)$ we use the L^2-bilinear form over the wire baskets, i.e.

$$b_j(v, w) := \langle v, w \rangle_{L^2(\mathcal{W})} \quad \text{for} \quad v, w \in \Psi_{\mathcal{W}}(\Gamma) \tag{5}$$

and

$$b_j(v, w) := \langle v, w \rangle_{L^2(W)} \quad \text{for} \quad v, w \in \Psi_W(\Gamma). \tag{6}$$

The right hand side of (4), $f_N = \sum_{j=1}^k P_j u_N$, can be computed without knowing the solution u_N of (2) by

$$b_j(P_j u_N, \varphi) = \langle f, \varphi \rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_j, \; j = 1, \dots, k.$$

Eq. (4) is the preconditioned linear system and an estimate of its condition number is given by the next theorem.

Theorem 1 *There exist positive constants c_1, c_2 which are independent of H_j, h_j, and p_j such that for all $v \in \Psi$ there holds*

$$c_1 \min_j (1 + \log \frac{H_j}{h_j} p_j)^{-2} \langle Dv, v \rangle_{L^2(\Gamma)} \leq \langle DPv, v \rangle_{L^2(\Gamma)} \leq c_2 \langle Dv, v \rangle_{L^2(\Gamma)}.$$

P is the additive Schwarz operator defined by the decomposition of the ansatz space Ψ and by the given bilinear forms.

4 Technical details.

In this section we present the technical lemmas which are needed to prove Theorem 1. First we need to introduce some Sobolev spaces and norms.

Let M be a polygon in $\mathbf{R}^2$ or a polyhedron in $\mathbf{R}^3$. By $L^2(M)$ and $H^1(M)$ we denote the usual Sobolev spaces with norms $\|\cdot\|_{L^2(M)}$, $\|\cdot\|_{H^1(M)}$ and semi-norm $|\cdot|_{H^1(M)}$. For scaling the H^1-norm on domains M_h of diameter h we use, instead of the standard H^1-norm, the norm with weighted L^2-term

$$\|\cdot\|^2_{H^1(M_h)} = \frac{1}{h^2}\|\cdot\|^2_{L^2(M_h)} + |\cdot|^2_{H^1(M_h)},$$

and analogously for domains with diameter H.

On an open surface piece Γ we introduce the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ where the latter space is most often denoted by $H_{00}^{1/2}(\Gamma)$ in the finite element literature. Let $\tilde{\Gamma}$ be a closed surface (in our case a polyhedral surface) with $\Gamma \subset \tilde{\Gamma}$. We define

$$H^{1/2}(\tilde{\Gamma}) := \{\phi|_{\tilde{\Gamma}};\ \phi \in H^1(\mathbf{R}^3)\}, \quad H^{1/2}(\Gamma) := \{\phi|_{\Gamma};\ \phi \in H^{1/2}(\tilde{\Gamma})\},$$

and

$$\tilde{H}^{1/2}(\Gamma) := \{\phi \in H^{1/2}(\Gamma);\ \tilde{\phi} \in H^{1/2}(\tilde{\Gamma})\},$$

where $\tilde{\phi}$ denotes the extension of ϕ by 0 from Γ onto $\tilde{\Gamma}$. A norm in $H^{1/2}(\Gamma)$ is given by (see [10])

$$\|\cdot\|^2_{H^{1/2}(\Gamma)} = \|\cdot\|^2_{L^2(\Gamma)} + |\cdot|^2_{H^{1/2}(\Gamma)}$$

with semi-norm

$$|v|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^3}\, dS_x dS_y.$$

On a local portion $\Gamma_j \subset \Gamma$ with diam $(\Gamma_j) = O(h)$ we use, analogously as before, a different norm with weighted L^2-term

$$\|\cdot\|^2_{H^{1/2}(\Gamma_j)} = \frac{1}{h}\|\cdot\|^2_{L^2(\Gamma_j)} + |\cdot|^2_{H^{1/2}(\Gamma_j)}.$$

Finally, in $\tilde{H}^{1/2}$ on Γ and correspondingly on local portions $\Gamma_j \subset \Gamma$, we use the norm

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} = |v|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{|v(x)|^2}{\operatorname{dist}(x, \partial\Gamma)}\, dS_x.$$

We also note that the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ can be equivalently defined as intermediate spaces between $L^2(\Gamma)$ and $H^1(\Gamma)$ or $H^1_0(\Gamma)$ ($H^1_0(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ within $H^1(\Gamma)$). By interpolation theory we have for real $s \in (0,1)$, see [1],

$$H^s(\Gamma) = [L^2(\Gamma), H^1(\Gamma)]_s \quad \text{and} \quad \tilde{H}^s(\Gamma) = [L^2(\Gamma), H^1_0(\Gamma)]_s.$$

This relation holds analogously for subdomains $\Gamma_j \subset \Gamma$.

Now let us collect all the needed technical results.

As in [2] (see also [4]) we define an interpolation operator $\Pi^0_h :\ L^2(\Gamma) \to \Psi_h(\Gamma)$ associated with the fine mesh. Let $\{q\}$ denote the nodes of the fine mesh and let Γ_q be the union of the elements adjacent to q. Then

$$\Pi^0_h v := \sum_q \alpha_q \phi_q$$

where ϕ_q is the piecewise bilinear nodal basis function associated with q and $\alpha_q = \tilde{v}(q)$. Here, $\tilde{v}$ is continuous and piecewise bilinear on Γ_q (zero on $\Gamma \setminus \Gamma_q$) and is defined by

$$\int_{\Gamma_q} (v - \tilde{v})\phi\, dS = 0$$

for all continuous, piecewise bilinear functions ϕ on Γ_q. Analogously, we define the interpolation operator $\Pi^0_H :\ L^2(\Gamma) \to \Psi^*_H(\Gamma)$ associated with the coarse mesh.

Lemma 1 *For any $v \in H^{1/2}(\Gamma)$ and locally quasi-uniform coarse and fine meshes,*

$$|\Pi_h^0 v - v|^2_{H^{1/2}(\Gamma)} + \sum_{j,i} h_j^{-1} \|\Pi_h^0 v - v\|^2_{L^2(\Gamma_{ji})} \le c|v|^2_{H^{1/2}(\Gamma)}, \tag{7}$$

$$|\Pi_H^0 v - v|^2_{H^{1/2}(\Gamma)} + \sum_{j} H_j^{-1} \|\Pi_H^0 v - v\|^2_{L^2(\Gamma_{j})} \le c|v|^2_{H^{1/2}(\Gamma)}. \tag{8}$$

Proof. This lemma is the special case of piecewise bilinear approximation polynomials in [2] where piecewise polynomials of arbitrary degree are considered, see also [4]. □

Lemma 2 *For any $v \in \Psi_H^*$,*

$$|\Pi_h v|_{H^l(\Gamma)} \le c|v|_{H^l(\Gamma)}, \quad 0 \le l \le 1.$$

Proof. For $l = 0, 1$ the assertion is proved by [4, Lemma 4.3]. For intermediate spaces we use interpolation theory. □

Lemma 3 *Let Γ_j be any subdomain and $\mathcal{W}_j = \mathcal{W} \cap \partial\Gamma_j$. Then for all $v \in \Psi_h$,*

$$\|v\|^2_{L^2(\mathcal{W}_j)} \le c(1 + \log \frac{H_j}{h_j})\|v\|^2_{H^{1/2}(\Gamma_j)}.$$

Proof. Lemma 4.3 in [3] proves

$$\|v\|^2_{L^2(\mathcal{W}_j)} \le c(1 + \log \frac{H_j}{h_j})\|V\|^2_{H^1(\Omega_j)}$$

where $\Omega_j \subset \mathbf{R}^3$ is of diameter $O(H_j)$ and is an extension of Γ_j. Furthermore, Ω_j is provided with a quasi-uniform mesh extending that of Γ_j and V is a piecewise polynomial of degree one with trace v on Γ_j. Choosing V to be the discrete harmonic extension of v we obtain

$$\|V\|^2_{H^1(\Omega_j)} \le c\|v\|^2_{H^{1/2}(\Gamma_j)}$$

which proves the assertion. For more details we refer to [8] where the pure h version is considered. □

Lemma 4 *Let Γ_j be any subdomain and $v \in \Psi_h$. Define $\tilde{v} \in \Psi_h$ on Γ_j by coincidence with v at all the nodes in Γ_j and by $\tilde{v}|_{\partial\Gamma_j} = 0$. Then $\tilde{v}$ satisfies*

$$\|\tilde{v}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le c(1 + \log \frac{H_j}{h_j})^2\|v\|^2_{H^{1/2}(\Gamma_j)}.$$

Proof. As in the proof of the previous lemma we extend Γ_j to Ω_j. Combining the trace theorem and Lemma 4.5 in [3] we obtain

$$\|\tilde{v}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le c|\tilde{V}|^2_{H^1(\Omega_j)} \le c(1 + \log \frac{H_j}{h_j})^2\|V\|^2_{H^1(\Omega_j)}.$$

Here, $\tilde{V}$ is a piecewise polynomial of degree one, extending $\tilde{v}$ and vanishing on $\partial\Omega_j \setminus \Gamma_j$. The function V is also a piecewise polynomial of degree one, extending v and coinciding with $\tilde{V}$ at the interior nodes of Ω_j. Now, choosing V to be the discrete harmonic extension of v onto Ω_j, the proof is finished as in the previous lemma. □

The following lemma is cited from [6, Lemma 3.4].

Lemma 5 *Let I be one side of the reference element Γ_{ref}. Then for any polynomial v of degree p*

$$\|v\|^2_{L^2(I)} \le c(1+\log p)\|v\|^2_{H^{1/2}(\Gamma_{\mathrm{ref}})}.$$

Lemma 6 *For any polynomial v of degree P (note the definition of the degree vector) with wire basket component $v_W \in \Psi^{[N]}_{\mathcal{P}}(\Gamma_{\mathrm{ref}}) + \cup_{l=1}^4 \Psi^{[\gamma_l]}_{\mathcal{P}}(\Gamma_{\mathrm{ref}})$, i.e. $v - v_W = 0$ on $\partial\Gamma_{\mathrm{ref}}$, there holds*

$$\|v - v_W\|^2_{\tilde{H}^{1/2}(\Gamma_{\mathrm{ref}})} \le c(1+\log p_{\max})^2\|v\|^2_{H^{1/2}(\Gamma_{\mathrm{ref}})}.$$

Proof. Using [12, Lemma 5.7] and the results in [13] we obtain

$$\|v - v_W\|^2_{\tilde{H}^{1/2}(\Gamma_{\mathrm{ref}})} \le c(1+\log p_{\max})\big(\max_{x_1}\|v(x_1,\cdot)\|^2_{L^2(-1,1)} + \max_{x_2}\|v(\cdot,x_2)\|^2_{L^2(-1,1)}\big)$$

and, referring to Lemma 5, this proves the assertion. □

5 Proof of Theorem 1.

Estimate of the minimum eigenvalue. For an arbitrary $u \in \Psi$ we define

$$\begin{array}{lll}
u_H \in \Psi_H & & u_H = \Pi_h \Pi^0_H u,\\
u_{\mathcal{W}} \in \Psi_{\mathcal{W}} & \text{such that} & u_{\mathcal{W}} = \Pi^0_h u - u_H \text{ on } \mathcal{W},\\
u_{\Gamma_j} \in \Psi_{\Gamma_j} & & u_{\Gamma_j} = \Pi^0_h u - (u_H + u_{\mathcal{W}}),\\
 & & \\
u_W \in \Psi_W & \text{such that} & u_W = u - \Pi^0_h u \text{ on } W,\\
u_{\Gamma_{ji}} \in \Psi_{\Gamma_{ji}} & & u_{\Gamma_{ji}} = u - (\Pi^0_h u + u_W).
\end{array}$$

- By Lemma 2 we obtain via interpolation between $L^2(\Gamma)$ and $H^1_0(\Gamma)$

$$\|u_H\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\|\Pi^0_H u\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\|u\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

- We use Lemma 3 and obtain

$$\begin{aligned}
\|u_{\mathcal{W}}\|^2_{L^2(\mathcal{W}_j)} &\le c(1+\log\frac{H_j}{h_j})\|\Pi^0_h u - \Pi_h\Pi^0_H u\|^2_{H^{1/2}(\Gamma_j)}\\
&\le c(1+\log\frac{H_j}{h_j}) \times\\
&\times \Big(\|\Pi^0_h u - u\|^2_{H^{1/2}(\Gamma_j)} + \|\Pi^0_H u - u\|^2_{H^{1/2}(\Gamma_j)} + \|\Pi^0_H u - \Pi_h\Pi^0_H u\|^2_{H^{1/2}(\Gamma_j)}\Big).
\end{aligned}$$

By (7)

$$\begin{aligned}
\sum_j \|\Pi_h^0 u - u\|^2_{H^{1/2}(\Gamma_j)} &= \sum_j \Big(|\Pi_h^0 u - u|^2_{H^{1/2}(\Gamma_j)} + H_j^{-1} \|\Pi_h^0 u - u\|^2_{L^2(\Gamma_j)} \Big) \\
&\le c|\Pi_h^0 u - u|^2_{H^{1/2}(\Gamma)} + \sum_{j,i} h_j^{-1} \|\Pi_h^0 u - u\|^2_{L^2(\Gamma_{ji})} \\
&\le c|u|^2_{H^{1/2}(\Gamma)}, \qquad (9)
\end{aligned}$$

and by (8)

$$\begin{aligned}
\sum_j \|\Pi_H^0 u - u\|^2_{H^{1/2}(\Gamma_j)} &= \sum_j |\Pi_H^0 u - u|^2_{H^{1/2}(\Gamma_j)} + \sum_j H_j^{-1} \|\Pi_H^0 u - u\|^2_{L^2(\Gamma_j)} \\
&\le c|\Pi_H^0 u - u|^2_{H^{1/2}(\Gamma)} + \sum_j H_j^{-1} \|\Pi_H^0 u - u\|^2_{L^2(\Gamma_j)} \\
&\le c|u|^2_{H^{1/2}(\Gamma)}. \qquad (10)
\end{aligned}$$

By Lemma 2 and a quotient space argument

$$\begin{aligned}
\|\Pi_H^0 u - \Pi_h \Pi_H^0 u\|^2_{H^{1/2}(\Gamma_j)} &= |\Pi_H^0 u - \Pi_h \Pi_H^0 u|^2_{H^{1/2}(\Gamma_j)} + H_j^{-1} \|\Pi_H^0 u - \Pi_h \Pi_H^0 u\|^2_{L^2(\Gamma_j)} \\
&\le c \inf_{\alpha \in \mathrm{R}} \big(|\Pi_H^0 u + \alpha|^2_{H^{1/2}(\Gamma_j)} + H_j^{-1} \|\Pi_H^0 u + \alpha\|^2_{L^2(\Gamma_j)} \big) \\
&\le c|\Pi_H^0 u|^2_{H^{1/2}(\Gamma_j)}.
\end{aligned}$$

Using (8) we therefore obtain

$$\sum_j \|\Pi_H^0 u - \Pi_h \Pi_H^0 u\|^2_{H^{1/2}(\Gamma_j)} \le c \sum_j |\Pi_H^0 u|^2_{H^{1/2}(\Gamma_j)} \le c|u|^2_{H^{1/2}(\Gamma)}. \qquad (11)$$

A combination of (9), (10), and (11) yields

$$\sum_j \|\Pi_h^0 u - \Pi_h \Pi_H^0 u\|^2_{H^{1/2}(\Gamma_j)} \le c|u|^2_{H^{1/2}(\Gamma)}, \qquad (12)$$

which proves

$$\|u_{\mathcal{W}}\|^2_{L^2(\mathcal{W})} \le c \max_j \big(1 + \log \frac{H_j}{h_j}\big) \|u\|^2_{H^{1/2}(\Gamma)}.$$

• Since $u_{\mathcal{W}} = 0$ at the nodes of the subdomains Γ_j there holds $u_{\Gamma_j} = \Pi_h^0 u - (u_H + u_{\mathcal{W}}) = \Pi_h^0 u - u_H$ at the nodes in Γ_j. We therefore conclude from Lemma 4 and by using (12)

$$\begin{aligned}
\sum_j \|u_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma_j)} &\le \sum_j c(1 + \log \frac{H_j}{h_j})^2 \|\Pi_h^0 u - u_H\|^2_{H^{1/2}(\Gamma_j)} \\
&\le c \max_j (1 + \log \frac{H_j}{h_j})^2 |u|^2_{H^{1/2}(\Gamma)}.
\end{aligned}$$

• By Lemma 5 and (7) we obtain

$$\|u_W\|^2_{L^2(W)} \le \sum_{j,i} \|u_W\|^2_{L^2(W\cap\partial\Gamma_{ji})} \le c\sum_{j,i}(1+\log p_j)\|u-\Pi_h^0 u\|^2_{H^{1/2}(\Gamma_{ji})}$$
$$\le c\max_j(1+\log p_j)|u|^2_{H^{1/2}(\Gamma)}.$$

• By Lemma 6 and (7)

$$\sum_{j,i}\|u_{\Gamma_{ji}}\|^2_{\tilde H^{1/2}(\Gamma_{ji})} = \sum_{j,i}\|u-\Pi_h^0 u - u_W\|^2_{\tilde H^{1/2}(\Gamma_{ji})}$$
$$\le c\max_j(1+\log p_j)^2 \sum_{j,i}\|u-\Pi_h^0 u\|^2_{H^{1/2}(\Gamma_{ji})}$$
$$\le c\max_j(1+\log p_j)^2|u|^2_{H^{1/2}(\Gamma)}.$$

Estimate of the maximum eigenvalue. First we note that there holds

$$\|u\|^2_{\tilde H^{1/2}(\Gamma)} \le c\sum_j \|u|_{\Gamma_j}\|^2_{\tilde H^{1/2}(\Gamma_j)}$$

for decompositions $\bar\Gamma = \cup_j\bar\Gamma_j$ into non-overlapping Lipschitz domains Γ_j and functions $u \in \tilde H^{1/2}(\Gamma)$ such that $u|_{\Gamma_j} \in \tilde H^{1/2}(\Gamma_j)$, see [17, Lemma 3.2]. Together with a coloring argument, this gives for $u \in \Psi$ an estimate like

$$\|u\|^2_{\tilde H^{1/2}(\Gamma)} \le$$
$$c\Big(\|u_H\|^2_{\tilde H^{1/2}(\Gamma)} + \|u_{\mathcal W}\|^2_{\tilde H^{1/2}(\Gamma)} + \sum_j \|u_{\Gamma_j}\|^2_{\tilde H^{1/2}(\Gamma)} + \|u_W\|^2_{\tilde H^{1/2}(\Gamma)} + \sum_{j,i}\|u_{\Gamma_{ji}}|^2_{\tilde H^{1/2}(\Gamma)}\Big).$$

The constant c does not depend on u, h, and p. Here,

$$u = u_H + u_{\mathcal W} + \sum_j u_{\Gamma_j} + u_W + \sum_{j,i} u_{\Gamma_{ji}}$$

is an arbitrary representation of u according to our decomposition of the ansatz space into subspaces. For the definition of the additive Schwarz preconditioner we used two different bilinear forms. For all but the wire basket components the inner product given by the integral operator is taken whereas, for the wire basket components, the L^2-inner product over the wire basket is used. It therefore remains to bound the energy norm of the wire basket components by the L^2-norm of the respective traces. In order to do so we apply the results in the three dimensional case by Guo and Cao [4]. Therefore, we need an extension procedure. In a straightforward manner we extend Γ to a domain $\Omega \subset \mathbf{R}^3$ such that Γ is part of the boundary $\partial\Omega$ of Ω. Moreover, we provide Ω with a locally quasi-uniform mesh of hexahedra such that the trace on Γ of the mesh coincides with the mesh on Γ. The coincidence of the meshes is also assumed to hold with respect to subdomains. On Ω these subdomains are tetrahedra, pentahedra, or hexahedra. Again, H_i and h_i refer to the diameters of the subdomains and elements. For more details see [4].

To be precise we denote the wire basket of the mesh on Ω by W_Ω and the wire basket formed by the subdomains is $\mathcal{W}_\Omega$. Note that there holds $W_\Omega|_\Gamma = W$ and $\mathcal{W}_\Omega|_\Gamma = \mathcal{W}$.

We now introduce two extension operators (using the same symbol $\mathcal{E}$) for wire basket functions associated with $\mathcal{W}$ and W:

$$\mathcal{E} : \{u_{\mathcal{W}} : \Gamma \to \mathbf{R}\} \mapsto \{\mathcal{E}u_{\mathcal{W}} : \Omega \to \mathbf{R}\},$$
$$\mathcal{E} : \{u_W : \Gamma \to \mathbf{R}\} \mapsto \{\mathcal{E}u_W : \Omega \to \mathbf{R}\}.$$

For $\mathcal{E}u_{\mathcal{W}}$ we simply extend the bilinear basis of $u_{\mathcal{W}}$ to a trilinear basis (with respect to the coarse mesh on Ω). For the definition of $\mathcal{E}u_W$ we extend the vertex and edge basis functions locally on the reference cube $\Omega_{\text{ref}} = \{(x_1, x_2, x_3);\ (x_1, x_2) \in \Gamma_{\text{ref}}, x_3 \in (-1, 1))\}$ with correspondence $\Gamma_{\text{ref}} = \Omega_{\text{ref}}|_{x_3=-1}$. The extensions of the nodal basis functions are, e.g. for the node $(-1, -1)$,

$$\varphi_0^{p_{\gamma_1}}(-x_1)\varphi_0^{p_{\gamma_4}}(-x_2) \mapsto \varphi_0^{p_{\gamma_1}}(-x_1)\varphi_0^{p_{\gamma_4}}(-x_2)\varphi_0^{\min\{p_{\gamma_1}, p_{\gamma_4}\}}(-x_3)$$

and for the edge basis functions, e.g. for the edge $x_2 = -1$, we take

$$\psi^{p_{\gamma_1}}(x_1)\varphi_0^{p_{I_2}}(-x_2) \mapsto \psi^{p_{\gamma_1}}(x_1)\varphi_0^{p_{I_2}}(-x_2)\varphi_0^{\min\{p_{\gamma_1}, p_{I_2}\}}(-x_3).$$

By the above relations $\mathcal{E}u_{\mathcal{W}}$ and $\mathcal{E}u_W$ are defined on the elements of Ω which have a face on Γ. We then extend by zero onto the remaining elements of Ω. We note that the extensions $\mathcal{E}u_{\mathcal{W}}$ and $\mathcal{E}u_W$ are wire basket functions on $\mathcal{W}$ and W, respectively, in the sense of [4]. That means they are uniquely defined by their values on $\mathcal{W}$ and W, respectively, and there holds

$$|\mathcal{E}u_{\mathcal{W}}|^2_{H^1(\Omega)} \le c\|\mathcal{E}u_{\mathcal{W}}\|_{L^2(\mathcal{W}_\Omega)} \simeq \|u_{\mathcal{W}}\|_{L^2(\mathcal{W})}$$

(see (4.7) in [4]) and

$$|\mathcal{E}u_W|^2_{H^1(\Omega)} \le c\|\mathcal{E}u_W\|_{L^2(W_\Omega)} \simeq \|u_W\|_{L^2(W)}$$

(see (4.9) in [4]). Now we use the trace theorem and obtain

$$\|u_{\mathcal{W}}\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\|u_{\mathcal{W}}\|^2_{L^2(\mathcal{W})} \quad \text{and} \quad \|u_W\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c\|u_W\|^2_{L^2(W)}.$$

Therefore, the maximum eigenvalue of the additive Schwarz operator is bounded by a constant which is independent of h and p.

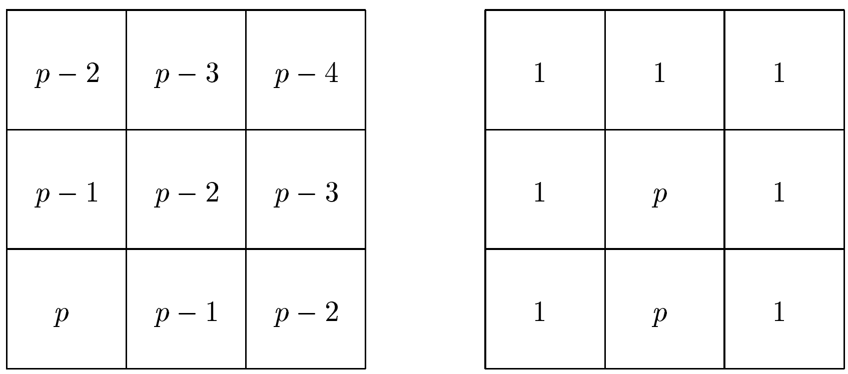
6 Numerical results.

To demonstrate the efficiency of our preconditioning method and to underline the theoretical estimates we present some experimental results for the extremum eigenvalues and the condition numbers of the preconditioned systems.

We emphasize that the boundary element method produces stiffness matrices which are in general fully occupied which means that even functions with disjoint supports are coupled via the integral operator. Therefore, when performing a domain decomposition to create a preconditioner, one not only decouples adjacent subdomains but also neglects the coupling of functions in subdomains which are not adjacent. The latter coupling is not present in the finite element method. Therefore, in the boundary element method, the theoretical bounds for the extremum eigenvalues are most often just asymptotically obtained and are not as obvious as in the finite element method from the experimental results.

The numerical experiments are carried out by implementing the entries of the stiffness matrices by analytical recurrence formulas, see [11]. The condition numbers are computed by reducing the matrices to tridiagonal form and by calculating the minimum and maximum eigenvalues via the QR-algorithm. For this we use standard LAPACK-routines.

For our model problem we choose the domain Γ to be the square plate $(-1/2, 1/2)^2 \times \{0\}$ and we use uniform meshes of squares of size h as indicated in the tables. For the polynomial degree distribution we consider a uniform, a locally uniform and a non-uniform method. For the uniform method the same polynomial degree is used everywhere and for the locally uniform method we use the mesh and polynomial degree distribution as indicated in Figure 1(a) (a number p means internal polynomial degrees $p_{I_1} = p_{I_2} = \max\{p, 1\}$). On the sides we take the minimum degree of the internal degrees in this direction from the neighboring elements. The non-uniform p version, for which we take the mesh and polynomial degrees of Figure 1(b), serves only as a test problem which does not solve the original problem efficiently (it even does not converge to the true solution). Moreover, our theory for the preconditioner does not cover this situation. This is due to the fact that all the sides (except one) are associated with degree $p = 1$. This does not give bounded ratios of maximum and minimum polynomial degrees on the two elements with internal degree p. However, the numerical results below show that our method works well in this case as well.



(a) (b)

Figure 1: The meshes and polynomial degrees for the locally uniform p version (a) and for the non-uniform p version (b).

Now let us present some numerical results. Due to Theorem 1 we expect for the uniform and locally uniform methods bounded maximum eigenvalues and minimum eigenvalues which behave like $(1 + \log \frac{p_{\max} H}{h})^{-2}$.

The polylogarithmic behavior in p of the condition number is checked with Figure 2. Here we consider the mesh $h = 1/3$ and $H/h = 1$. Both cases, uniform and locally uniform p version, as well as the results for the non-uniform p version are shown. In the uniform case

$p = 7$ corresponds to $N = 400$ number of unknowns. In the locally uniform case we have only $N = 202$ for $p = 7$ and in the non-uniform example $p = 7$ means $N = 82$. However, we observe that all curves a quite close which means that the condition number essentially depends on the maximum polynomial degree. The efficiency of the preconditioner seems to be independent of the actual distribution of the polynomial degrees, which is restricted in our theory. Moreover, in all cases the theoretical bound $(1 + \log p)^2$ is numerically fulfilled. Further, let us note that sometimes it is natural, e.g. when the full stiffness matrix is available, to use the original bilinear form instead of the L^2-bilinear form on the wire baskets, cf. (5) and (6). This replacement yields a different preconditioner whose implementation does not require additional inner products. Although this method in not covered by our theory the results in Figure 2 for this preconditioner (indicated by (1)) show the same asymptotic behavior as the theoretically justified method.

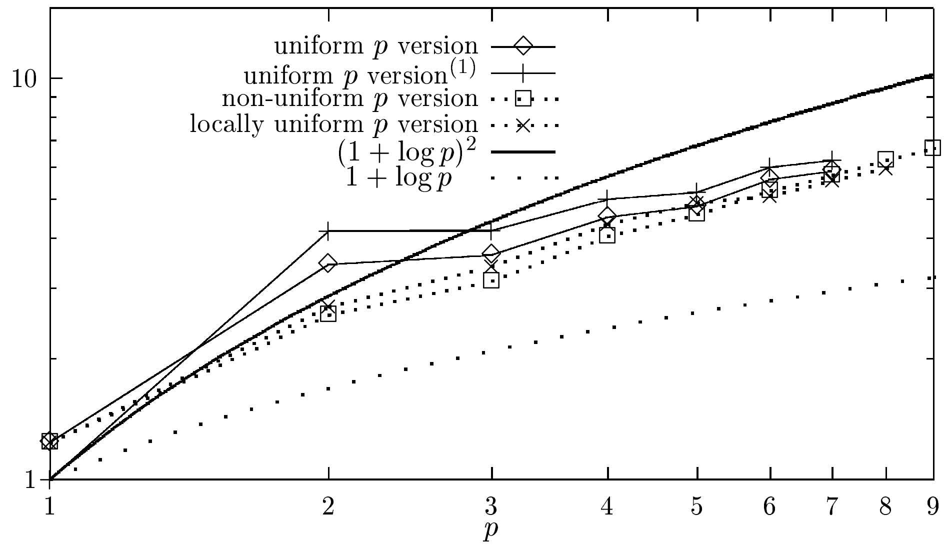


Figure 2: Condition numbers for the preconditioned uniform and non-uniform/locally uniform p version ($1/h = 3$, $H/h = 1$). The results marked by (1) (only the uniform p version) are obtained by using the original bilinear form instead of the L^2-bilinear form in the definition of the preconditioner.

Now let us check the dependence of the condition numbers on the parameters H and h. In Table 1 we list the results for different mesh sizes and fixed ratios H/h. Here, the case $p = 3$ (uniform distribution) is considered. The results seem to support bounded minimum and maximum eigenvalues. Table 2 analogously presents the results in the case $p = 2$. For this rather low degree we are able to further refine the mesh down to $h = 1/16$. Then the boundedness of the extremum eigenvalues becomes more obvious, cf. $H/h = 2$. Moreover, in that case there are some results for $H/h = 4$.

To underline the polylogarithmic dependence of the condition number on the ratio H/h we present Figure 3. The condition numbers are shown for the uniform polynomial degree $p = 2$ and where either H or h is fixed. The curve for fixed h corresponds to $h = 1/16$, i.e. 256 elements and 961 unknowns. The results for fixed H stem from the meshes $H = 1/2$ where

		$H/h=1$			$H/h=1^{(1)}$			$H/h=2$		
$1/h$	N	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
2	25	3.291	0.6234	2.052	3.987	0.6001	2.392	2.483	0.6234	1.548
3	64	3.633	0.6154	2.236	4.169	0.5981	2.493			
4	121	3.712	0.6106	2.267	4.236	0.5966	2.527	2.930	0.6106	1.789
5	196	3.739	0.6078	2.273	4.268	0.5957	2.542			
6	289	3.751	0.6061	2.273	4.285	0.5952	2.551	3.297	0.6061	1.998
7	400	3.761	0.6050	2.275	4.296	0.5950	2.556			
8	529	3.768	0.6040	2.276	4.303	0.5948	2.559	3.435	0.6038	2.074
9	676	3.774	0.6033	2.277	4.308	0.5947	2.562			
10	841	3.778	0.6028	2.277	4.312	0.5945	2.563	3.501	0.6027	2.110

Table 1: Condition numbers and eigenvalues for different ratios H/h ($p = 3$ uniformly). The columns marked by (1) are the results when the L^2-bilinear forms are replaced by the original bilinear form.

		$H/h=1$			$H/h=2$			$H/h=4$		
$1/h$	N	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$	κ	$\lambda_{\min}$	$\lambda_{\max}$
2	9	2.739	0.8280	2.268	2.109	0.8270	1.744			
4	49	3.583	0.7376	2.643	2.775	0.7355	2.041	2.677	0.7360	1.970
6	121	3.857	0.7168	2.765	3.019	0.7118	2.149			
8	225	3.998	0.7039	2.814	3.194	0.6977	2.229	3.111	0.6942	2.160
10	361	4.067	0.6977	2.837	3.286	0.6909	2.270			
12	529	4.104	0.6945	2.850	3.338	0.6872	2.294	3.265	0.6717	2.193
14	729	4.127	0.6925	2.858	3.372	0.6848	2.309			
16	961	4.141	0.6913	2.863	3.394	0.6832	2.319	3.304	0.6675	2.206

Table 2: Condition numbers and eigenvalues for different ratios H/h ($p = 2$ uniformly).

h decreases from $1/4$ to $1/16$. In that case the number of unknowns increases from $N = 49$ to $N = 961$. The figure also shows the theoretical asymptotic bound $(1 + \log H/h)^2$ as well as the function $1 + \log H/h$. The results seem to be better than the predicted asymptotic bound as they are closer to $1 + \log H/h$ (in the considered range of H/h).

Eventually Tables 3–5 present the number of iterations of the conjugate gradient method which are necessary to reduce the initial residual by a factor of 10^{-6}. The results demonstrate the efficiency of the preconditioner as the iteration numbers are substantially reduced in any situation. The method where the L^2-bilinear forms on the wire baskets are replaced by the original bilinear form (again indicated by (1)) requires iteration numbers which are very close to those of the proposed method. It appears that the iteration numbers depend only slightly on the ratio H/h and on p. This all holds for the uniform h and p versions (Table 3), for the locally uniform p version (Table 4) and for the non-uniform p version (Table 5) which is not guaranteed by the theory. The columns denoted by "–" refer to the un-preconditioned conjugate gradient method. In the standard case we use the basis functions as introduced in §2. To be precise, we did not specify the basis functions on the sides. Only the vertical

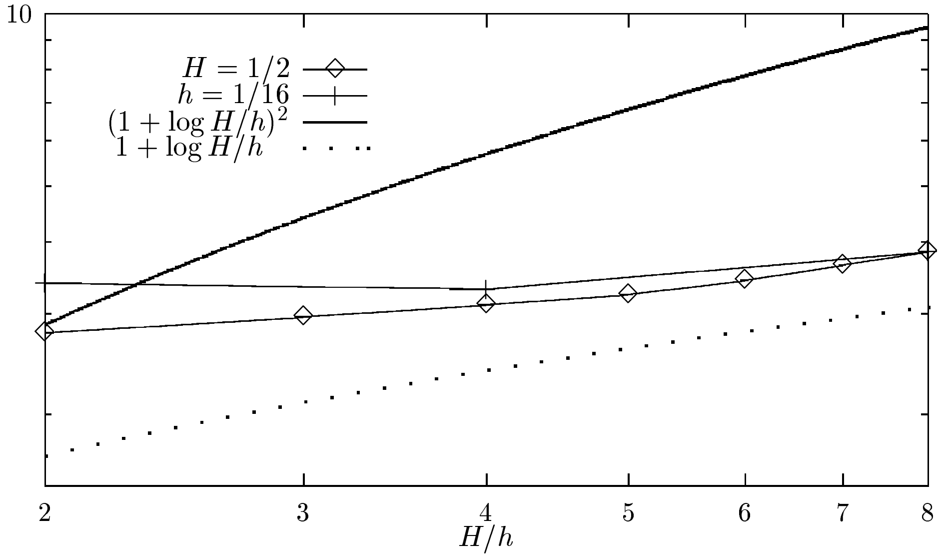


Figure 3: Condition numbers for the preconditioned h version ($p = 2$ uniformly).

components of the side functions are defined. For our implementation we use antiderivatives of Legendre polynomials on the sides that vanish at the endpoints of the sides. The basis functions given in §2 are particularly chosen to ensure the efficiency of our preconditioner. In a standard h-p boundary element method, where the convergence does not depend on the basis functions (only on the approximation spaces), one would use simpler basis functions. These basis functions are referred to as the standard basis functions. They differ from the basis functions in §2 in replacing the φ_0^p components by affine functions. That means, on the reference interval $(-1, 1)$ we use for the standard basis the affine functions $\varphi_0^1(x)$ and $\varphi_0^1(-x)$ instead of the polynomials $\varphi_0^p(x)$ and $\varphi_0^p(-x)$ of full degree. The corresponding results in Tables 3–5 are indicated by (2). Obviously, the iteration numbers for the special basis functions of §2 are already smaller than when using the standard basis functions. In Table 5 this is not evident but we note that in that example only one set of side functions is present, cf. Figure 1.

$1/h$	p	N	H/h					
			$-^{(2)}$	$-$	1	$1^{(1)}$	2	4
2	2	9	3	3	3	3	3	
4	2	49	10	10	10	9	10	9
6	2	121	25	16	12	13	10	
8	2	225	40	20	13	13	12	11
10	2	361	49	24	13	14	11	
12	2	529	56	27	13	14	12	12
14	2	729	58	29	13	14	12	
16	2	961	64	32	13	14	12	12
3	2	25	6	6	6	6		
3	3	64	11	10	9	8		
3	4	121	40	27	10	10		
3	5	196	67	43	11	10		
3	6	289	118	65	12	11		
3	7	400	161	89	13	13		

Table 3: Iteration numbers, (2) indicates standard basis functions and (1) refers to the preconditioner with L^2-bilinear form replaced by the original bilinear form.

$1/h$	$p_{\max}$	N	$-^{(2)}$	$-$	$H/h = 1$	$H/h = 1^{(1)}$
3	2	5	4	4	4	3
3	3	12	8	8	7	7
3	4	32	25	22	10	10
3	5	70	52	47	12	12
3	6	127	104	80	13	13
3	7	202	156	111	14	14
3	8	295	227	151	14	14

Table 4: Iteration numbers for the locally uniform p version, (2) indicates standard basis functions and (1) refers to the replacement of the L^2-bilinear form, as before.

$1/h$	$p_{\max}$	N	$-^{(2)}$	$-$	$H/h = 1$	$H/h = 1^{(1)}$
3	2	7	5	5	5	5
3	3	14	7	7	7	7
3	4	25	21	20	8	8
3	5	40	29	30	9	9
3	6	59	54	58	9	9
3	7	82	74	63	9	10
3	8	109	130	110	9	10
3	9	140	151	150	10	11

Table 5: Iteration numbers for the non-uniform p version, (2) indicates standard basis functions and (1) refers to the replacement of the L^2-bilinear form, as before.

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