Crouzeix–Raviart Boundary Elements

Norbert Heuer *   Francisco–Javier Sayas †

Abstract

This paper establishes a foundation of non-conforming boundary elements. We present a discrete weak formulation of hypersingular integral operator equations that uses Crouzeix–Raviart elements for the approximation. The cases of closed and open polyhedral surfaces are dealt with. We prove that, for shape regular elements, this non-conforming boundary element method converges and that the usual convergence rates of conforming elements are achieved. Key ingredient of the analysis is a discrete Poincaré-Friedrichs inequality in fractional order Sobolev spaces. A numerical experiment confirms the predicted convergence of Crouzeix–Raviart boundary elements.

Key words: Crouzeix–Raviart elements, boundary element method, non-conforming Galerkin method, hypersingular operator
AMS Subject Classification: 65N55, 65N38

1 Introduction

Discontinuous finite elements are widely used and well analysed. In this paper we demonstrate that there is some hope to develop such techniques also for boundary integral equations of the first kind with hypersingular operators.

The analysis of finite elements for the discretisation of boundary integral equations of the first kind goes back to Nédélec and Planchard [13], and Hsiao and Wendland [10]. Stephan [14] studied boundary elements for singular problems on open surfaces. Hypersingular boundary integral equations are well-posed in fractional Sobolev spaces of order 1/2 and conforming Galerkin discretisations require continuous basis functions. In [8] we showed that the homogeneous essential condition for approximating functions on the boundary of an (open) surface can be efficiently implemented by a Lagrangian multiplier. This was the first step towards the analysis of non-conforming boundary

*Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile, email: nheuer@mat.puc.cl This author is supported by Fondecyt-Chile under grant no. 1080044.

†Departamento de Matemática Aplicada, Universidad de Zaragoza, Centro Politécnico Superior, María de Luna, 3 - 50018 Zaragoza, Spain, and School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA, email: jsayas@unizar.es The second author is partially supported by MEC-FEDER Project MTM2007–63204 and Gobierno de Aragón (Grupo Consolidado PDIE).

elements. In this paper we analyse a non-conforming boundary element method (BEM) with discontinuous elements. More precisely, we study the use of piecewise linear Crouzeix–Raviart elements [5] where jumps of basis functions across element boundaries have integral mean zero. In this way, our meshes are triangular and regular, but basis functions are not continuous. We prove that, for sufficiently smooth given data and on closed surfaces, this method converges like $O(h^{1/2})$ in a discrete $H^{1/2}$-norm (see Theorem 2) whereas the conforming method has the same rate of convergence in the energy norm (see [1]). Here, h indicates the parameter of the triangular meshes. (On open surfaces there is a slight loss in the convergence order of both methods when using standard Sobolev regularity.) General discontinuous methods for hypersingular operators have not yet been analysed.

Our model problem is that of the Laplacian exterior to a polyhedral domain or exterior to an open polyhedral surface. When considering Dirichlet boundary conditions this problem reduces to the integral equation

$$Wu := -\partial_\nu \int_\Gamma \partial_{\nu(\mathbf{y})} \Big[\frac{1}{4\pi|\,\cdot\,-\mathbf{y}|}\Big] u(\mathbf{y}) \mathrm{d}\Gamma(\mathbf{y}) = f \qquad \text{on} \quad \Gamma. \tag{1}$$

Here, Γ is the open or closed surface, W is the hypersingular operator and f is a given function. For the analysis we will need the regularity $f \in L^2(\Gamma)$. We will consider the two cases of closed and open surfaces separately, and for ease of presentation we assume that Γ is plane with a polygonal boundary in the open case.

Hypersingular integral operators are not well-posed on spaces of discontinuous functions. We therefore consider a particular weak formulation of (1) that is equivalent to (1) when considering the energy space of W and that is well-posed also for discontinuous elements. Following [8] we use integration by parts to rewrite (1) in the weak form: *Find* $u \in \widetilde{H}^{1/2}(\Gamma)$ *such that*

$$a(u, v) := \langle V \mathbf{curl}\, u, \mathbf{curl}\, v\rangle = \langle f, v\rangle \quad \forall v \in \widetilde{H}^{1/2}(\Gamma). \tag{2}$$

Here, V is the single-layer operator defined by

$$V\psi := \int_\Gamma \frac{\psi(\mathbf{y})}{4\pi|\,\cdot\,-\mathbf{y}|}\, \mathrm{d}\Gamma(\mathbf{y}), \tag{3}$$

and **curl** is the surface curl operator. For sufficiently smooth surface Γ and function Φ in $\mathbb{R}^3$ with $\Phi = \phi$ on Γ, $\mathbf{curl}\,\phi = \nu \cdot \operatorname{curl}\Phi$ on Γ with ν being the exterior unit normal vector on Γ and curl being the standard curl operator. For the definition and properties of **curl** on closed Lipschitz surfaces Γ and operating on functions of $H^{1/2}(\Gamma)$, see [4]. For open surfaces we refer to [8]. A definition of the Sobolev spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$ is given at the end of this section.

In [8] we introduced a Lagrangian multiplier to deal with the homogeneous essential boundary condition for basis functions on the boundary of Γ (in the case of open surfaces). In this paper we extend this idea to deal with all inter-element discontinuities and prove quasi-optimal convergence of the discrete scheme.

One key issue in both this paper and [8], is to prove ellipticity of the bilinear form in (2). This requires, on one hand, the definition of appropriate "energy spaces" and, on the other hand, the availability of a corresponding Poincaré–Friedrichs inequality. The latter inequality is needed since (2) defines only a positive semi-definite bilinear form. In [8] this situation is relatively simple: the appropriate space is $H^{1/2}(\Gamma)$ (a precise definition is given below) and the Poincaré–Friedrichs inequality (needed to make the semi-norm in $H^{1/2}(\Gamma)$ a norm on approximation spaces satisfying a Lagrangian multiplier condition) is needed on the fixed domain Γ. In our case the "energy space" is $H^{1/2}(\Gamma, \mathcal{T}_h)$ (to be defined below) which requires only piecewise $H^{1/2}$-regularity with respect to the given mesh. Therefore, we need to establish a discrete Poincaré–Friedrichs inequality in fractional order Sobolev spaces. To this end we will follow the analysis by Brenner [2] who considered the piecewise H^1-case. In this paper, we will use it only for the particular case of discrete functions which allows us to prove the optimal rate of convergence of the non-conforming boundary element method. As this result is of interest in itself, we state and prove it in its full generality in piecewise fractional order Sobolev spaces.

Another technical difficulty in this paper is that, in order to deal with inter-element discontinuities in a discrete weak sense, one faces the problem that the trace operator (needed to analyse jumps of functions across element edges) is not well-defined in the energy space of hypersingular operators (it is $\widetilde{H}^{1/2}(\Gamma)$ which will also be defined below). In [8] we solved this problem by switching to slightly more regular spaces and by using inverse properties to return to the energy space. We use the same technique in this paper but have to consider the case where the domains (individual elements) are not fixed but h-dependent (h being the mesh parameter). We therefore need to consider carefully the scaling properties of fractional order Sobolev norms.

An overview of the remainder of this paper is as follows. Below we define the needed Sobolev spaces and specify some notations. For the case of a closed surface, the boundary element method with Crouzeix–Raviart elements is introduced in the next section. In Section 3 we present our main result Theorem 2 which states a quasi-optimal a priori error estimate for the discrete scheme on a closed surface. Some technical results follow and the proof of Theorem 2 is given at the end of the section. The case of an open surface is dealt with in Section 4. Numerical results which confirm our theory on open surfaces are presented in Section 5. In an appendix (Section A) we study a discrete Poincaré–Friedrichs inequality for functions of fractional order Sobolev spaces and for piecewise linear functions.

Sobolev spaces. Let $\Gamma \subset \mathbb{R}^3$ be the boundary of a closed Lipschitz polyhedron. The spaces $H^r(\Gamma)$ for $-1 \le r \le 1$ can be defined in a standard way (see, e.g., [12] for the general Lipschitz case and [11] for smooth, open and closed, surfaces). For positive r we will consider the Aronszain–Slobodecki seminorms

$$|u|_{r,\Gamma} := \Big[\int_\Gamma \int_\Gamma \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2r+2}} d\Gamma(\mathbf{x}) d\Gamma(\mathbf{y}) \Big]^{1/2}, \qquad 0 < r < 1,$$

and the norms $\|\cdot\|^2_{r,\Gamma} := \|\cdot\|^2_{0,\Gamma} + |\cdot|^2_{r,\Gamma}$, where the zero index is used for the $L^2 = H^0$ norm. The norms for negative order spaces are the corresponding dual norms. Let Θ be any flat surface with polygonal boundary and let Γ be an arbitrary polyhedral surface that includes Θ in one of its faces. For $r \in [0, 1]$ we define the space $H^r(\Theta)$ of functions that admit an extension to a function in $H^r(\Gamma)$, endowed with the norm $\|\cdot\|_{r,\Theta}$ defined as above. We also consider the space $\widetilde{H}^r(\Theta)$ of functions whose extension by zero belongs to $H^r(\Gamma)$. For $0<r<1$ a norm in this space is given by

$$\|u\|^2_{r,\sim,\Theta} := |u|^2_{r,\Theta} + \int_\Theta \frac{|u|^2}{\mathrm{dist}(\,\cdot\,,\partial\Theta)}.$$

For $r=1$ we take the usual Sobolev norm. Notice that for $0<r<1/2$, $\widetilde{H}^r(\Theta)=H^r(\Theta)$ and for $1/2<r<1$, the norm $\|\cdot\|_{r,\Theta}$ is equivalent to $\|\cdot\|_{r,\sim,\Theta}$ on the subspace of $H^r(\Theta)$-functions whose trace on the boundary of Θ vanishes. Though, we maintain this last notation for duality. The corresponding dual spaces have the tilde sign interchanged,

$$H^{-r}(\Theta) := (\widetilde{H}^r(\Theta))', \qquad \widetilde{H}^{-r}(\Theta) := (H^r(\Theta))'.$$

The norms of the negative index spaces $\widetilde{H}^r(\Theta)$ will be also denoted $\|\cdot\|_{r,\sim,\Theta}$.

The tangential vector **curl** operator can be easily defined in $H^1(\Gamma)$ and $H^1(\Theta)$. Extension to fractional order spaces on Γ can be accomplished following the construction of [3] extended in [4]. For the case of a flat open surface, we refer to [8]. Properties will be referred to as needed in the text.

Notations. Given any space X, we will write $\mathbf{X} := X^3$ endowed with the product norm. The symbol $\lesssim$ will be used in the usual sense, as for instance in [2]. In short, $a_h \lesssim b_h$ when there exists a constant $C>0$ independent of the discretisation parameter h such that $a_h \le C b_h$. The double inequality $a_h \lesssim b_h \lesssim a_h$ is simplified to $a_h \approx b_h$. In our case, the constants are also independent of the fractional Sobolev index ϵ whenever this is present.

2 Crouzeix–Raviart boundary elements

Let $\Gamma \subset \mathbb{R}^3$ be the boundary of a closed Lipschitz polyhedron. Consider a sequence of shape-regular triangular meshes $\mathcal{T}_h$ of Γ without hanging nodes and the corresponding sets of edges $\mathcal{E}_h$. We will use the spaces

$$H^r(\mathcal{T}_h) := \prod_{T\in\mathcal{T}_h} H^r(T), \qquad L^2(\mathcal{E}_h) := \prod_{e\in\mathcal{E}_h} L^2(e),$$

as well as the broken Sobolev norms and seminorms, defined for the appropriate indices,

$$\|\cdot\|^2_{r,\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} \|\cdot\|^2_{r,T}, \qquad |\cdot|^2_{r,\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} |\cdot|^2_{r,T}.$$

Let us define the discrete spaces

$$H_h := \prod_{T\in\mathcal{T}_h} \mathbb{P}_1(T), \qquad M_h := \prod_{e\in\mathcal{E}_h} \mathbb{P}_0(e),$$

and, denoting by $\mathbf{m}_e$ the midpoint of $e \in \mathcal{E}_h$, the Crouzeix–Raviart space is

$$V_h := \{v_h \in H_h \,|\, v_h \text{ continuous in } \mathbf{m}_e \;\forall e \in \mathcal{E}_h\}.$$

Here, for integer r, $\mathbb{P}_r(T)$ denotes the space of polynomials on T up to total degree r, and correspondingly for other geometric objects. For any function $v \in H^{1/2+\epsilon}(\mathcal{T}_h)$ with $\epsilon > 0$ we define the inter-element jumps

$$[v] \in L^2(\mathcal{E}_h) := \prod_{e\in\mathcal{E}_h} L^2(e),$$

by assigning an orientation to the normal vectors on edges so that we have a positive sign on the side of e pointed out by the normal vector.

Notice that if we denote

$$b(v,\lambda) := \sum_{e\in\mathcal{E}_h} \int_e [\,v\,]\,\lambda$$

for $v \in H^1(\mathcal{T}_h)$ and $\lambda \in L^2(\mathcal{E}_h)$ then, given $v_h \in H_h$,

$$v_h \in V_h \qquad \Longleftrightarrow \qquad b(v_h,\mu_h) = 0 \quad \forall \mu_h \in M_h.$$

Let $\mathbf{curl}_h : H^1(\mathcal{T}_h) \to \mathbf{L}^2(\Gamma)$ be the piecewise tangential curl operator and

$$a_h(u,v) := \langle V\mathbf{curl}_h u, \mathbf{curl}_h v\rangle$$

where V is the single-layer operator defined in (3). Here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(\Gamma)$ inner product and its extension to duality between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ (or their vector valued versions). All these bilinear forms we denote by the same symbol. Then consider the problem

$$\left[\begin{array}{ll} u_h \in V_h, \quad \int_\Gamma u_h = 0, & \\ a_h(u_h,v_h) = \langle f, v_h\rangle & \forall v_h \in V_h. \end{array}\right. \tag{4}$$

This problem can be equivalently stated as

$$\left[\begin{array}{ll} u_h \in V_h^0, & \\ a_h(u_h,v_h) = \langle f, v_h\rangle & \forall v_h \in V_h^0, \end{array}\right.$$

with $V_h^0 := \{v_h \in V_h \mid \int_\Gamma v_h = 0\}$.

Proposition 1 *Problem* (4) *is equivalent to*

$$\left[\begin{array}{lll} (u_h,\lambda_h) \in H_h \times M_h, & \int_\Gamma u_h = 0, & \\ a(u_h,v_h) + b(v_h,\lambda_h) & = \langle f, v_h\rangle & \forall v_h \in H_h, \\ b(u_h,\mu_h) & = 0 & \forall \mu_h \in M_h. \end{array}\right.$$

Both problems are uniquely solvable.

Proof. Since V is elliptic in $H^{-1/2}(\Gamma)$, the only point to prove is the implication

$$b(v_h, \lambda_h) = 0 \quad \forall v_h \in H_h \qquad \Longrightarrow \qquad \lambda_h = 0.$$

To prove this, take a triangle T and let $N_T^\alpha \in \mathbb{P}_1(T)$ ($\alpha \in \{1,2,3\}$) be the Lagrange basis functions associated with its vertices. If $\mathcal{E}(T)$ is the set of edges of T and λ_e is the value of λ_h on e then it is simple to see that

$$\sum_{e\in\mathcal{E}(T)} \lambda_e \int_e N_T^\alpha = 0, \quad \alpha = 1,2,3$$

implies $\lambda_e = 0$ for all $e \in \mathcal{E}(T)$. The remainder of the proof is straightforward. □

3 An a priori error estimate for Crouzeix–Raviart boundary elements

In this section we present an a priori error estimate for the BEM with Crouzeix–Raviart elements.

Henceforth we denote by h_T the diameter of $T \in \mathcal{T}_h$, by h_e the length of $e \in \mathcal{E}_h$ and

$$h := \max\{h_T \,|\, T \in \mathcal{T}_h\}, \qquad h_{\min} := \min\{h_T \,|\, T \in \mathcal{T}_h\}.$$

We also assume shape regularity of the triangulation so that $h_e \approx h_T$ for all $e \in \mathcal{E}(T)$. Given $T \in \mathcal{T}_h$ we denote by F_T an affine bijective transformation from the reference triangle $\widehat{T} := \{(x_1,x_2) \,|\, 0 \le x_1, x_2 \le 1,\ x_1 + x_2 \le 1\}$ onto T. Also, given $v : T \to \mathbb{R}$ we write $\widehat{v} := v \circ F_T$.

We consider the discrete norm

$$\|v_h\|_h^2 := |v_h|^2_{1/2,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1} \Big| \int_e [\, v_h \,] \Big|^2 + \Big| \int_\Gamma v_h \Big|^2, \qquad v_h \in H_h.$$

Notice that

$$\|v_h\|_h = |v_h|_{1/2,\mathcal{T}_h} \qquad \forall v_h \in V_h^0. \tag{5}$$

The main result, Theorem 2 below, is the quasi-optimal convergence of the BEM with Crouzeix–Raviart elements, with respect to the discrete norm just introduced. Improved estimates assuming additional regularity will be indicated at the end of this section.

Theorem 2 *Assume that $f \in L^2(\Gamma)$. Then there holds*

$$\|u - u_h\|_{1/2,\mathcal{T}_h} \lesssim h^{1/2} \|u\|_{1,\Gamma}.$$

Here, u and u_h are the solutions of (1) *subject to $\int_\Gamma u = 0$ and* (4)*, respectively.*

A proof of this theorem is given in the remainder of this section. We will need, however, a discrete Poincaré–Friedrichs inequality (Theorem 8) which will be presented and proved in Section A. The case of an open screen Γ will be studied in Section 4.

3.1 Technical details and the proof of Theorem 2

Before proving Theorem 2 we collect some technical results.

Proposition 3 *There holds*

$$a_h(v_h, v_h) \gtrsim \|v_h\|^2_{1/2,\mathcal{T}_h} \qquad \forall v_h \in V_h^0. \tag{6}$$

Proof. Since $H^{-1/2}(\Gamma) \hookrightarrow \prod_T H^{-1/2}(T)$ with injection constant being independent of the triangulation, we obtain

$$\Big[\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}\,(v_h|_T)\|^2_{-1/2,T}\Big]^{1/2} \lesssim \|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}.$$

Also, there holds

$$\|\mathbf{curl}\,\widehat{p}\|_{-1/2,\widehat{T}} \approx |\widehat{p}|_{1/2,\widehat{T}} \qquad \forall \widehat{p} \in \mathbb{P}_1(\widehat{T}). \tag{7}$$

Recalling the equivalences

$$h^{1/2}|\widehat{v}|_{1/2,\widehat{T}} \approx |v|_{1/2,T}, \qquad h^{3/2}\|\widehat{v}\|_{-1/2,\widehat{T}} \approx \|v\|_{-1/2,T}, \tag{8}$$

(note the scaling property of the negative order Sobolev norm, see [9, Lemma 2]) and taking into account the change of variables in the **curl** operator, (7) implies that

$$\|\mathbf{curl}\ p\|_{-1/2,T} \approx |p|_{1/2,T} \qquad \forall p \in \mathbb{P}_1(T), \quad \forall T \in \mathcal{T}_h \tag{9}$$

and therefore, by (5)

$$\Big[\sum_{T\in\mathcal{T}_h} \|\mathbf{curl}(v_h|_T)\|^2_{-1/2,T}\Big]^{1/2} \approx |v_h|_{1/2,\mathcal{T}_h} = \|v_h\|_h \qquad \forall v_h \in V_h^0.$$

Then, Theorem 8 implies that

$$\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma} \gtrsim |v_h|_{1/2,\mathcal{T}_h} + \|v_h\|_{0,\Gamma} \qquad \forall v_h \in V_h^0. \tag{10}$$

By the ellipticity of the single-layer operator V there holds

$$a_h(v_h, v_h) \gtrsim \|\mathbf{curl}_h v_h\|^2_{-1/2,\Gamma}$$

and therefore, (10) proves the statement. □

Consider the spaces of conforming elements

$$C_h := H_h \cap \mathcal{C}(\Gamma), \qquad C_h^0 := \{\psi_h \in C_h \mid \int_\Gamma \psi_h = 0\} \subset V_h^0.$$

Comparison with this space gives the following Strang-type error estimate.

Proposition 4 *There holds*

$$|u-u_h|_{1/2,\mathcal{T}_h}+\|u-u_h\|_{0,\Gamma}\lesssim \inf_{\psi_h\in C_h^0}\|u-\psi_h\|_{1/2,\Gamma}+\sup_{v_h\in V_h^0}\frac{|a(u-u_h,v_h)|}{\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}}.$$

Proof. Let $\psi_h\in C_h^0$. By the discrete ellipticity (6) we have the bound

$$\begin{aligned}|u_h-\psi_h|_{1/2,\mathcal{T}_h}+\|u_h-\psi_h\|_{0,\Gamma} &\lesssim \sup_{v_h\in V_h^0}\frac{|a_h(u_h-\psi_h,v_h)|}{\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}}\\ &\le \sup_{v_h\in V_h^0}\frac{|a_h(u-u_h,v_h)|}{\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}}+\sup_{v_h\in V_h^0}\frac{|a_h(u-\psi_h,v_h)|}{\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}}.\end{aligned}$$

Since $u-\psi_h\in H^{1/2}(\Gamma)$ we have $\mathbf{curl}_h(u-\psi_h)=\mathbf{curl}(u-\psi_h)$ and by the continuity of $\mathbf{curl}: H^{1/2}(\Gamma)\to\mathbf{H}^{-1/2}(\Gamma)$,

$$\begin{aligned}|a_h(u-\psi_h,v_h)| &= |\langle V\,\mathbf{curl}\,(u-\psi_h),\mathbf{curl}_h v_h\rangle|\\ &\lesssim \|\mathbf{curl}\,(u-\psi_h)\|_{-1/2,\Gamma}\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}\\ &\lesssim \|u-\psi_h\|_{1/2,\Gamma}\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}.\end{aligned}$$

The result follows straightforwardly from the preceding bounds by adding and subtracting an arbitrary $\psi_h\in C_h^0$ in the error term on the left-hand side of the statement. □

Let $\mathbf{t}_e$ be the unit tangential vector that arises from a positive $\pi/2$ rotation of the normal vector taken to define signs of the jumps. The consistency term can be bounded by using the following result.

Proposition 5

$$\sup_{v_h\in V_h^0}\frac{|a(u-u_h,v_h)|}{\|\mathbf{curl}_h v_h\|_{-1/2,\Gamma}}\lesssim \inf_{\mu_h\in M_h}\Big[\sum_{e\in\mathcal{E}_h}\|\mathbf{t}_e\cdot V\,\mathbf{curl}\,u-\mu_h\|_{0,e}^2\Big]^{1/2}$$

Proof. By integrating by parts over each triangle, we obtain for $v_h\in V_h$

$$\begin{aligned}a(u-u_h,v_h) &= \langle V\,\mathbf{curl}\,u,\mathbf{curl}_h v_h\rangle-\langle f,v_h\rangle\\ &= \sum_{e\in\mathcal{E}_h}\langle\mathbf{t}_e\cdot V\,\mathbf{curl}\,u,[v_h]\rangle_e=\sum_{e\in\mathcal{E}_h}\langle\mathbf{t}_e\cdot V\,\mathbf{curl}\,u-\mu_h,[v_h]\rangle_e.\end{aligned}\tag{11}$$

Here, μ_h is an arbitrary function of M_h. Now consider the reference elements (see Figure 3.1)

$$\widehat{T}_1:=\widehat{T},\qquad \widehat{T}_2:=\{(\widehat{x},\widehat{y})\in[0,1]^2\,|\,\widehat{x}+\widehat{y}\ge 1\},$$

$$\widehat{e}:=\{(\widehat{x},\widehat{y})\in[0,1]^2\,|\,\widehat{x}+\widehat{y}=1\}=\widehat{T}_1\cap\widehat{T}_2$$

and the finite-dimensional space

$$\widehat{\mathbb{P}} := \Big\{ \widehat{p} : [0,1]^2 \to \mathbb{R} \,\Big|\, \widehat{p}|_{\widehat{T}_i} \in \mathbb{P}_1, \quad i = 1,2, \qquad \int_{\widehat{e}} [\widehat{p}] = 0 \Big\},$$

where we have the bound

$$\|[\widehat{p}]\|_{0,\widehat{e}}^2 \le \widehat{C} \Big[|\widehat{p}|_{1/2,\widehat{T}_1}^2 + |\widehat{p}|_{1/2,\widehat{T}_2}^2 \Big] \qquad \forall \widehat{p} \in \widehat{\mathbb{P}}. \tag{12}$$

Let $e \in \mathcal{E}_h$ and let T_1, T_2 be the triangles that share e as an edge. With a continuous piecewise affine map we transform bijectively the reference triplet $(\widehat{T}_1, \widehat{T}_2, \widehat{e})$ onto (T_1, T_2, e). Notice that $v_h \in V_h$, restricted to $T_1 \cup T_2$, is transformed back onto an element of $\widehat{\mathbb{P}}$. Using (8) and (12) we obtain

$$\|[v_h]\|_{0,e}^2 \approx h \|[\widehat{v}_h]\|_{0,\widehat{e}}^2 \lesssim h \Big(|\widehat{v}_h|_{1/2,\widehat{T}_1}^2 + |\widehat{v}_h|_{1/2,\widehat{T}_2}^2 \Big) \approx |v_h|_{1/2,T_1}^2 + |v_h|_{1/2,T_2}^2 \quad \forall v_h \in V_h,$$

whence

$$\Big[\sum_{e \in \mathcal{E}_h} \|[v_h]\|_{0,e}^2 \Big]^{1/2} \lesssim |v_h|_{1/2,\mathcal{T}_h}.$$

Together with (9) this proves that

$$\Big[\sum_{e \in \mathcal{E}_h} \|[v_h]\|_{0,e}^2 \Big]^{1/2} \lesssim \|\mathbf{curl}_h v_h\|_{-1/2,\Gamma} \qquad \forall v_h \in V_h. \tag{13}$$

This bound and (11) imply the statement of the proposition. $\square$

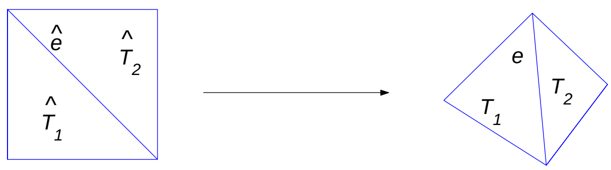

Figure 1: The reference double triangle of the proof of Proposition 5 and its transformed image.

Making use of Propositions 4 and 5, the proof of Theorem 2 is reduced to approximation and regularity arguments. Before giving the proof we present some approximation results.

Recall the equivalence, obtained by changing to the reference element (the first inequality of (8) is just a particular case of this):

$$|v|_{r,T}^2 \approx h_T^{2-2r} |\widehat{v}|_{r,\widehat{T}}^2, \qquad 0 \le r \le 1. \tag{14}$$

Recall also the trace lemma on a fixed domain

$$\|v\|_{0,\partial\widehat{T}} \le \frac{C}{\epsilon^{1/2}} \|v\|_{1/2+\epsilon,\widehat{T}} \qquad \forall v \in H^{1/2+\epsilon}(\widehat{T}), \quad \forall \epsilon \in (0,1/2]. \tag{15}$$

This formulation is proven as Lemma 4.3 in [8], based on bounds that can be found in [12] for example. Also, for the best $L^2(\widehat{T})$ approximation by constants, we have by interpolation

$$\Big\| v - \frac{1}{|\widehat{T}|}\int_{\widehat{T}} v \Big\|_{0,\widehat{T}} \le C|v|_{r,\widehat{T}} \qquad \forall v \in H^r(\widehat{T}), \quad \forall r \in [0,1]. \tag{16}$$

Therefore

$$\Big\| v - \frac{1}{|\widehat{T}|}\int_{\widehat{T}} v \Big\|_{r,\widehat{T}} \le C|v|_{r,\widehat{T}} \qquad \forall v \in H^r(\widehat{T}), \quad \forall r \in [0,1].$$

With (15) and (16) it is simple to prove that there holds

$$\Big\| v - \frac{1}{|\widehat{T}|}\int_{\widehat{T}} v \Big\|_{0,\partial\widehat{T}} \le \frac{C}{\epsilon^{1/2}}|v|_{1/2+\epsilon,\widehat{T}} \qquad \forall v \in H^{1/2+\epsilon}(\widehat{T}), \quad \forall \epsilon \in (0,1/2]. \tag{17}$$

Lemma 6 *For all* $\epsilon \in (0,1/2]$,

$$\inf_{\mu_h \in M_h} \Big[\sum_{e\in\mathcal{E}_h} \|v - \mu_h\|^2_{0,e} \Big]^{1/2} \lesssim \epsilon^{-1/2} h^{\epsilon} |v|_{1/2+\epsilon,\Gamma} \qquad \forall v \in H^{1/2+\epsilon}(\Gamma).$$

Proof. If $e \in \mathcal{E}(T)$ (it is immaterial which of the two possible triangles sharing e we choose), it is simple to see that

$$\begin{aligned} \inf_{c\in\mathbb{R}} \|v - c\|^2_{0,e} &\approx h_e \inf_{c\in\mathbb{R}} \|\widehat{v} - c\|^2_{0,\widehat{e}} \\ &\lesssim \epsilon^{-1} h_e |\widehat{v}|^2_{1/2+\epsilon,\widehat{T}} \approx \epsilon^{-1} h_T^{2\epsilon} |v|^2_{1/2+\epsilon,T}, \end{aligned}$$

where in the last two inequalities we have respectively applied (17) and (14), as well as the fact that $h_e \approx h_T$. The result follows by adding the contribution from all edges. $\square$

Proof of Theorem 2. Combining Propositions 4 and 5 we obtain

$$\begin{aligned} &|u - u_h|_{1/2,\mathcal{T}_h} + \|u - u_h\|_{0,\Gamma} \\ &\quad \lesssim \inf_{\psi_h \in C_h^0} \|u - \psi_h\|_{1/2,\Gamma} + \inf_{\mu_h\in M_h} \Big[\sum_{e\in\mathcal{E}_h} \|\mathbf{t}_e \cdot V\,\mathbf{curl}\, u - \mu_h\|^2_{0,e} \Big]^{1/2}. \end{aligned} \tag{18}$$

Note that, since by assumption $f \in L^2(\Gamma)$, there holds $u \in H^1(\Gamma)$. By well-known approximation properties by piecewise polynomial functions the first term on the right-hand side of (18) can be bounded like

$$\inf_{\psi_h \in C_h^0} \|u - \psi_h\|_{1/2,\Gamma} \lesssim h^{1/2} |u|_{1,\Gamma}. \tag{19}$$

For the second term we use Lemma 6 with $\epsilon = 1/2$ and recall the continuity of $V\,\mathbf{curl} : H^1(\Gamma) \to \mathbf{H}^1(\Gamma)$. This yields

$$\inf_{\mu_h\in M_h} \Big[\sum_{e\in\mathcal{E}_h} \|\mathbf{t}_e \cdot V\,\mathbf{curl}\, u - \mu_h\|^2_{0,e} \Big]^{1/2} \lesssim h^{1/2} \|u\|_{1,\Gamma}.$$

This finishes the proof. $\square$

Some improvement on the order of convergence can be obtained assuming additional regularity of the solution. For $1 < s < 3/2$ consider

$$H^s(\Gamma) := \{u \in H^1(\Gamma) \,|\, u|_L \in H^s(L) \quad \forall L \,\text{face of}\, \Gamma\},$$

endowed with the product norm of the norms on the faces and the $H^1(\Gamma)$ norm. The spaces $H^s(L)$ are the traditional Sobolev spaces on the faces. If $u \in H^s(\Gamma)$, the bound (19) can be improved to

$$\inf_{\psi_h \in C_h^0} \|u - \psi_h\|_{1/2,\Gamma} \lesssim C_1(s)\, h^{s-1/2} |u|_{s,\Gamma}.$$

Similarly, it is possible to obtain a bound

$$\inf_{\mu_h \in M_h} \left[\sum_{e \in \mathcal{E}_h} \|\mathbf{t}_e \cdot V \,\mathbf{curl}\, u - \mu_h\|_{0,e}^2 \right]^{1/2} \lesssim C_2(s) h^{s-1/2} \|V \,\mathbf{curl}\, u\|_{s,\Gamma}, \tag{20}$$

assuming enough regularity of $V \,\mathbf{curl}\, u$.

4 The case of an open flat screen

Let now Γ be a flat open screen with polygonal boundary. We assume that it is placed in horizontal position and we construct a closed cubical surface S including Γ in its top face and a copy of it in the lower face, as shown in Figure 4. Since we will make occasional reference to this closed surface and since there are significant differences between the closed and the open surface cases, we will indicate with a sub- or superscript S any continuous or discrete element related to the closed surface S. Symbols related to the open screen Γ do not have an additional index.

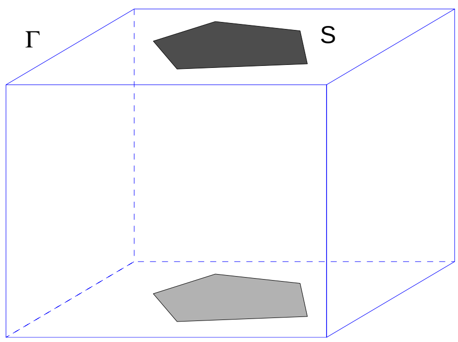

Figure 2: Geometrical configuration. The screen Γ is inserted in a face of a cube S, with a copy of it on the opposite face. The discretisation of the cubical surface will keep this symmetry.

We are given a triangulation of Γ, $\mathcal{T}_h$, and consider the set of all sides of this triangulation, $\mathcal{E}_h$. The triangulation of Γ is extended to a triangulation of S, respecting the symmetry of the top and low faces, i.e., the inherited triangulations of these faces are assumed to be identical. For orientation, we will always assume that an edge on $\partial\Gamma$ takes as positive sense the interior triangle. We then define the jump operator $H^1(\mathcal{T}_h) \to L^2(\mathcal{E}_h)$ as follows: $v_h \in H^1(\mathcal{T}_h)$ is extended by zero to all remaining triangles outside Γ and the jumps are taken only on $\mathcal{E}_h \subset \mathcal{E}_h^S$. Obviously, with this definition, jumps over edges on $\partial\Gamma$ are simply restrictions to these edges. The spaces H_h and M_h are defined as before, whereas now

$$V_h := \{v_h \in H_h \mid \int_e [v_h] = 0 \quad \forall e \in \mathcal{E}_h\}.$$

This definition implies continuity at midpoints of interior edges and zero value at midpoints of edges on $\partial\Gamma$. The discrete problem is

$$\left[\begin{array}{l} u_h \in V_h, \\ a_h(u_h, v_h) = \langle f, v_h\rangle_\Gamma \quad \forall v_h \in V_h, \end{array}\right. \tag{21}$$

where $a_h(u_h, v_h) := \langle V\mathbf{curl}_h u_h, \mathbf{curl}_h v_h\rangle_\Gamma$. Notice that since the surface is now open, we do not impose the condition of cancellation of the integral of u_h over the surface. Finally, the discrete norm is

$$\|v_h\|_{h,\Gamma}^2 := |v_h|_{1/2,\mathcal{T}_h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\Big|\int_e [v_h]\Big|^2, \qquad v_h \in H_h.$$

Similarly to Theorem 8 we have

$$\|v_h\|_{0,\Gamma} \lesssim \|v_h\|_{h,\Gamma} \qquad \forall v_h \in H_h,$$

as can be seen as follows. We extend $v_h \in H_h$ to $\widetilde{v}_h \in H_h^S$ in a particular way. On the copy of Γ lying on the opposite face of S we take $-v_h$ (translated to the corresponding positions) and extend by zero onto the remaining elements. In this way

$$\int_S \widetilde{v}_h = 0, \qquad \|\widetilde{v}_h\|_{0,S}^2 = 2\|v_h\|_{0,\Gamma}^2, \qquad \|\widetilde{v}_h\|_h^2 = 2\|v_h\|_{h,\Gamma}^2,$$

and the result follows from Theorem 8. From the ellipticity of V in $\widetilde{H}^{-1/2}(\Gamma)$, the uniform continuity of the injection $\widetilde{H}^{-1/2}(\Gamma) \hookrightarrow \prod_{T\in\mathcal{T}_h} H^{-1/2}(T)$, (9) and Theorem 8, we obtain

$$a_h(v_h, v_h) \gtrsim \|v_h\|_{1/2,\mathcal{T}_h}^2 \qquad \forall v_h \in V_h.$$

This corresponds to (6). The discrete space of conforming functions is

$$C_h := \{\psi_h \in H_h \mid \psi_h \in \mathcal{C}(\Gamma), \quad \psi_h|_{\partial\Gamma} = 0\} = H_h \cap \widetilde{H}^{1/2}(\Gamma) \subset V_h.$$

In order to be able to repeat the arguments in Proposition 4 we note that $V : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\mathbf{curl} : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\Gamma)$ are bounded (for the latter property see Lemma 2.2 in [8]). Also, modifying (13), we get

$$\Big[\sum_{e\in\mathcal{E}_h} \|[v_h]\|_{0,e}^2\Big]^{1/2} \lesssim \|\mathbf{curl}_h v_h\|_{-1/2,\sim,\Gamma} \qquad \forall v_h \in V_h.$$

Analogously to (18) one proves that

$$\|u-u_h\|_{1/2,\mathcal{T}_h} \lesssim \inf_{\psi_h\in C_h}\|u-\psi_h\|_{1/2,\Gamma} + \inf_{\mu_h\in M_h}\Big[\sum_{e\in\mathcal{E}_h}\|\mathbf{t}_e\cdot V\,\mathbf{curl}\,u-\mu_h\|_{0,e}^2\Big]^{1/2}.$$

For $f\in L^2(\Gamma)$ the expected regularity is $u\in\widetilde{H}^{1-\epsilon}(\Gamma)$ ($\epsilon>0$). This eventually proves the following error estimate for the boundary element method with Crouzeix–Raviart elements on open surfaces.

Theorem 7 *Let Γ be an open plane screen with polygonal boundary and assume that $f\in L^2(\Gamma)$. Then there holds*

$$\|u-u_h\|_{1/2,\mathcal{T}_h} \lesssim h^{1/2-\epsilon}\|u\|_{1-\epsilon,\sim,\Gamma}.$$

Here, u and u_h are the solutions of (1) *and* (21)*, respectively.*

5 Numerical results

We consider the model problem (1) with $\Gamma=(0,1)\times(0,1)$ and $f=1$, and use uniform triangular meshes $\mathcal{T}_h$. In this case there holds $u\in\widetilde{H}^{1-\epsilon}(\Gamma)$ for any $\epsilon>0$. Theorem 7 proves that the Crouzeix–Raviart boundary element method converges like $O(h^{1/2-\epsilon})$ for any $\epsilon>0$ whereas the conforming method converges like $O(h^{1/2})$, see [1]. (Note, however, that the missing ϵ in the convergence order of the conforming method is due to a refined error analysis for singularities which, in principle, should also be applicable to the non-conforming method.)

The exact solution u is unknown and we proceed similarly as in [8] to calculate an upper bound for the error in the semi-norm $|u-u_h|_{H^{1/2}(\Gamma,\mathcal{T}_h)}$. There holds

$$a(u-u_h,u-u_h)\ge C|u-u_h|^2_{H^{1/2}(\Gamma,\mathcal{T}_h)}.$$

Since u and u_h solve (1) and (21), respectively, one finds that

$$a(u-u_h,u-u_h)=a(u,u)-2a(u,u_h)+a(u_h,u_h)=\langle Wu,u\rangle+\langle f,u_h\rangle-2a(u,u_h).$$

Integration by parts (cf. [8, Lemma 4.2]) proves

$$\begin{aligned} a(u,u_h) &= \langle V\,\mathbf{curl}\,u,\mathbf{curl}_h u_h\rangle = \sum_{T\in\mathcal{T}_h}\big(\langle Wu,u_h\rangle_T-\langle\mathbf{t}\cdot V\,\mathbf{curl}\,u,u_h\rangle_{\partial T}\big)\\ &= \langle Wu,u_h\rangle-\sum_{e\in\mathcal{E}_h}\langle\mathbf{t}\cdot V\,\mathbf{curl}\,u-\mu_h,[u_h]\rangle_e \qquad \forall\mu_h\in M_h. \end{aligned}$$

Here, we made use of the integral-mean-zero property of the Crouzeix–Raviart elements on the edges and $[u_h]$ is the trace of $u_h|_T$ onto e if $e\subset\partial T\cap\partial\Gamma$. Therefore, we obtain by also using the approximation property (20) together with $V\,\mathbf{curl}\,u \in H^{1-\epsilon}(\Gamma)$ (see [8])

$$
\begin{aligned}
&|u-u_h|^2_{H^{1/2}(\Gamma,\mathcal{T}_h)}\\
&\lesssim\ |\langle Wu,u\rangle - \langle f,u_h\rangle| + 2\inf_{\mu_h\in M_h}\Big(\sum_{e\in\mathcal{E}_h}\|\mathbf{t}\cdot V\,\mathbf{curl}\,u-\mu_h\|^2_{0,e}\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}\|[u_h]\|^2_{0,e}\Big)^{1/2}\\
&\lesssim\ |\langle Wu,u\rangle - \langle f,u_h\rangle| + h^{1/2-\epsilon}\|V\,\mathbf{curl}\,u\|_{1-\epsilon,\sim,\Gamma}\Big(\sum_{e\in\mathcal{E}_h}\|[u_h]\|^2_{0,e}\Big)^{1/2}\quad \forall\epsilon>0.
\end{aligned}
$$

That is, for $\epsilon > 0$ there holds

$$
|u-u_h|_{H^{1/2}(\Gamma,\mathcal{T}_h)} \lesssim |\langle Wu,u\rangle - \langle f,u_h\rangle|^{1/2} + C(\epsilon)\,h^{1/4-\epsilon}\Big(\sum_{e\in\mathcal{E}_h}\|[u_h]\|^2_{0,e}\Big)^{1/4} \qquad (22)
$$

with $C(\epsilon) = \|V\,\mathbf{curl}\,u\|_{1-2\epsilon,\sim,\Gamma}$. The terms $\langle f,u_h\rangle$ and $\|[u_h]\|_{0,e}$ can be easily calculated and we approximate $\langle Wu,u\rangle$ by an extrapolated value that we denote by $\|u\|^2_{\text{ex}}$ (cf. [7]). Therefore, instead of the relative error $\|u-u_h\|_{H^{1/2}(\Gamma,\mathcal{T}_h)}/\|u\|_{1/2,\sim,\Gamma}$, we present results for the terms on the right-hand side of (22), normalised by $\|u\|_{\text{ex}}$.

The results are presented in Figure 3 in a double-logarithmic scale, and are plotted versus h. The curve (1) represents the values of $|\langle Wu,u\rangle - \langle f,u_h\rangle|^{1/2}$ (normalised by $\|u\|_{\text{ex}}$) and (2) gives the values of $h^{1/4}\Big(\sum_{e\in\mathcal{E}_h}\|[u_h]\|^2_{0,e}\Big)^{1/4}$ (again normalised) measuring the jumps of the approximation. The curve (2) is parallel to the line $h^{1/2}$ which is also given, and the curve (1) is of higher order in this range of unknowns. Therefore, the theoretical result of about $O(h^{1/2})$-convergence of the Crouzeix–Raviart boundary element method is confirmed. Moreover, the errors are dominated by the jumps of the discontinuous approximations. For comparison, we also give the normalised $\widetilde{H}^{1/2}(\Gamma)$-errors of the conforming boundary element method which also behave like $O(h^{1/2})$.

A A fractional order discrete Poincaré–Friedrichs inequality

The following result is central to the proof of our main theorem (Theorem 2). It is a discrete Poincaré–Friedrichs inequality in fractional order Sobolev spaces. A corresponding result for the integer order space H^1 has been proved by Brenner [2, Theorem 5.1].

Theorem 8 *There hold*

$$
\|v\|^2_{0,\Gamma} \lesssim \epsilon^{-1}|v|^2_{1/2+\epsilon,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1-2\epsilon}\Big|\int_e [\,v\,]\Big|^2 + \Big|\int_\Gamma v\ \Big|^2 \qquad \forall v\in H^{1/2+\epsilon}(\mathcal{T}_h),\ \forall\epsilon\in(0,1/2)
$$

and

$$
\|v\|_{0,\Gamma} \lesssim \|v\|_h \qquad \forall v\in H_h.
$$

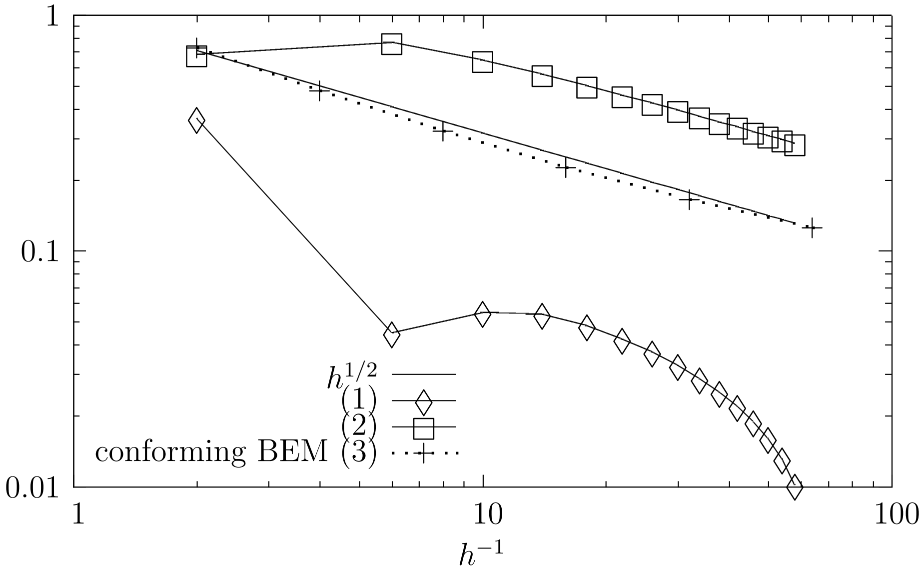

Figure 3: Relative error curves (normalised by $\|u\|_{\text{ex}}$): (1) $\left|\|u\|_{\text{ex}} - \langle f, u_h\rangle\right|^{1/2}$, (2) $h^{1/4}\Big(\sum_{e\in\mathcal{E}_h} \|[u_h]\|^2_{0,e}\Big)^{1/4}$, (3) error in $\widetilde{H}^{1/2}(\Gamma)$ for conforming BEM.

Our proof follows closely the steps of Brenner [2] who considered piecewise H^1-functions. First let us collect some technical results.

Consider the interpolant $\mathcal{I} : H^1(\mathcal{T}_h) \to V_h$ defined by

$$(\mathcal{I}v)(\mathbf{m}_e) := \frac{1}{h_e}\int_e \{v\},$$

$\mathbf{m}_e$ being the midpoint of e and $\{v\}$ being the average of the values of v approaching e from the triangles sharing e. Also, let $\Pi : H^1(\mathcal{T}_h) \to H_h$ be defined triangle by triangle as

$$(\Pi_T v)(\mathbf{m}_e) := \frac{1}{h_e}\int_e v|_T.$$

Note that there holds

$$(\mathcal{I}v - \Pi v)|_T(\mathbf{m}_e) = \frac{1}{2h_e}\int_e [\,v\,].$$

Lemma 9 *For all $\epsilon \in (0, 1/2)$, $v \in H^{1/2+\epsilon}(\mathcal{T}_h)$ and $T \in \mathcal{T}_h$,*

$$|\mathcal{I}v - \Pi v|^2_{1/2+\epsilon,T} \;\lesssim\; \sum_{e\in\mathcal{E}(T)} h_e^{-1-2\epsilon}\Big|\int_e [\,v\,]\Big|^2, \tag{23}$$

$$\|\mathcal{I}v - \Pi v\|_{0,T}^2 \lesssim \sum_{e\in\mathcal{E}(T)} \Big|\int_e [\,v\,]\Big|^2, \tag{24}$$

$$\|v - \Pi v\|_{0,T}^2 \lesssim \epsilon^{-1} h_T^{1+2\epsilon} |v|_{1/2+\epsilon,T}^2, \tag{25}$$

$$|\Pi v|_{1/2+\epsilon,T}^2 \lesssim \epsilon^{-1} |v|_{1/2+\epsilon,T}^2. \tag{26}$$

Proof. It is a simple transformation to the reference triangle. For $e \in \mathcal{E}(\widehat{T})$ let $\widehat{N}_e$ be the basis function of the Crouzeix–Raviart element associated with the edge e of the reference triangle. Let $F_T : \widehat{T} \to T$ be an affine bijection. Then

$$2\,(\mathcal{I}v - \Pi v)|_T \circ F_T = \sum_{e\in\mathcal{E}(T)} \Big(h_e^{-1} \int_e [\,v\,]\Big) \widehat{N}_e$$

so that, by (14),

$$\begin{aligned}
|\mathcal{I}v - \Pi v|_{1/2+\epsilon,T}^2 &\approx h_T^{1-2\epsilon} |(\mathcal{I}v - \Pi v)\circ F_T|_{1/2+\epsilon,\widehat{T}}^2 \\
&\lesssim h_T^{1-2\epsilon} h_T^{-2} \sum_{e\in\mathcal{E}(T)} \Big|\int_e [\,v\,]\Big|^2 \max_{e\in\mathcal{E}(\widehat{T})} |\widehat{N}_e|_{1/2+\epsilon,\widehat{T}}^2.
\end{aligned}$$

By interpolation we see that

$$|\widehat{N}_e|_{1/2+\epsilon,\widehat{T}} \lesssim \|\widehat{N}_e\|_{1,\widehat{T}} \lesssim 1,$$

and the first bound is proved. For the second bound we apply exactly the same ideas to obtain

$$\|\mathcal{I}v - \Pi v\|_{0,T}^2 \approx h_T^2 \|(\mathcal{I}v - \Pi v)\circ F_T\|_{0,\widehat{T}}^2 \lesssim \sum_{e\in\mathcal{E}(T)} \Big|\int_e [\,v\,]\Big|^2.$$

Let $\widehat{v} := v \circ F_T$ and

$$c := \frac{1}{|\widehat{T}|} \int_{\widehat{T}} \widehat{v}.$$

Then, shortening $\widehat{\Pi} := \Pi_{\widehat{T}}$ and using (16) and (17), we prove that

$$\begin{aligned}
\|\widehat{v} - \widehat{\Pi}\widehat{v}\|_{0,\widehat{T}} &\le \|\widehat{v} - c\|_{0,\widehat{T}} + \|\widehat{\Pi}(\widehat{v} - c)\|_{0,\widehat{T}} \\
&\lesssim \|\widehat{v} - c\|_{0,\widehat{T}} + \|\widehat{v} - c\|_{0,\partial\widehat{T}} \\
&\lesssim |\widehat{v}|_{1/2+\epsilon,\widehat{T}} + \epsilon^{-1/2} |\widehat{v}|_{1/2+\epsilon,\widehat{T}}.
\end{aligned}$$

Now (25) follows from a transformation to the reference triangle:

$$\|v - \Pi v\|_{0,T}^2 \lesssim h_T^2 \|\widehat{v} - \widehat{\Pi}\widehat{v}\|_{0,\widehat{T}}^2 \lesssim \epsilon^{-1} h_T^2 |\widehat{v}|_{1/2+\epsilon,\widehat{T}}^2 \lesssim \epsilon^{-1} h_T^{1+2\epsilon} |v|_{1/2+\epsilon,T}^2.$$

On the other hand, by (17)

$$|\widehat{\Pi}\widehat{v}|_{1/2+\epsilon,\widehat{T}} = |\widehat{\Pi}(\widehat{v} - c)|_{1/2+\epsilon,\widehat{T}} \lesssim \|\widehat{v} - c\|_{0,\partial T} \lesssim \epsilon^{-1/2} |\widehat{v}|_{1/2+\epsilon,\widehat{T}}.$$

Therefore, transforming to the reference triangle proves

$$|\Pi v|^2_{1/2+\epsilon,T} \approx h_T^{1-2\epsilon}|\widehat{\Pi}\widehat{v}|^2_{1/2+\epsilon,\widehat{T}} \lesssim \epsilon^{-1}h_T^{1-2\epsilon}|\widehat{v}|^2_{1/2+\epsilon,\widehat{T}} \approx \epsilon^{-1}|v|^2_{1/2+\epsilon,T},$$

which is (26). □

Lemma 10 *For all $\epsilon \in (0,1/2)$ and $v \in H^{1/2+\epsilon}(\mathcal{T}_h)$,*

$$|\mathcal{I}v|^2_{1/2+\epsilon,\mathcal{T}_h} \lesssim \epsilon^{-1}|v|^2_{1/2+\epsilon,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1-2\epsilon}\Big|\int_e [\,v\,]\Big|^2, \tag{27}$$

$$\|v-\mathcal{I}v\|^2_{0,\Gamma} \lesssim \epsilon^{-1}h^{1+2\epsilon}|v|^2_{1/2+\epsilon,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h}\Big|\int_e [\,v\,]\Big|^2. \tag{28}$$

Moreover, for $v \in H_h$,

$$|\mathcal{I}v|^2_{1/2,\mathcal{T}_h} \lesssim |v|^2_{1/2,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\Big|\int_e [\,v\,]\Big|^2, \tag{29}$$

$$\|v-\mathcal{I}v\|^2_{0,\Gamma} \lesssim \sum_{e\in\mathcal{E}_h}\Big|\int_e [\,v\,]\Big|^2. \tag{30}$$

Proof. By adding and subtracting Πv, (27) is a straightforward consequence of (23) and (26). To prove (28), add and subtract again Πv, and use (24) and (25) to derive the required bound. Estimates (29) and (30) are analogously proved by noting that (23) and (24) hold also for $\epsilon = 0$. Moreover, since $\Pi v = v$ for any $v \in H_h$, (25) and (26) are not needed.

□

Let W_h be the space of continuous $\mathbb{P}_2$ finite elements on $\mathcal{T}_h$. Let $E_h : V_h \to W_h$ be the interpolation operator that takes values at midpoints of sides and average values at vertices where the average at a node is calculated by taking the values of the function there from the surrounding triangles.

Lemma 11 *For all $r \in (0,1)$,*

$$\|E_h v_h - v_h\|^2_{0,\Gamma} \lesssim \sum_{T\in\mathcal{T}_h} h_T^{2r}|v_h|^2_{r,T} \qquad \forall v_h \in V_h. \tag{31}$$

$$\|E_h v_h\|_{0,\Gamma} \approx \|v_h\|_{0,\Gamma} \qquad \forall v_h \in V_h. \tag{32}$$

$$|E_h v_h|_{r,\Gamma} \lesssim |v_h|_{r,\mathcal{T}_h} \qquad \forall v_h \in V_h, \tag{33}$$

with omitted constants being independent of r.

Proof. Bound (31) follows from Lemma 3.2 in [2] with element-wise inverse inequalities. From (31) and further applications of inverse inequalities we obtain

$$\|E_h v_h\|_{0,\Gamma} \lesssim \|v_h\|_{0,\Gamma} \qquad \forall v_h \in V_h. \tag{34}$$

Let then $F_h : W_h \to V_h$ be the interpolation operator that takes values only on midpoints of the sides. Note that $F_h E_h = I$ in V_h. Then, similarly as before,

$$\|F_h w_h - w_h\|_{0,\Gamma}^2 \lesssim \sum_{T\in\mathcal{T}_h} h_T^{2r} |w_h|_{r,T}^2 \qquad \forall w_h \in W_h$$

and

$$\|F_h w_h\|_{0,\Gamma} \lesssim \|w_h\|_{0,\Gamma} \qquad \forall w_h \in W_h. \tag{35}$$

To see the reverse bound in (32) we use the fact that $v_h = F_h E_h v_h$ and apply (35).

Corollary 3.3. in [2] proves that

$$|E_h v_h|_{1,\Gamma} \lesssim |v_h|_{1,\mathcal{T}_h} \qquad \forall v_h \in V_h.$$

Then, (33) follows by interpolation between this inequality and (34). $\square$

To shorten some forthcoming expressions let us denote

$$\Phi(v) := \Big|\int_\Gamma v\Big|.$$

Proposition 12 *For all $r \in (0,1)$*

$$\|v_h\|_{0,\Gamma} \lesssim |v_h|_{r,\mathcal{T}_h} + \Phi(v_h) \qquad \forall v_h \in V_h. \tag{36}$$

Proof. Making use of the Poincaré–Friedrichs inequality

$$\|v\|_{0,\Gamma} \le C\Big[|v|_{1,\Gamma} + \Phi(v)\Big] \qquad \forall v \in H^1(\Gamma)$$

we find by interpolation that

$$\|v\|_{0,\Gamma} \le C'\Big[|v|_{r,\Gamma} + \Phi(v)\Big] \qquad \forall v \in H^r(\Gamma), \quad \forall r \in (0,1). \tag{37}$$

By (31) there holds

$$\Phi(E_h v_h - v_h) \lesssim \|E_h v_h - v_h\|_{0,\Gamma} \lesssim h^r |v_h|_{r,\mathcal{T}_h} \qquad \forall v_h \in V_h. \tag{38}$$

Therefore, applying successively (32), (37), (33) and (38), one verifies that

$$\begin{aligned}\|v_h\|_{0,\Gamma} &\lesssim \|E_h v_h\|_{0,\Gamma} \lesssim |E_h v_h|_{r,\Gamma} + \Phi(E_h v_h)\\ &\lesssim |v_h|_{r,\mathcal{T}_h} + \Phi(v_h) + \Phi(E_h v_h - v_h) \lesssim |v_h|_{r,\mathcal{T}_h} + \Phi(v_h)\end{aligned}$$

$\square$

Proof of Theorem 8. By (36), the triangular inequality and the fact that $\Phi(v) \lesssim \|v\|_{0,\Gamma}$, it follows that, for $\epsilon \in [0, 1/2)$,

$$\begin{aligned} \|v\|_{0,\Gamma} &\lesssim \|v - \mathcal{I}v\|_{0,\Gamma} + |\mathcal{I}v|_{1/2+\epsilon,\mathcal{T}_h} + \Phi(\mathcal{I}v) \\ &\lesssim \|v - \mathcal{I}v\|_{0,\Gamma} + |\mathcal{I}v|_{1/2+\epsilon,\mathcal{T}_h} + \Phi(v). \end{aligned}$$

The first assertion of the theorem is now a straightforward consequence of (27) and (28). Recalling the definition

$$\|v\|_h^2 = |v|^2_{1/2,\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1} \Big| \int_e [\, v \,] \Big|^2 + \Big| \int_\Gamma v \ \Big|^2$$

the second assertion is consequence of (29) and (30). $\square$

Acknowledgement. We thank an anonymous referee for comments that helped to improve results of this paper.

References

[1] Bespalov, A.; Heuer, N. The *hp*-version of the boundary element method with quasi-uniform meshes in three dimensions. *ESAIM Math. Model. Num. Anal.* **42** (2008), 821–849.

[2] Brenner, S. Poincaré–Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.* **41** (2003) 306–324.

[3] Buffa, A.; Ciarlet Jr., P. On traces for functional spaces related to Maxwell's equations I. An integration by parts formula in Lipschitz polyhedra. *Math. Methods Appl. Sci.* **24** (2001) 9–30.

[4] Buffa, A; Costabel, M.; Sheen, D. On traces for $H(curl, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.* **276** (2002) 845–867.

[5] Crouzeix, M.; Raviart, P.-A. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I, *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge* **7** (1973) 33–75.

[6] Costabel, M. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.* **19** (1988) 613–626.

[7] Ervin, V. J.; Heuer, N.; Stephan, E. P. On the h-p version of the boundary element method for Symm's integral equation on polygons, *Comput. Methods Appl. Mech. Engrg.* **110** (1993) 25–38.

[8] Gatica, G.; Healey, M.; Heuer, N. The Boundary Element Method with Lagrangian multipliers. *Numer. Methods Partial Differential Eq.*, in print.

[9] Heuer, N. Additive Schwarz method for the *p*-version of the boundary element method for the single layer potential operator on a plane screen. *Numer. Math.* **88** (2001), 485–511.

[10] Hsiao, G. C.; Wendland, W. L. A finite element method for some integral equations of the first kind, *J. Math. Anal. Appl.* **58** (1977) 449–481.

[11] Lions, J.-L.; Magenes E. *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, New-York, 1972.

[12] McLean, W. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, 2000.

[13] Nédélec, J.-C.; Planchard, J. Une méthode variationnelle d'éléments finis pour la résolution numérique d'un problème extérieur dans R^3, *RAIRO* **7** (no. R-3) (1973) 105–129.

[14] Stephan, E. P. A boundary integral equation method for three-dimensional crack problems in elasticity, *Math. Methods Appl. Sci.* **8** (1986) 609–623.