An overlapping domain decomposition preconditioner for the p-version of the BEM with anisotropic elements

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Abstract

We study a preconditioner for the p-version of the boundary element method with anisotropic quadrilateral elements for hypersingular integral equations in three dimensions. Our preconditioner is based on an overlapping domain decomposition which is assumed to be locally quasi-uniform. Essentially, for quasi-uniform subdomain decompositions with mesh size H and overlap δ, we prove that the condition number of the preconditioned system is bounded by $O(1+\log H/\delta)^2$. Numerical results supporting our theory are reported.

Key words. p-version of the boundary element method, Schwarz method, preconditioner

AMS subject classifications. 65N55, 65N38

1 Introduction.

In this paper we are concerned with the preconditioning of linear systems which stem from the p-version of the boundary element Galerkin method (BEM) for hypersingular integral equations of the first kind on surfaces in $\mathbf{R}^3$. We restrict our considerations to quadrilateral meshes and we assume that the surface under consideration is such that it can be discretized by these elements. Our attention is focused on additive Schwarz type preconditioners which use subdomain decompositions with small overlap. For the h-version of the finite element method, this strategy and corresponding analysis are given by Dryja and Widlund in [5]. They consider also a Schur complement method by eliminating unknowns interior to subdomains, called Smith' method [15]. In [5] all the details are provided in two dimensions and the three dimensional situation is just briefly sketched. In this paper we analyze the BEM for three dimensional problems, thus dealing with trace spaces on surfaces. We consider piecewise polynomial spaces with arbitrary but uniform polynomial degrees. Our results are valid for anisotropic elements as long as

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the subdomain decomposition is locally quasi-uniform. The amount of overlap may vary from subdomain to subdomain. Our proofs are based in part also on results by Guo and Cao [6] (*hp*-version of the FEM) and our recent paper [9] (*hp*-version of the BEM). The strategies in the latter references are based on direct subspace decompositions. Therefore, the results in [6, 9] depend slightly (poly-logarithmically) on polynomial degrees. Here, we consider overlapping subspace decompositions which lead to optimal results concerning polynomial degrees (the condition numbers do not depend on them). Let us also mention that investigations similar to ours but for the h-version are in progress [17]. Moreover, additive Schwarz methods with generous overlap for the p-version of the FEM and BEM are analyzed in [13, 14] and [7], respectively.

Our model problem is as follows. We consider the weak form of the hypersingular integral equation

$$\langle Du, v\rangle = \langle f, v\rangle \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on a plane polygonal surface piece $\Gamma \subset \mathbf{R}^3$ where $f \in H^{-1/2}(\Gamma)$ is a given function. The Sobolev space $\tilde{H}^{1/2}(\Gamma)$ is defined in Section 3 and $H^{-1/2}(\Gamma)$ is its dual space. The notation $\langle \cdot, \cdot \rangle$ refers to the $L^2(\Gamma)$-duality. D is the hypersingular integral operator

$$Du(x) = \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma$$

which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [16]. Hence, there holds the equivalence of norms

$$\langle Dv, v\rangle_{L^2(\Gamma)} \simeq \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma). \tag{2}$$

The solution u of (1) is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathbf{R}^3 \setminus \bar{\Gamma}$, cf. [16].

We approximate u by the p-version of the Galerkin method. To this end we introduce a mesh consisting of quadrilaterals. Later, for the construction of preconditioners, we need a coarse mesh which consists in subdomains Γ_j, $j = 1, \ldots, J$. We assume that the subdomains are again quadrilaterals forming a regular, locally quasi-uniform mesh. That is, there are no hanging nodes in the coarse mesh and the subdomains are shape regular:

(i) for any subdomain the ratio of maximum to minimum side length is bounded by a constant < 1,

(ii) every subdomain contains a circle with radius larger than a constant > 0 times the diameter of the subdomain,

(iii) all angles are smaller than $\pi - \phi_0$ for some constant $\phi_0 > 0$.

The longest side length of Γ_j is denoted by H_j. We assume that the subdomains are formed by a set of individual elements Γ_{ji}. The fine mesh defined by the elements Γ_{ji} is assumed to be regular, i.e. no hanging nodes are present. Moreover, the elements must satisfy a minimum angle condition but may become distorted:

(iv) every element satisfies the minimum angle condition (iii) above,

(v) for every element the lengths of opposite sides are proportional.

Condition (v) allows that the ratio of adjacent side lengths grows during mesh refinement. For simplicity we use uniform polynomial degrees. The treatment of piecewise polynomial spaces with variable degrees by overlapping preconditioners is still under investigation. Some generalizations to non-uniform polynomial degrees for the FEM are given in [14]. Fixing a polynomial degree p, our ansatz space is given by

$$\Psi(\Gamma) = \{\psi \in C^0(\bar{\Gamma});\ \psi|_{\Gamma_{ji}} \circ T_{ji} \in \mathcal{P}_p((0,1)^2),\ \psi|_{\partial\Gamma} = 0\}.$$

Here, $T_{ji} :\ (0,1)^2 \to \Gamma_{ji}$ are mappings which are bilinear in both components and $\mathcal{P}_p$ denotes the space of polynomials up to degree p in both variables. As usual, the Galerkin scheme for (1) then reads as follows. *Find* $u_N \in \Psi(\Gamma)$ *such that*

$$\langle Du_N, v\rangle = \langle f, v\rangle \quad \textit{for all} \quad v \in \Psi(\Gamma). \tag{3}$$

Here, N refers to the dimension of $\Psi(\Gamma)$.

Since our operator D is symmetric and positive definite, the appearing matrices of the linear systems are symmetric and positive definite as well. The method of choice for solving positive definite linear systems is the conjugate gradient method. However, the p-version of the BEM yields ill-conditioned matrices and a preconditioner is necessary for an efficient solution. The goal of this paper is to present a preconditioner which guarantees moderate bounds for the spectral condition number of these matrices.

On outline of the paper is as follows. In the next section we define the preconditioning method and state the main result (Theorem 1). In Section 3 we provide the technical results and the proof of the theorem. Some numerical experiments are reported in Section 4.

Throughout the paper c denotes a generic constant which may take on different values at different occurrences.

2 Additive Schwarz preconditioner.

In order to specify our additive Schwarz preconditioner we define an overlapping decomposition of the domain Γ. To this end we extend each subdomain Γ_j (defined previously) to a subdomain Γ'_j where we do not cut through elements. When a subdomain meets the boundary of Γ then we perform the same construction but cut the part outside Γ. Each subdomain Γ'_j overlaps with neighboring subdomains, the overlap forming a strip covering $\partial\Gamma_j$ (the part not lying on $\partial\Gamma$). The minimum width of this strip is denoted by δ_j. We allow that $\Gamma_j = \Gamma'_j$ for some subdomains but with the restriction that $\delta_j > 0$. Moreover, we assume that the extended subdomains are again quadrilaterals. In this way we define the decomposition

$$\Psi(\Gamma) = \Psi_H(\Gamma) + \Psi(\Gamma'_1) + \cdots + \Psi(\Gamma'_J) \tag{4}$$

Here, $\Psi_H(\Gamma) = \Pi_h \Psi_H^*(\Gamma)$ where $\Psi_H^*(\Gamma)$ is the space of piecewise bilinear functions on the coarse mesh and $\Pi_h : \Psi_H^*(\Gamma) \to \Psi_h(\Gamma)$ is the interpolation operator onto the space of piecewise bilinear functions $\Psi_h(\Gamma)$ on the fine mesh. Furthermore, $\Psi(\Gamma'_j)$ is the local space

$$\Psi(\Gamma'_j) = \{\psi \in \Psi(\Gamma);\ \mathrm{supp}\,(\psi) \subset \bar{\Gamma}'_j\}.$$

For ease of presentation we use, instead of (4), also the notation

$$\Psi(\Gamma) = H_0 + \ldots + H_J. \tag{5}$$

The additive Schwarz method consists in solving, by an iterative method, the equation

$$Pu_N := (P_0 + P_1 + \cdots + P_J)u_N = f_N, \tag{6}$$

where the projections $P_j : \Psi(\Gamma) \to H_j$, $j = 0, \ldots, J$, are defined for any $v \in \Psi(\Gamma)$ by

$$\langle DP_j v, \varphi\rangle = \langle Dv, \varphi\rangle \quad \text{for any } \varphi \in H_j. \tag{7}$$

The right hand side of (6), $f_N = \sum_{j=0}^J P_j u_N$, can be computed without knowing the solution u_N of (3) by

$$\langle DP_j u_N, \varphi\rangle = \langle f, \varphi\rangle \quad \text{for any } \varphi \in H_j,\ j = 0, \ldots, J.$$

Eq. (6) is the preconditioned linear system and an estimate of its condition number is given by the next theorem.

Theorem 1 *There exist positive constants c_1, c_2 which are independent of H_j, δ_j, p and the fine mesh (under the above mentioned restrictions) such that for all $v \in \Psi(\Gamma)$ there holds*

$$c_1 \min_j \left(1 + \log \frac{H_j}{\delta_j}\right)^{-2} \min_{i,j:\,\Gamma_i \cap \Gamma'_j \neq \emptyset} \frac{H_i}{H_j}\ \langle Dv, v\rangle \le \langle DPv, v\rangle \le c_2 \langle Dv, v\rangle.$$

Therefore, the spectral condition number of P is bounded by

$$\kappa(P) \le \frac{c_2}{c_1} \max_j \left(1 + \log \frac{H_j}{\delta_j}\right)^2 \max_{i,j:\,\Gamma_i \cap \Gamma'_j \neq \emptyset} \frac{H_i}{H_j}.$$

Here, P is the additive Schwarz operator defined by the decomposition (4) *of the ansatz space* $\Psi(\Gamma)$.

3 Technical results and proof of the main theorem.

In this section we present preliminary technical lemmas and the proof of Theorem 1. First we need to define some Sobolev spaces and norms.

On an open surface piece Γ we introduce the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ where the latter space is most often denoted by $H_{00}^{1/2}(\Gamma)$ in the finite element literature. Let $\tilde{\Gamma}$ be a closed surface (in our case a polyhedral surface) with $\Gamma \subset \tilde{\Gamma}$. We define

$$H^{1/2}(\tilde{\Gamma}) := \{\phi|_{\tilde{\Gamma}};\ \phi \in H^1(\mathbf{R}^3)\}, \quad H^{1/2}(\Gamma) := \{\phi|_{\Gamma};\ \phi \in H^{1/2}(\tilde{\Gamma})\},$$

and

$$\tilde{H}^{1/2}(\Gamma) := \{\phi \in H^{1/2}(\Gamma);\ \tilde{\phi} \in H^{1/2}(\tilde{\Gamma})\},$$

where $\tilde{\phi}$ denotes the extension of ϕ by 0 from Γ onto $\tilde{\Gamma}$. A norm in $H^{1/2}(\Gamma)$ is given by (see [10])

$$\|\cdot\|^2_{H^{1/2}(\Gamma)} = \|\cdot\|^2_{L^2(\Gamma)} + |\cdot|^2_{H^{1/2}(\Gamma)}$$

with semi-norm

$$|v|^2_{H^{1/2}(\Gamma)} := \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^3}\, dS_x dS_y.$$

On a subdomain $S \subset \Gamma$ of diameter $O(H)$ we use the norm with weighted L^2-term

$$\|\cdot\|^2_{H^{1/2}(S)} = \frac{1}{H}\|\cdot\|^2_{L^2(S)} + |\cdot|^2_{H^{1/2}(S)}. \tag{8}$$

For the special case of a square with side length H we will make use of the equivalence (see [4] and [12, Lemma 5.3, Chap. 2])

$$\begin{aligned} |v|^2_{H^{1/2}((0,H)^2)} \simeq & \int_0^H \int_0^H \frac{\|v(s_1,\cdot) - v(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1 - t_1|^2}\, ds_1\, dt_1 \\ & + \int_0^H \int_0^H \frac{\|v(\cdot, s_2) - v(\cdot, t_2)\|^2_{L^2(0,H)}}{|s_2 - t_2|^2}\, ds_2\, dt_2. \end{aligned} \tag{9}$$

The energy norm induced by the hypersingular integral operator is equivalent to the norm

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} = \|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{1/2}(\Gamma)} + \int_\Gamma \frac{|v(x)|^2}{\operatorname{dist}(x, \partial\Gamma)}\, dS_x. \tag{10}$$

On local portions $\Gamma_j \subset \Gamma$ we weight the L^2-term as in (8). We also note that the spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ can be equivalently defined as intermediate spaces between $L^2(\Gamma)$ and $H^1(\Gamma)$ or $H^1_0(\Gamma)$ ($H^1_0(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ within $H^1(\Gamma)$). By interpolation theory we have for real $s \in (0,1)$, see [2],

$$H^s(\Gamma) = [L^2(\Gamma), H^1(\Gamma)]_s \quad \text{and} \quad \tilde{H}^s(\Gamma) = [L^2(\Gamma), H^1_0(\Gamma)]_s.$$

This relation is used for subdomains $\Gamma_j \subset \Gamma$ as well.

Now let us collect some technical results. Recalling the procedure from [3] (see also [6]) we define an interpolation operator

$$\Pi^0_H :\ L^2(\Gamma) \to \Psi^*_H(\Gamma) \tag{11}$$

associated with the coarse mesh. Let $\{q\}$ denote the nodes of the coarse mesh and let Γ_q be the union of subdomains Γ_j adjacent to q. Then

$$\Pi_H^0 v := \sum_q \alpha_q \phi_q$$

where ϕ_q is the piecewise bilinear nodal basis function associated with q (and the coarse mesh) and $\alpha_q = \tilde{v}(q)$. Here, $\tilde{v}$ is continuous and piecewise bilinear on Γ_q (zero on $\Gamma \setminus \Gamma_q$) and is defined by

$$\int_{\Gamma_q} (v - \tilde{v})\phi \, dS = 0$$

for all continuous, piecewise bilinear functions ϕ on Γ_q.

The following result can be derived by interpolation from the analysis given by Clément [3], and has previously been given in [9, Lemma 1].

Lemma 1 *For any $v \in H^{1/2}(\Gamma)$ and locally quasi-uniform coarse meshes,*

$$|\Pi_H^0 v - v|^2_{H^{1/2}(\Gamma)} + \sum_j H_j^{-1} \|\Pi_H^0 v - v\|^2_{L^2(\Gamma_j)} \le c|v|^2_{H^{1/2}(\Gamma)}.$$

Moreover, for $v \in \tilde{H}^{1/2}(\Gamma)$,

$$\|\Pi_H^0 v\|_{\tilde{H}^{1/2}(\Gamma)} \le c\|v\|_{\tilde{H}^{1/2}(\Gamma)}.$$

Since piecewise bilinear functions on the coarse mesh in general are not piecewise bilinear functions on the fine mesh, we need to consider boundedness of the interpolation operator Π_h mentioned in Section 2.

Lemma 2 *For any $v \in \Psi_H^*$,*

$$|\Pi_h v|_{H^l(\Gamma)} \le c|v|_{H^l(\Gamma)}, \quad 0 \le l \le 1.$$

Proof. For non-distorted elements this result follows by interpolation, from [6, Lemma 4.3], see [9, Lemma 2]. The proof is based on smooth element mappings and uses, on the reference element, the equivalence of norms on finite dimensional spaces. For distorted elements, which we are considering, one only has to modify the element mappings. Considering a distorted quadrilateral Γ_{ji} we use a smooth mapping $F_{ji} : (0, h_1) \times (0, h_2) \to \Gamma_{ji}$ from a distorted reference rectangle. The mapping can be chosen to be bilinear in both components. Taking h_1 and h_2 such that the ratio of maximum and minimum side lengths of Γ_{ji} equals h_2/h_1 one has boundedness of the Jacobians of F_{ji}, F_{ji}^{-1} and of their determinants. Without loss of generality we may assume that $h_2 \ge h_1$. For $v \in \Psi_H^*$ we obtain with $\hat{v} = v \circ F_{ji}$ and $\widehat{\Pi_h v} = \Pi_h v \circ F_{ji}$

$$|\Pi_h v|^2_{H^1(\Gamma_{ji})} = \int_0^{h_2} \int_0^{h_1} |J_{F_{ji}^{-1}} \nabla \widehat{\Pi_h v}(\xi_1, \xi_2)|^2 \det(J_{F_{ji}}) \, d\xi_1 \, d\xi_2 \simeq |\widehat{\Pi_h v}|^2_{H^1((0,h_1)\times(0,h_2))}.$$

Using the affine mapping $G_{h_1,h_2} : (0,1)^2 \to (0,h_1) \times (0,h_2)$ and defining $\widetilde{\Pi_h v} = \Pi_h v \circ F_{ji} \circ G_{h_1,h_2}$ we continue estimating

$$|\Pi_h v|^2_{H^1(\Gamma_{ji})} \le c|\widehat{\Pi_h v}|^2_{H^1((0,h_1)\times(0,h_2))} \le c\frac{h_2}{h_1}|\widetilde{\Pi_h v}|^2_{H^1((0,1)^2)} \tag{12}$$

where we used that $h_2 \ge h_1$. Now we take a smooth transformation $F_j : (0,1)^2 \to \Gamma_j$ from the reference square onto the subdomain Γ_j which covers the element Γ_{ji} and denote $\tilde{\Gamma}_{ji} = F_j^{-1}(\Gamma_{ji})$. Then, $\tilde{\Gamma}_{ji}$ is a smooth transformation of the distorted element $(0, h_1/H_j) \times (0, h_2/H_j)$. Note that the subdomain Γ_j is shape regular and of diameter $O(H_j)$. Since $v \circ F_j|_{\tilde{\Gamma}_{ji}}$ depends only on its values at the vertices of $\tilde{\Gamma}_{ji}$ and since norms on finite dimensional spaces are equivalent, we get an upper bound for the semi-norm

$$|\widetilde{\Pi_h v}|_{H^1((0,1)^2)} \le c\|v \circ F_j\|_{L^\infty(\tilde{\Gamma}_{ji})}.$$

Then we conclude by a scaling argument as in the proof of Lemma 4.3 of [6] that there holds

$$|\widetilde{\Pi_h v}|^2_{H^1((0,1)^2)} \le c\|v \circ F_j\|^2_{L^\infty(\tilde{\Gamma}_{ji})} \le c\left(\frac{H_j^2}{h_1 h_2}\|v \circ F_j\|^2_{L^2(\tilde{\Gamma}_{ji})} + \frac{h_1/H_j}{h_2/H_j}|v \circ F_j|^2_{H^1(\tilde{\Gamma}_{ji})}\right). \tag{13}$$

Combining (12) and (13) and using a quotient space argument as in [6] this yields

$$|\Pi_h v|^2_{H^1(\Gamma_{ji})} \le c|v \circ F_j|^2_{H^1(\tilde{\Gamma}_{ji})} \le c|v|^2_{H^1(\Gamma_{ji})}.$$

Summing over all the elements we obtain the assertion for $l = 1$. The estimate for the L^2-norm is analogous and for intermediate spaces we use interpolation. □

The next two results are Lemmas 3.4, 3.5 in [5].

Lemma 3 *There exists a constant C such that, for $0 < \delta < H$,*

$$\|u\|^2_{L^2((0,H)\times(0,\delta))} \le C\delta(1 + \log\frac{H}{\delta})^2\|u\|^2_{H^{1/2}((0,H)^2)} \quad \forall u \in H^{1/2}((0,H)^2).$$

Lemma 4 *There exists a constant C such that, for $0 < \delta < H$,*

$$\int_0^H \int_\delta^H \frac{|u(s,t)|^2}{s}\, ds\, dt \le C(1 + \log\frac{H}{\delta})^2\|u\|^2_{H^{1/2}((0,H)^2)} \quad \forall u \in H^{1/2}((0,H)^2).$$

3.1 Proof of Theorem 1.

Since we use the original bilinear form, i.e. the one given by the hypersingular integral operator, for the definition of the projections (7), the bound for the maximum eigenvalue of P is straight forward. We have to prove that there exists a constant C such that

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le C\sum_{j=0}^{J}\|v_j\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

for any $v \in \Psi(\Gamma)$ and any representation $v = \sum_{j=0}^{J} v_j$, according to the decomposition of $\Psi(\Gamma)$. Since we use an overlapping decomposition where only a limited number of subspaces overlap the same region, this estimate follows from the boundedness

$$\|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \le c \sum_j \|v|_{\Gamma_j}\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

for functions $v \in \tilde{H}^{1/2}(\Gamma)$ such that $v|_{\Gamma_j} \in \tilde{H}^{1/2}(\Gamma_j)$, see, e.g., [1, 8].

Now we bound the minimum eigenvalue of the additive Schwarz operator P. Due to Lions' lemma we have to find a constant $c > 0$ such that, for any $v \in \Psi(\Gamma)$, there exists a representation $v = \sum_{j=0}^{J} v_j$ (according to (4),(5)) such that

$$c \min_j (1 + \log \frac{H_j}{\delta_j})^{-2} \min_{i,j:\, \Gamma_i \cap \Gamma'_j \neq \emptyset} \frac{H_i}{H_j} \sum_{j=0}^{J} \|v_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le \|v\|^2_{\tilde{H}^{1/2}(\Gamma)}.$$

Therefore, let $v \in \Psi(\Gamma)$ be given. We define the coarse grid contribution $v_0 \in \Psi_H(\Gamma)$ by Clément's interpolation and use a partition of unity to construct the local contributions $v_j \in \Psi(\Gamma'_j)$. More precisely, let $v_0 := \Pi_h \Pi_H^0 v$ where $\Pi_h : \Psi_H^*(\Gamma) \to \Psi_h(\Gamma) \subset \Psi(\Gamma)$ is the piecewise bilinear interpolation operator mentioned in Section 2 and Π_H^0 is the operator from (11). Combining Lemmas 2 and 1 we deduce

$$\|v_0\|_{\tilde{H}^{1/2}(\Gamma)} \le c\|\Pi_H^0 v\|_{\tilde{H}^{1/2}(\Gamma)} \le c\|v\|_{\tilde{H}^{1/2}(\Gamma)}. \tag{14}$$

Now, in order to define the local contributions $v_j \in \tilde{H}^{1/2}(\Gamma'_j)$ of v, we use a partition of unity $\{\Theta_j; j = 1, \ldots, J\}$, i.e. $\sum_{j=1}^{J} \Theta_j(x) = 1$, such that $0 \le \Theta_j(x) \le 1$ and $\operatorname{supp}(\Theta_j) = \bar{\Gamma}'_j$. Then $\Theta_j(x) = 1$ for x in the interior part of Γ'_j which does not intersect with neighboring subdomains. The gradient of Θ_j is bounded by

$$|\nabla \Theta_j| \le C/\delta_j.$$

We define $w = v - v_0$ and consider $w_j = \Theta_j w$. Then there holds $v = v_0 + \sum_{j=1}^{J} w_j$ but $w_j \notin \Psi(\Gamma'_j)$ in general since these local functions are of degree $p+1$. We therefore use the piecewise Legendre-Gauss-Lobatto interpolations

$$\Pi_p w_j \in \Psi(\Gamma'_j)$$

by transforming w_j, individually on every element, back to the reference element $(-1,1)^2$ and defining $\Pi_p w_j$ by the polynomial of degree p that coincides with w_j in the points (ξ_k, ξ_l). Here, ξ_k, $k = 1, \ldots, p+1$, are the zeroes of the antiderivative of the Legendre polynomial of degree p, $\int_{-1}^{\xi} L_p(t)\, dt$. This interpolation operator is bounded with respect to the L^2-norm and the H^1-semi-norm on every element and the bounds do not depend on the polynomial degree. For shape regular elements the proofs are given in [14, 13]. For distorted elements we use transformations as in the proof of Lemma 2. Therefore,

$$\|\Pi_p w_j\|_{\tilde{H}^{1/2}(\Gamma'_j)} \le c\|w_j\|_{\tilde{H}^{1/2}(\Gamma'_j)}. \tag{15}$$

By the linearity of Π_p, and since $w \in \Psi(\Gamma)$, there holds

$$\sum_{j=1}^{J} \Pi_p w_j = \Pi_p \sum_{j=1}^{J} \Theta_j w = \Pi_p w = w.$$

Therefore $v = \sum_{j=0}^{J} v_j$ for $v_j := \Pi_p w_j$ $(j > 0)$ and it remains to bound the norms $\|w_j\|_{\tilde{H}^{1/2}(\Gamma'_j)}$. We use the definition (10), (9) of the norm. The subdomain Γ'_j is a shape regular quadrilateral with side lengths proportional to $H_j + \delta_j$ and we are interested in small values of δ_j. Therefore it suffices to consider, instead of Γ'_j, the square $(0, \tilde{H}_j)^2$ with $\tilde{H}_j = H_j + \delta_j$ and boundary strip of width δ_j. The results we obtain depend on the ratio $\tilde{H}_j/\delta_j$ which is asymptotically the same as H_j/δ_j. We therefore may simplify the presentation by considering the square $(0, H_j)^2$ instead of $(0, \tilde{H}_j)^2$ and also we may assume that δ_j is small enough, e.g. that $0 < \delta_j < H_j/3$. Then δ_j does not get close to $H_j/2$. The proof for larger δ_j provides no new difficulties.

Moreover, we assume for simplicity that, transformed onto $(0, H_j)^2$, Θ_j can be written as a tensor product $\Theta_j(s_1, s_2) = \theta_j(s_1)\theta_j(s_2)$. We skip the index j for δ_j, H_j and θ_j and write

$$\begin{aligned}
\|w_j\|^2_{\tilde{H}^{1/2}(\Gamma'_j)} \simeq \|\Theta_j w\|^2_{\tilde{H}^{1/2}((0,H)^2)} \simeq\ & \int_0^H \int_0^H \frac{\|w_j(s_1, \cdot) - w_j(t_1, \cdot)\|^2_{L^2(0,H)}}{|s_1 - t_1|^2} \, ds_1 \, dt_1 \\
&+ \int_0^H \int_0^H \frac{\|w_j(\cdot, s_2) - w_j(\cdot, t_2)\|^2_{L^2(0,H)}}{|s_2 - t_2|^2} \, ds_2 \, dt_2 \\
&+ \int_0^H \int_0^H \frac{|w_j(s_1, s_2)|^2}{\operatorname{dist}(s, \partial(0, H)^2)} \, ds_1 \, ds_2. \qquad (16)
\end{aligned}$$

Next we follow Dryja and Widlund [4, Proof of Theorem 5.1] who provided the details in one space dimension. The first two double integrals on the right hand side of (16) can be estimated in the same way. We consider the first one. Denoting $S_1 = (0, \delta)$, $S_2 = (\delta, H - \delta)$, $S_3 = (H - \delta, H)$ we write

$$\begin{aligned}
&\int_0^H \int_0^H \frac{\|w_j(s_1, \cdot) - w_j(t_1, \cdot)\|^2_{L^2(0,H)}}{|s_1 - t_1|^2} \, ds_1 \, dt_1 \\
&= \sum_{k,l=1}^{3} \int_{S_k} \int_{S_l} \frac{\|w_j(s_1, \cdot) - w_j(t_1, \cdot)\|^2_{L^2(0,H)}}{|s_1 - t_1|^2} \, ds_1 \, dt_1.
\end{aligned}$$

There appear nine integrals. We use the notation $S_k \times S_l$ to refer to the double integrals given above with $s_1 \in S_l$ and $t_1 \in S_k$.

$S_1 \times S_1$: Here we write

$$\begin{aligned}
w_j(s_1, \cdot) - w_j(t_1, \cdot) &= \frac{s_1\theta(\cdot)w(s_1, \cdot) - t_1\theta(\cdot)w(t_1, \cdot)}{\delta} \\
&= \frac{(s_1 + t_1)\theta(\cdot)(w(s_1, \cdot) - w(t_1, \cdot))}{2\delta} + \frac{(s_1 - t_1)\theta(\cdot)(w(s_1, \cdot) + w(t_1, \cdot))}{2\delta}
\end{aligned}$$

Then, using $\|\theta\|_{L^\infty} \leq 1$, we obtain

$$\begin{aligned}
&\|w_j(s_1,\cdot) - w_j(t_1,\cdot)\|^2_{L^2(0,H)} \\
&\quad \leq \frac{1}{2\delta^2}\left(\|(s_1+t_1)(w(s_1,\cdot) - w(t_1,\cdot))\|^2_{L^2(0,H)} + \|(s_1-t_1)(w(s_1,\cdot) + w(t_1,\cdot))\|^2_{L^2(0,H)}\right).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_{S_1}\int_{S_1} \frac{\|w_j(s_1,\cdot) - w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \\
&\quad \leq \frac{1}{2\delta^2}\int_{S_1}\int_{S_1} \frac{(s_1+t_1)^2}{(s_1-t_1)^2}\|w(s_1,\cdot) - w(t_1,\cdot)\|^2_{L^2(0,H)}\,ds_1\,dt_1 \\
&\qquad + \frac{1}{2\delta^2}\int_{S_1}\int_{S_1} \|w(s_1,\cdot) + w(t_1,\cdot)\|^2_{L^2(0,H)}\,ds_1\,dt_1 \\
&\quad \leq 2\int_{S_1}\int_{S_1} \frac{\|w(s_1,\cdot) - w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + \frac{2}{\delta}\|w\|^2_{L^2(S_1\times(0,H))}
\end{aligned}$$

where we used that $(s_1+t_1)^2/(2\delta^2) \leq 2$ on $S_1 \times S_1$ for the last estimate.

$S_3 \times S_3$: This case is analogous to that of $S_1 \times S_1$.

$S_2 \times S_2$: Here, there holds $w_j(s_1,\cdot) = \theta(s_1)\theta(\cdot)w(s_1,\cdot) = \theta(\cdot)w(s_1,\cdot)$ such that, using $\|\theta\|_{L^\infty} \leq 1$, we obtain

$$\int_{S_2}\int_{S_2} \frac{\|w_j(s_1,\cdot) - w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \leq \int_{S_2}\int_{S_2} \frac{\|w(s_1,\cdot) - w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1.$$

$S_1 \times S_2$: We represent

$$\begin{aligned}
w_j(s_1,\cdot) - w_j(t_1,\cdot) &= \theta(\cdot)w(s_1,\cdot) - \frac{t_1}{\delta}\theta(\cdot)w(t_1,\cdot) \\
&= \theta(\cdot)w(s_1,\cdot) - \theta(\cdot)w(t_1,\cdot) + \frac{\delta\theta(\cdot)w(t_1,\cdot) - t_1\theta(\cdot)w(t_1,\cdot)}{\delta}
\end{aligned}$$

and obtain

$$\begin{aligned}
&\int_{S_1}\int_{S_2} \frac{\|w_j(s_1,\cdot) - w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \\
&\quad \leq 2\int_{S_1}\int_{S_2} \frac{\|w(s_1,\cdot) - w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + 2\int_{S_1}\int_{S_2} \frac{(\delta-t_1)^2}{\delta^2}\frac{\|w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1.
\end{aligned}$$

Since

$$\int_{S_2} \frac{(\delta-t_1)^2}{(s_1-t_1)^2}\,ds_1 \leq c\delta \quad (t_1 \in S_1)$$

we can estimate the last integral by

$$\int_{S_1}\int_{S_2} \frac{(\delta-t_1)^2}{\delta^2}\frac{\|w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \leq \frac{c}{\delta}\int_{S_1} \|w(t_1,\cdot)\|^2_{L^2(0,H)}\,dt_1 = \frac{c}{\delta}\|w\|^2_{L^2(S_1\times(0,H))}.$$

Therefore, on $S_1 \times S_2$ we finally have

$$\begin{aligned}
&\int_{S_1}\int_{S_2} \frac{\|w_j(s_1,\cdot)-w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \\
&\quad\leq\ 2\int_{S_1}\int_{S_2} \frac{\|w(s_1,\cdot)-w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + \frac{c}{\delta}\|w\|^2_{L^2(S_1\times(0,H))}.
\end{aligned}$$

$S_1 \times S_3$: We use that $\|\theta\|_{L^\infty} \leq 1$. It follows

$$\begin{aligned}
&\int_{S_1}\int_{S_3} \frac{\|w_j(s_1,\cdot)-w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \\
&\quad\leq\ 2\int_{S_1}\int_{S_3} \frac{\|w(s_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + 2\int_{S_1}\int_{S_3} \frac{\|w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 \\
&\quad\leq\ 2\frac{\delta}{(H-2\delta)^2}\|w\|^2_{L^2(S_3\times(0,H))} + 2\frac{\delta}{(H-2\delta)^2}\|w\|^2_{L^2(S_1\times(0,H))} \\
&\quad\leq\ \frac{c}{\delta}\left(\|w\|^2_{L^2(S_3\times(0,H))} + \|w\|^2_{L^2(S_1\times(0,H))}\right)
\end{aligned}$$

where we used that $|s_1 - t_1| \geq H - 2\delta \geq H/3$ for $(t_1, s_1) \in S_1 \times S_3$ and that $\delta < H/3$ by assumption. The last estimate is just a crude bound to get similar expressions for all cases.

All the other appearing double integrals are analogous to the preceding cases such that for the first two integrals on the right hand side of (16) we obtain

$$\begin{aligned}
&\int_0^H\int_0^H \frac{\|w_j(s_1,\cdot)-w_j(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + \int_0^H\int_0^H \frac{\|w_j(\cdot,s_2)-w_j(\cdot,t_2)\|^2_{L^2(0,H)}}{|s_2-t_2|^2}\,ds_2\,dt_2 \\
&\leq\ c\int_0^H\int_0^H \frac{\|w(s_1,\cdot)-w(t_1,\cdot)\|^2_{L^2(0,H)}}{|s_1-t_1|^2}\,ds_1\,dt_1 + c\int_0^H\int_0^H \frac{\|w(\cdot,s_2)-w(\cdot,t_2)\|^2_{L^2(0,H)}}{|s_2-t_2|^2}\,ds_2\,dt_2 \\
&\quad + \frac{c}{\delta}\left(\|w\|^2_{L^2(S_1\times(0,H))} + \|w\|^2_{L^2(S_3\times(0,H))} + \|w\|^2_{L^2((0,H)\times S_1)} + \delta\|w\|^2_{L^2((0,H)\times S_3)}\right). \qquad (17)
\end{aligned}$$

Now we deal with the third term on the right hand side of (16):

$$\int_0^H\int_0^H \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\,ds_1\,ds_2 = \sum_{k,l=1}^{3}\int_{S_k}\int_{S_l} \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\,ds_1\,ds_2. \qquad (18)$$

On $S_2 \times S_2$ there holds $w_j = w$ and we can use Lemma 4 to bound

$$\begin{aligned}
\int_{S_2}\int_{S_2} \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\,ds_1\,ds_2 &= \int_\delta^{H-\delta}\int_\delta^{H-\delta} \frac{|w(s_1,s_2)|^2}{\min\{s_1,s_2,H-s_1,H-s_2\}}\,ds_1\,ds_2 \\
&\leq\ c(1+\log\frac{H}{\delta})^2\|w\|^2_{H^{1/2}((0,H)^2)} \qquad (19)
\end{aligned}$$

To bound the other integrals in (18) it suffices to consider the cases $S_1 \times S_1$, $S_2 \times S_1$ and $S_3 \times S_1$. The remaining cases are analogous.

On $S_1 \times S_1$ there holds

$$\frac{\theta^2(s_1)\theta^2(s_2)}{\operatorname{dist}(s, \partial(0,H)^2)} = \frac{s_1^2 s_2^2}{\delta^4 \min\{s_1, s_2\}} = \frac{\max\{s_1, s_2\} s_1 s_2}{\delta^4} \le \frac{1}{\delta}$$

such that

$$\int_{S_1}\int_{S_1} \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\, ds_1\, ds_2 = \int_0^\delta \int_0^\delta \frac{|\theta(s_1)\theta(s_2)w(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\, ds_1\, ds_2 \le \frac{1}{\delta}\|w\|^2_{L^2(S_1\times S_1)}. \tag{20}$$

On $S_2 \times S_1$ there holds $\operatorname{dist}(s,\partial(0,H)^2) = s_1$, $\theta(s_1) = s_1/\delta$ and $\theta(s_2) = 1$ such that

$$\int_{S_2}\int_{S_1} \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\, ds_1\, ds_2 = \int_\delta^{H-\delta}\int_0^\delta \frac{s_1}{\delta^2}|w(s_1,s_2)|^2\, ds_1\, ds_2 \le \frac{1}{\delta}\|w\|^2_{L^2(S_1\times S_2)}. \tag{21}$$

Finally, on $S_3 \times S_1$, there holds $\operatorname{dist}(s,\partial(0,H)^2) = \min\{s_1, H-s_2\}$, $\theta(s_1) = s_1/\delta$ and $\theta(s_2) = (H-s_2)/\delta$ such that

$$\frac{\theta^2(s_1)\theta^2(s_2)}{\operatorname{dist}(s,\partial(0,H)^2)} = \frac{s_1^2(H-s_2)^2}{\delta^4\min\{s_1, H-s_2\}} = \frac{\max\{s_1, H-s_2\}s_1(H-s_2)}{\delta^4} \le \frac{1}{\delta}.$$

This yields

$$\int_{S_3}\int_{S_1} \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\, ds_1\, ds_2 \le \frac{1}{\delta}\|w\|^2_{L^2(S_1\times S_3)}. \tag{22}$$

Therefore, by using (19)–(22), we obtain for (18) the estimate

$$\begin{aligned}\int_0^H\int_0^H \frac{|w_j(s_1,s_2)|^2}{\operatorname{dist}(s,\partial(0,H)^2)}\, ds_1\, ds_2 &\le c(1+\log\frac{H}{\delta})^2\|w\|^2_{H^{1/2}((0,H)^2)} \\ &+\frac{1}{\delta}\left(\|w\|^2_{L^2(S_1\times(0,H))} + \|w\|^2_{L^2(S_3\times(0,H))} + \|w\|^2_{L^2((0,H)\times S_1)} + \|w\|^2_{L^2((0,H)\times S_3)}\right).\end{aligned} \tag{23}$$

Now we combine the bounds (17), (23) and use Lemma 3 to conclude that (16) is bounded by

$$\|w_j\|^2_{\tilde H^{1/2}((0,H)^2)} \le c(1+\log\frac{H}{\delta})^2\|w\|^2_{H^{1/2}((0,H)^2)}.$$

Recalling the definition of the local $H^{1/2}$-norm with weighted L^2-term (8) and transforming $(0,H)^2 \to \Gamma'_j$ we obtain

$$\|w_j\|^2_{\tilde H^{1/2}(\Gamma'_j)} \le c(1+\log\frac{H_j}{\delta_j})^2\left(\frac{1}{H_j}\|w\|^2_{L^2(\Gamma'_j)} + |w|^2_{H^{1/2}(\Gamma'_j)}\right). \tag{24}$$

Now we recall the definition of $w = v - v_0 = v - \Pi_h\Pi_H^0 v$ and use the triangle inequality to bound the term in brackets:

$$\begin{aligned}\frac{1}{H_j}\|w\|^2_{L^2(\Gamma'_j)} + |w|^2_{H^{1/2}(\Gamma'_j)} \;\le\;& \frac{2}{H_j}\left(\|v-\Pi_H^0 v\|^2_{L^2(\Gamma'_j)} + \|\Pi_H^0 v - \Pi_h\Pi_H^0 v\|^2_{L^2(\Gamma'_j)}\right) \\ &+2|v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma'_j)} + 2|\Pi_H^0 v - \Pi_h\Pi_H^0 v|^2_{H^{1/2}(\Gamma'_j)}.\end{aligned} \tag{25}$$

We estimate the terms on the right hand side of (25). The subdomain Γ_j' overlaps with neighboring subdomains. We estimate

$$\begin{aligned}\frac{1}{H_j}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j')} &= \frac{1}{H_j}(\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j)} + \|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j'\setminus\Gamma_j)})\\ &\le \frac{1}{H_j}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j)} + \sum_{i:\, i\neq j,\, \Gamma_i\cap\Gamma_j'\neq\emptyset} \frac{H_i}{H_j}\frac{1}{H_i}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_i)}. \quad (26)\end{aligned}$$

Second, by Lemma 2 and a quotient space argument we obtain

$$\begin{aligned}&\frac{2}{H_j}\|\Pi_H^0 v - \Pi_h\Pi_H^0 v\|^2_{L^2(\Gamma_j')} + 2|\Pi_H^0 v - \Pi_h\Pi_H^0 v|^2_{H^{1/2}(\Gamma_j')}\\ &\le 4\inf_{\alpha\in\mathrm{R}}\left(\frac{1}{H_j}\|\Pi_H^0 v+\alpha\|^2_{L^2(\Gamma_j')} + |\Pi_H^0 v+\alpha|^2_{H^{1/2}(\Gamma_j')}\right) \le c|\Pi_H^0 v|^2_{H^{1/2}(\Gamma_j')}. \quad (27)\end{aligned}$$

Now, combination of (25)–(27) yields

$$\begin{aligned}\frac{1}{H_j}\|w\|^2_{L^2(\Gamma_j')} + |w|^2_{H^{1/2}(\Gamma_j')} &\le \frac{c}{H_j}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j)} + c\sum_{i:\, i\neq j,\, \Gamma_i\cap\Gamma_j'\neq\emptyset}\frac{H_i}{H_j}\frac{1}{H_i}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_i)}\\ &\quad + c|v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma_j')} + c|\Pi_H^0 v|^2_{H^{1/2}(\Gamma_j')}.\end{aligned}$$

Since each subdomain overlaps only with finitely many other subdomains one can bound

$$\sum_{j=1}^{J}|v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma_j')} \le c|v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma)}$$

and analogously for $\Pi_H^0 v$. Therefore, (24) and the previous two estimates lead to

$$\begin{aligned}\sum_{j=1}^{J}\|w_j\|^2_{\tilde H^{1/2}(\Gamma_j')} &\le c\max_j(1+\log\frac{H_j}{\delta_j})^2\max_j\left(\max_{i:\, i\neq j,\, \Gamma_i\cap\Gamma_j'\neq\emptyset}\frac{H_i}{H_j}\right)\\ &\quad\left(\sum_{j=1}^{J}\frac{1}{H_j}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j)} + |v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma)} + |\Pi_H^0 v|^2_{H^{1/2}(\Gamma)}\right). \quad (28)\end{aligned}$$

By Lemma 1 we can bound

$$\sum_{j=1}^{J}\frac{1}{H_j}\|v-\Pi_H^0 v\|^2_{L^2(\Gamma_j)} + |v-\Pi_H^0 v|^2_{H^{1/2}(\Gamma)} + |\Pi_H^0 v|^2_{H^{1/2}(\Gamma)} \le c|v|^2_{H^{1/2}(\Gamma)}.$$

Therefore, recalling the continuity of the operator Π_p (15) from the beginning of the proof and the definition $v_j = \Pi_p w_j$, we finally conclude from (28) that there holds

$$\sum_{j=1}^{J}\|v_j\|^2_{\tilde H^{1/2}(\Gamma_j')} \le c\max_j(1+\log\frac{H_j}{\delta_j})^2\max_j\left(\max_{i:\, \Gamma_i\cap\Gamma_j'\neq\emptyset}\frac{H_i}{H_j}\right)|v|^2_{H^{1/2}(\Gamma)}.$$

Together with (14) this gives the lower bound for the minimum eigenvalue of P and the proof of Theorem 1 is finished.

4 Numerical results.

To underline the theoretical estimates we present some experimental results for the condition numbers of the preconditioned boundary element systems. For our model problem we choose the domain Γ to be the square plate $(-1/2, 1/2)^2 \times \{0\}$ and we use two different sequences of test meshes. These meshes are especially chosen to test the dependence of the condition number on the ratio of H/δ where H is proportional to the diameter of the subdomains and δ is a measure for the overlap of neighboring subdomains. Our theory covers the case of subdomains of different sizes and with different overlapping distances, cf. Theorem 1. For our experiments, however, we only consider subdomains of identical size and identical overlap.

The entries of the stiffness matrices are calculated by analytical inner integration and numerical outer integration with a Gaussian quadrature, see [11]. Integrals are always calculated by using monomials which then have to be combined to generate standard basis functions (tensor products of integrated Legendre polynomials). This summing of integrals of monomials leads to a loss of precision which prevents us from considering higher polynomial degrees for high aspect ratios of elements, than we did. The condition numbers are computed by reducing the matrices to tridiagonal form and by calculating the minimum and maximum eigenvalues via the QR-algorithm. For this we use standard LAPACK-routines.

Now let us present the numerical results. Our first example considers a sequence of meshes as in Figure 1. We keep the number of elements fixed at 9 and choose different values for δ. With the notation given in the figure (i.e. $H_1 + H_3 = 1$, $H_1 = H_2 = H_4 = H_3 + \delta$), the subdomains and corresponding extensions we have chosen are

$$\begin{aligned}
\Gamma_1 &= (-\frac{1}{2}, -\frac{1}{2} + H_1) \times (-\frac{1}{2}, -\frac{1}{2} + H_1 - \delta), \\
\Gamma_1' &= (-\frac{1}{2}, -\frac{1}{2} + H_1) \times (-\frac{1}{2}, -\frac{1}{2} + H_1), \\
\Gamma_2 = \Gamma_2' &= (-\frac{1}{2}, -\frac{1}{2} + H_2) \times (-\frac{1}{2} + H_1 - \delta, \frac{1}{2}), \\
\Gamma_3 &= (-\frac{1}{2} + H_1, \frac{1}{2}) \times (-\frac{1}{2}, -\frac{1}{2} + H_3), \\
\Gamma_3' &= (-\frac{1}{2} + H_1 - \delta, \frac{1}{2}) \times (-\frac{1}{2}, -\frac{1}{2} + H_3 + \delta), \\
\Gamma_4 &= (-\frac{1}{2} + H_1, \frac{1}{2}) \times (-\frac{1}{2} + H_3, \frac{1}{2}). \\
\Gamma_4' &= (-\frac{1}{2} + H_1 - \delta, \frac{1}{2}) \times (-\frac{1}{2} + H_3, \frac{1}{2}).
\end{aligned}$$

Then there are 4 subdomains Γ_j' each of them consisting of 4 elements adjacent to an interior node. The overlap is δ for all extended subdomains as indicated in the figure. The subdomains form a quasi-uniform coarse mesh with mesh size H which is bounded from below. By Theorem 1 we expect a condition number of the preconditioned stiffness matrix which behaves at most like $c(1 + \log(H/\delta))^2$ with a constant that is independent of H, δ, and the polynomial degree p. The corresponding results (spectral condition number versus H/δ) as well as the curves for

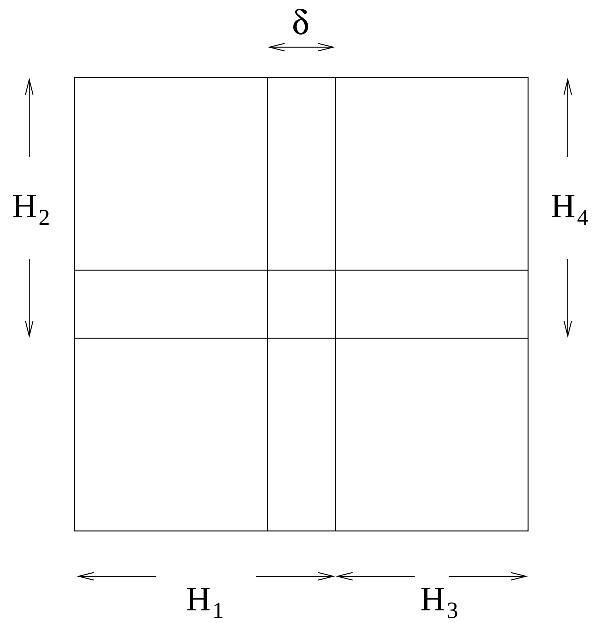

Figure 1: Special mesh of 9 elements with varying ratios H_j/δ ($\delta \to 0$, $H_j \to 1/2$, $j = 1, 2, 3, 4$).

$(1 + \log(H/\delta))$ and $(1 + \log(H/\delta))^2$ are given by Figure 2 in a double logarithmic scale (we plot the results for $H := H_3$). Results for different polynomial degrees are given ($p = 2, 3, 4, 5$) and they confirm the theoretical result.

Now, as a second example, we choose a sequence of meshes like the one given in Figure 3. Here, a geometrical mesh grading towards the left side of the domain is performed and the number of horizontal slices is kept at 3. In each step of our sequence of meshes we cut the elements at the left side vertically in the middle. Again we use uniform polynomial degrees. In this case there are only 2 subdomains Γ'_j, the shaded one in the lower two thirds and the corresponding one in the upper two thirds. Hence, H and δ are constant but the aspect ratio of the elements adjacent to the left side gets smaller in our sequence (and the dimension of the ansatz space increases). Theorem 1 says that the condition number of the preconditioned stiffness matrix in this example is independent of the aspect ratio and independent of the polynomial degree. In Table 1 we give the corresponding numerical results and they confirm the assertion.

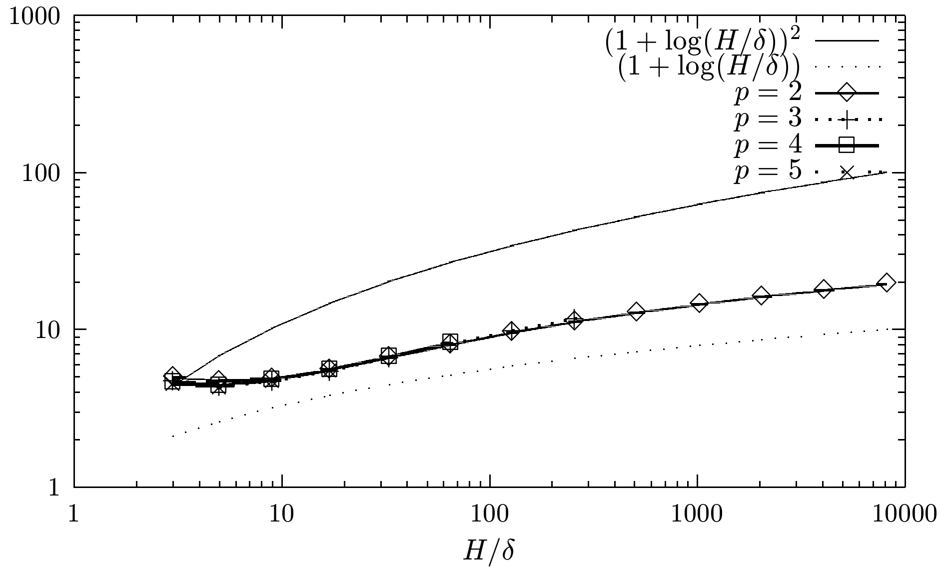

Figure 2: Condition numbers for different polynomial degrees using meshes of the type given in Figure 1 with varying aspect ratio of elements and varying ratio H/δ ($H := H_3$ is taken).

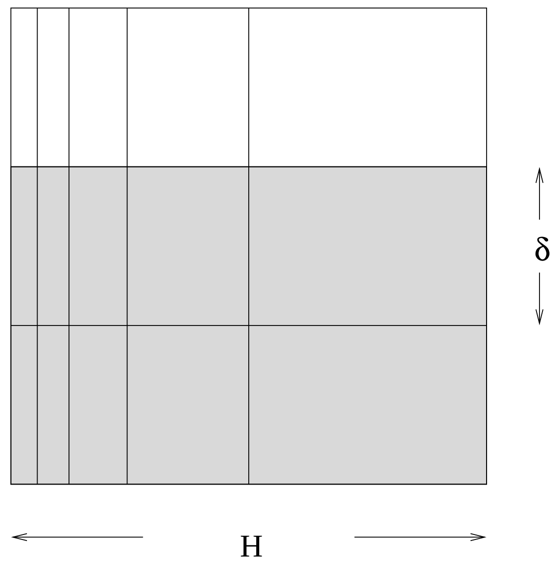

Figure 3: Special mesh with constant H/δ and varying aspect ratio of elements.

n	ratio	$p=2$	$p=3$	$p=4$	$p=5$
2	1.5	2.437	2.296	2.341	2.311
3	1.5	2.439	2.297	2.342	2.312
4	2.7	2.440	2.298	2.343	2.312
5	5.3	2.441	2.298	2.343	2.312
6	11	2.441	2.298	2.343	2.312
7	21	2.441	2.299	2.343	
8	43	2.442	2.299	2.343	
9	85	2.442	2.299	2.343	
10	171	2.442	2.299		
11	341	2.442	2.299		
12	683	2.442	2.299		

Table 1: Condition numbers for different polynomial degrees using meshes of the type given in Figure 3 with varying aspect ratio of elements and constant ratio H/δ.

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