Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen.

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Dedicated to Prof. Dr. Ernst P. Stephan on the occasion of his 50th birthday.

Abstract

We analyze an additive Schwarz preconditioner for the p-version of the boundary element method for the single layer potential operator on a plane screen in the three-dimensional Euclidean space. We decompose the ansatz space, which consists of piecewise polynomials of degree p on a mesh of size h, by introducing a coarse mesh of size $H \geq h$. After subtraction of the coarse subspace of piecewise constant functions on the coarse mesh this results in local subspaces of piecewise polynomials living only on elements of size H. This decomposition yields a preconditioner which bounds the spectral condition number of the stiffness matrix by $O(\log \frac{H}{h} p)^2$. Numerical results supporting the theory are presented.

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1 Introduction.

In this paper we analyze a preconditioner of the additive Schwarz type for linear systems arising from the Galerkin method for symmetric, positive definite pseudo-differential operators of order minus one on surfaces in the three-dimensional Euclidean space. As a model problem we consider the single layer potential operator V on a plane screen $\Gamma \subset \mathbb{R}^3$:

For given $g \in H^{1/2}(\Gamma)$ find $u \in \tilde{H}^{-1/2}(\Gamma)$ such that $Vu = g$, i.e.

$$a(u,v) := \langle Vu, v\rangle_{L^2(\Gamma)} = \langle g, v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \tilde{H}^{-1/2}(\Gamma) \tag{1}$$

where

$$Vu(x) := \frac{1}{4\pi}\int_\Gamma \frac{u(y)}{|x-y|}\,dy,\ x \in \Gamma.$$

For the definition of the Sobolev spaces $H^{1/2}(\Gamma)$ and $\tilde{H}^{-1/2}(\Gamma)$ see §3. V is a continuous and symmetric, positive definite mapping from $\tilde{H}^{-1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, cf. [13, 37], and therefore V defines a norm which is equivalent to the $\tilde{H}^{-1/2}$-norm.

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Physically the solution u of (1) represents the charge density of a thin, electrified plate Γ loaded with some given potential. We note that the results in this paper hold for polyhedral surfaces Γ as well.

The Galerkin scheme for (1) reads: For a given finite-dimensional subspace $X_N \subset \tilde{H}^{-1/2}(\Gamma)$ find $u_N \in X_N$ such that for every $v \in X_N$

$$a(u_N, v) = \langle g, v \rangle_{L^2(\Gamma)}. \tag{2}$$

Here we focus on the p-version of the boundary element method which reduces the approximation error $\|u - u_N\|_V$ by increasing the polynomial degrees of the ansatz functions on a given mesh Γ_h. We use uniform meshes which consist of rectangles of length h. Since the conformity condition $X_N \subset \tilde{H}^{-1/2}(\Gamma)$ does not require continuous functions we simply use affine images of tensor products of L^2-normed Legendre polynomials as basis functions, i.e. on the reference element $(-1,1)^2$ we take the functions

$$\sqrt{(2q_1+1)(2q_2+1)}\, L_{q_1}(x_1)\, L_{q_2}(x_2), \quad \max\{q_1, q_2\} \le p. \tag{3}$$

We note that only for ease of presentation and in order to give full details of the proofs we deal with rectangular meshes and basis functions of the tensor product type. The main ingredients in the proof of the main theorem are estimates of different Sobolev norms for general functions on Lipschitz domains, scaling properties of Sobolev norms, and the inverse property of basis functions. All the needed details we expect to hold for more general, quasi-uniform and regular, quadrilateral and triangular meshes and other basis functions as well.

It is well known that the solution u of (1) behaves singularly at the edges and corners of Γ, cf. [39]. Therefore the p-version converges twice as fast as the h-version, where one uses piecewise polynomials of low degree on a sequence of refined meshes, cf. [38]. On the other hand the spectral condition number κ of the stiffness matrix A in (2) behaves like $O(h^{-1}p^2)$ and, therefore, increases much faster for the p-version than for the h-version. This behavior of the condition number occurs when taking basis functions that are orthonormal with respect to the L^2-inner product. A proof of this fact uses the characterization of the extreme eigenvalues by the Rayleigh quotient and the inverse property of the basis functions (see Lemma 4 below). For condition number estimates concerning the single layer potential operator on curves see also [24, 20].

The method of choice for solving positive definite linear systems is the conjugate gradient method. A bound on the decrease of the energy norm of the error, after k steps, is given by

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \quad \text{where} \quad \kappa = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

In this paper we present a preconditioner for A that yields a spectral condition number κ which, in its optimal form, is bounded by a constant that does not depend on the mesh size h and that grows only moderately in the polynomial degree p of the ansatz functions. Our method fits in the additive Schwarz framework where we use a direct sum decomposition of the ansatz space X_N. This decomposition is induced by the decomposition of Γ into subdomains by a coarse mesh. The preconditioner therefore amounts to a simple block Jacobi step if one numbers the basis functions appropriately.

For an overview of domain decomposition methods with regard to finite elements we refer the reader to [11]. For the application of these methods to the boundary element method we refer to [17, 20, 40, 41, 23] for problems in the two-dimensional Euclidean space and to [21, 22] for operators of order one on surfaces in the three-dimensional space. First results for additive Schwarz methods for operators of order minus one on surfaces in $\mathbb{R}^3$ are given in [19]. However, those results are not optimal as the bounds for the condition numbers behave as cp^ε for a positive ε where the constant c depends on ε and h. For the pure h-version we also refer to [31] where a two-level method has been investigated. There, a bound like $O(H/h)^2$ for the spectral condition number of the additive Schwarz operator has been proved. We note that our method for the p-version reduces to the method proposed in [31] when taking $p = 0$. However, our results then give an improved bound for the spectral condition number which behaves like $O(\log H/h)^2$ and which give the same result only for fixed ratio H/h.

In two dimensions, where one has to deal with operators on curves, it is straight forward to show that if one has an additive Schwarz method for an operator of order one then one obtains a related method for an operator of order minus one which gives the same behavior for the condition number. One just has to take the derivatives with respect to the arc length of the basis functions for the operator of positive order to define new basis functions for the operator of negative order. The appropriate subspace decomposition is given accordingly. This corresponding method works due to the fact that standard integration and differentiation can be extended to mappings which are inverses of each other and which are isomorphisms between the spaces $\tilde{H}^{1/2}$ and $\tilde{H}^{-1/2}$ when identifying functions which differ only by constants, see [17]. In the three-dimensional Euclidean space, where one has to deal with operators on surfaces, we do not know of such an easy mechanism. Following an idea of Bramble et al. in [9] one can use the positive square root $(-\Delta)^{1/2}$ to define a norm in $\tilde{H}^{1/2}$ and one has the inverse operator $(-\Delta)^{-1/2}$. However, these operators in general do not map piecewise polynomials to piecewise polynomials which are needed as basis functions for the usual boundary element method.

We note that multigrid methods for operators of negative order solving problems in $\mathbb{R}^3$ have been analyzed in [8]. We further note that it is also possible first to perform domain decompositions to split a given boundary value problem and then to discretize the local problems by the finite element or the boundary element method as appropriate, see [25, 36, 27]. A preconditioning strategy for first kind integral operators, that is different from the additive Schwarz framework considered here, is proposed by Khoromskij and Wendland in [26]. They use some artificial boundary integral operators on canonical domains where the corresponding discrete operators are easily invertible and, thus, can be used as preconditioners.

An outline of this paper is as follows. In §2 we define the preconditioner for our model problem and state the main result (Theorem 1). In §3 we introduce the needed Sobolev spaces and prove various technical details dealing, e.g., with decompositions in Sobolev spaces, scaling properties, and the inverse property of piecewise polynomials. The proof of the main theorem is given in §3.1. Numerical results which support our theory are presented in §4.

Throughout the paper c and C denote positive constants which may take on different values at different occurrences.

2 Preconditioner.

In this section we define the preconditioner which can be considered as a non-overlapping additive Schwarz method and we present a bound for the resulting condition number. Before doing so we need to give the abstract setting of additive Schwarz methods. To this end we decompose the ansatz space X_N:

$$X_N = H_0 \cup \cdots \cup H_k \tag{4}$$

The additive Schwarz method consists in solving, by an iterative method, the equation

$$Pu_N := (P_0 + P_1 + \cdots + P_k)u_N = f_N, \tag{5}$$

where the projections $P_j : X_N \to H_j$, $j = 0, \ldots, k$, are defined for any $v \in X_N$ by

$$a(P_j v, \varphi) = a(v, \varphi) \quad \text{for any } \varphi \in H_j.$$

Here, $a(\cdot,\cdot)$ is a bilinear form which is given by the operator under consideration or which is almost spectrally equivalent to it. For our model problem, we take

$$a(u,v) = \langle Vu, v\rangle_{L^2(\Gamma)} \quad \text{for all} \quad u,v \in \tilde{H}^{-1/2}(\Gamma).$$

The right hand side of (5), $f_N = \sum_{j=0}^k P_j u_N$, can be computed without knowing the solution u_N of (2) by

$$a(P_j u_N, \varphi) = \langle g, \varphi\rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_j,\ j = 0, \ldots, k.$$

Equation (5) is the preconditioned linear system and the aim is to bound its condition number for a specific decomposition (4). This will be done by estimating the extreme eigenvalues of the additive Schwarz operator where we make use of the following theoretical bounds. The proofs can be found, e.g., in [32, 33, 43].

If there exists a constant C_1 such that for any $v \in X_N$ there exist $v_j \in H_j$, $j = 0, \ldots, k$, satisfying $v = \sum_{j=0}^k v_j$ and

$$\sum_{j=0}^{k} a(v_j, v_j) \le C_1^{-1} a(v,v)$$

then $\lambda_{\min}(P) \ge C_1$. If there exists a constant C_2 such that for any $v \in X_N$ and $v_j \in H_j$, $j = 0, \ldots, k$, satisfying $v = \sum_{j=0}^k v_j$ and

$$a(v,v) \le C_2 \sum_{j=0}^{k} a(v_j, v_j)$$

then $\lambda_{\max}(P) \le C_2$.

To define the preconditioner for our model problem let $\bar{\Gamma}_h = \cup_{j=1}^J \bar{\Gamma}_j$ be a given mesh of J rectangles which implicitly define the space X_N of piecewise polynomials on Γ_h by specifying the polynomial degrees. We simply use the same degree p everywhere but our method works for non-uniform degree distributions as well. For the decomposition of X_N we choose a coarse mesh $\bar{\Gamma}_H = \cup_{j=1}^n \bar{G}_j$ of size $H \ge h$. We assume that the coarse mesh Γ_H is compatible with the boundary element mesh Γ_h. The extreme case $H = h$ of identical meshes is included. The subdomains G_j are implicitly meshed by the restrictions $\Gamma_h|_{G_j}$ and the respective elements will be denoted by $\Gamma_{j,i}$, $i = 1, \ldots, J_j$. We decompose

$$X_N = H_0 \oplus H_1 \oplus \cdots \oplus H_n \tag{6}$$

where H_0 is the space of piecewise constant functions on the mesh Γ_H and

$$H_j := \{v|_{G_j};\ v \in X_N,\ \langle v, 1\rangle_{L^2(G_j)} = 0\}, \quad j = 1, \ldots, n.$$

Note that (6) is a direct sum decomposition which gives a unique representation for any $v \in X_N$. Further, in the case $H > h$, the subspaces H_j ($j > 0$) also comprise piecewise constant functions, the pieces understood with respect to the local fine meshes $\Gamma_h|_{G_j}$. The functions being constant on the whole pieces G_j are excluded from these subspaces. They are covered by H_0.

The following theorem is the main result of this paper.

Theorem 1 *There exists a constant $c > 0$ independent of h, H, and p such that for the condition number of the additive Schwarz operator P implicitly defined by the decomposition* (6) *there holds*

$$\kappa(P) \le c\Big(1 + \log\big(\frac{H}{h}(p+1)\big)\Big)^2.$$

The above theorem can be directly applied to the h-version (using piecewise constant ansatz functions). For completeness we give the decomposition and the result for the corresponding additive Schwarz preconditioner. Our theory improves the result by Mund et al. [31] when dealing with varying ratio H/h of the coarse and fine mesh sizes.

Given two mesh sizes H and h ($H > h$, H/h integer) we decompose

$$X_N = S_h^0 = S_H^0 \cup S_{h,1}^0 \cup \cdots \cup S_{h,n}^0 \tag{7}$$

where S_H^0 ($= H_0$ from above) is the space of piecewise constant functions on the coarse mesh Γ_H of domains G_j, $j = 1, \ldots, n$, and $S_{h,j}^0$ is the space of piecewise constant functions on the mesh $\Gamma_h|_{G_j}$ which are L^2-orthogonal to 1. This decomposition defines the two-level additive Schwarz operator P and the above theorem gives the following result.

Corollary 1 *There exists a constant $c > 0$ such that for any two mesh sizes $H > h$ (H/h integer) there holds for the two-level additive Schwarz operator P implicitly defined by the decomposition* (7)

$$\kappa(P) \le c(1 + \log\frac{H}{h})^2.$$

Before giving the proof of the main theorem we need to derive a couple of technical results which are collected in the next section.

3 Sobolev norms and polynomials.

We define the Sobolev spaces that will be used and prove some technical lemmas.

We take the usual unweighted L^2- and H^1-norms and the H^1-seminorm. When not otherwise noted, the Sobolev spaces of non-integral orders are introduced as interpolation spaces. Here we use the real K-method, cf. [7]. For two normed spaces A_0 and A_1 the interpolation space $A_s = [A_0, A_1]_s$ $(0 < s < 1)$ is equipped with the norm

$$\|a\|_{[A_0,A_1]_s} := \left(\int_0^\infty \left(t^{-s} \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}) \right)^2 \frac{dt}{t} \right)^{1/2}.$$

For $0 < s < 1$ we define

$$H^s(\Gamma) = \left[L^2(\Gamma), H^1(\Gamma)\right]_s, \quad \tilde{H}^s(\Gamma) = \left[L^2(\Gamma), H_0^1(\Gamma)\right]_s.$$

The notation $\tilde{H}^s$ is used by Grisvard and is common in the boundary element literature, whereas the notation $H_{00}^s = \tilde{H}^s$ is used by Lions and Magenes and is common in the finite element literature.

For higher $s > 1$ the interpolation is analogously defined by using Sobolev spaces of higher integral orders. The spaces $H^{-s}(\Gamma)$ (resp. $\tilde{H}^{-s}(\Gamma)$) for $s > 0$ are the dual spaces of $\tilde{H}^s(\Gamma)$ (resp. $H^s(\Gamma)$) with respect to the L^2-inner product.

We emphasize that the K-method of interpolation is exact of exponent s (see [7]), i.e.,

$$\|T\|_{[A_0,A_1]_s \to [B_0,B_1]_s} \le \|T\|_{A_0 \to B_0}^{1-s} \|T\|_{A_1 \to B_1}^{s} \tag{8}$$

for bounded linear operators $T:\ A_0 \to B_0$, $T:\ A_1 \to B_1$. That means that, for given spaces A_0, A_1, B_0, B_1, whenever we use the K-method of interpolation to prove boundedness of linear operators on interpolation spaces, then the boundedness holds uniformly for $0 < s < 1$.

Some technical results will be proved by using different equivalent Sobolev norms. When applying those results, where norms different from the interpolation norm are used, we have to check uniform equivalence of the definitions. Our standard norms are the

- interpolation norm obtained by the **K-method** for $s > 0$ and the
- dual norm for $s < 0$ ("**duality**").

Other definitions are the

- **Sobolev-Slobodeckij** norm $(0 < s < 1/2)$

$$\|v\|_{H^s(\Gamma)} = \left(\|v\|_{L^2(\Gamma)}^2 + \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \right)^{1/2} \quad (n = 2), \tag{9}$$

$$\|v\|_{\tilde{H}^s(\Gamma)} = \left(\|v\|_{H^s(\Gamma)}^2 + \|\frac{v(x)}{\operatorname{dist}(x, \partial\Gamma)^s}\|_{L^2(\Gamma)}^2 \right)^{1/2}$$

and the

- **K-method** ($s < 0$)

$$H^s(\Gamma) = [H^{-1}(\Gamma), L^2(\Gamma)]_{1+s}, \quad \tilde{H}^s(\Gamma) = [\tilde{H}^{-1}(\Gamma), L^2(\Gamma)]_{1+s}$$

  for $-1 < s < 0$ and analogously for $s < -1$.

For clarity we mention the methods used to prove the technical lemmata. The notation (method1, method2) indicates that method1 and method2 are used for $s > 0$ and $s < 0$, respectively.

The next lemma will be used to localize Sobolev norms of non-integral orders which is straight forward for norms of integral orders. This localization is essential when performing domain decompositions for integral operators, e.g., of orders plus and minus one.

Lemma 1 (K-method, duality/K-method) *Let Γ be a Lipschitz domain in $\mathbb{R}^2$ and let $\{\Gamma_j;\ j = 1, \ldots, J\}$ be a decomposition of Γ into Lipschitz domains, i.e.,*

$$\cup_{j=1}^J \bar{\Gamma}_j = \bar{\Gamma} \quad \textit{and} \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \textit{if} \quad i \neq j.$$

Moreover we assume that

(i) for each of the domains Γ_j, $j = 1, \ldots, J$, a finite number (being independent of j and J) of Lipschitz mappings can be used to describe its boundary and that

(ii) the Lipschitz constants of all the boundary mappings are uniformly bounded.

Then, for $s \in [-1, 1]$ and $u \in \tilde{H}^s(\Gamma)$ with $u_j := u|_{\Gamma_j} \in \tilde{H}^s(\Gamma_j)$, $j = 1, \ldots, J$, there exists a constant $c > 0$ which is independent of s, u and J such that

$$\|u\|^2_{\tilde{H}^s(\Gamma)} \leq c \sum_{j=1}^J \|u_j\|^2_{\tilde{H}^s(\Gamma_j)} \quad \textit{(K-method, duality).} \tag{10}$$

Further, for arbitrary $u \in H^s(\Gamma)$,

$$\sum_{j=1}^J \|u_j\|^2_{H^s(\Gamma_j)} \leq c \|u\|^2_{H^s(\Gamma)} \quad \textit{(K-method, K-method)} \tag{11}$$

for a constant c that is independent of s, u and J.

Proof. For the complex interpolation method a proof of this lemma is given by von Petersdorff (see [42, Lemma 3.2]) and can be used in part also here. Indeed, for positive s, the assertion (10) follows by interpolating the boundedness of the composition operator $T:\ \Pi_{j=1}^J \tilde{H}^s(\Gamma_j) \to \tilde{H}^s(\Gamma)$ between $s = 0$ and $s = 1$. Here we also use that, for the K-method, the product space of interpolated spaces is equivalent to the corresponding interpolated product spaces, see [22, Lemma 1]. In the same manner assertion (11) follows for positive s by interpolating between $s = 0$ and $s = 1$. For $s \in [-1, 0]$ inequality (10) then is consequence of (11) by duality and using the Cauchy-Schwarz inequality.

It remains to prove (11) for $s \in [-1, 0)$. First let us consider the case $s = -1$, i.e., $u \in H^{-1}(\Gamma)$ and $u_j = u|_{\Gamma_j} \in H^{-1}(\Gamma_j)$. We define a function $w \in H^1(\Gamma)$ to be the solution of the homogeneous Dirichlet problem $-\Delta w = u$ in Γ and $w = 0$ on the boundary of Γ.

Correspondingly we define $w_j \in H^1(\Gamma_j)$ $(j = 1, \dots, J)$ by the local problem $-\Delta w_j = u_j$ in Γ_j and $w_j = w$ on the boundary of Γ_j. By standard regularity we have $\|w\|_{H^1(\Gamma)} \le C\|u\|_{H^{-1}(\Gamma)}$ and by the continuity of the Laplacian there holds $\|u_j\|_{H^{-1}(\Gamma_j)} \le c_j \|w_j\|_{H^1(\Gamma_j)}$ $(j = 1, \dots, J)$, see, e.g., [16, Theorem 1.4.4.6] for Lipschitz domains. The functions w_j are uniquely defined and by construction there holds $w_j = w|_{\Gamma_j}$, $j = 1, \dots, J$. Therefore,

$$\sum_{j=1}^{J} \|u_j\|^2_{H^{-1}(\Gamma_j)} \le \max_j c_j \sum_{j=1}^{J} \|w_j\|^2_{H^1(\Gamma_j)} = \max_j c_j \|w\|^2_{H^1(\Gamma)} \le C\|u\|^2_{H^{-1}(\Gamma)}.$$

We still have to check that the constants c_j in the above estimate can be bounded independently of J, the number of subdomains. Then, for $s \in (-1, 0)$, (11) follows by interpolation and by considering the product norm of the interpolated spaces instead of the interpolation norm of the product spaces, as above.

Now, as the last step of the proof, we prove that there holds

$$\|u_j\|_{H^{-1}(\Gamma_j)} \le c\|w_j\|_{H^1(\Gamma_j)} \tag{12}$$

for a constant c that is independent of the subdomain Γ_j. Here w_j is the solution to the local Dirichlet problem for the Laplacian as considered above. Let $\tilde{u}_j$ be an arbitrary extension of u_j onto $\mathbb{R}^2$, i.e., $\tilde{u}_j|_{\Gamma_j} = u_j$. Further, for a function ψ on Γ_j let $\tilde{\psi}^0$ denote its extension by 0. Then we obtain

$$\begin{aligned} \|u_j\|_{H^{-1}(\Gamma_j)} &= \sup_{\psi \in H_0^1(\Gamma_j)} \frac{\langle u_j, \psi\rangle_{L^2(\Gamma_j)}}{\|\psi\|_{H^1(\Gamma_j)}} = \sup_{\psi \in H_0^1(\Gamma_j)} \frac{\langle \tilde{u}_j, \tilde{\psi}^0\rangle_{L^2(\mathbb{R}^2)}}{\|\tilde{\psi}^0\|_{H^1(\mathbb{R}^2)}} \\ &\le \sup_{\psi \in H^1(\mathbb{R}^2)} \frac{\langle \tilde{u}_j, \psi\rangle_{L^2(\mathbb{R}^2)}}{\|\psi\|_{H^1(\mathbb{R}^2)}} = \|\tilde{u}_j\|_{H^{-1}(\mathbb{R}^2)}. \end{aligned} \tag{13}$$

We also need an extension operator by Stein $E_S : \ H^1(\Gamma_j) \to H^1(\mathbb{R}^2)$, see Section 3 in Chapter VI of [35]. In our situation (domain in $\mathbb{R}^2$, order of differentiation is one) the bound of E_S only depends on the number of local Lipschitz mappings used to describe the boundary of Γ_j, and on their Lipschitz constants (see Theorem 5' in Section 3.2 and the proof of Theorem 5 in Section 3.3 both in [35, Chapter VI]). Thus we have

$$\|E_S w_j\|_{H^1(\mathbb{R}^2)} \le c\|w_j\|_{H^1(\Gamma_j)} \tag{14}$$

for a constant c that is independent of the diameter of Γ_j. The constant only depends on the Lipschitz constants of the boundary of Γ_j as long as a bounded number of Lipschitz mappings is used.

Now we use the extension operator by Stein to extend w_j and we define an extension $\tilde{u}_j$ of u_j by $-\Delta E_S w_j = \tilde{u}_j$. Then, denoting by $\mathcal{F}$ the Fourier transformation operator and using (13) and (14), we obtain

$$\begin{aligned} \|u_j\|^2_{H^{-1}(\Gamma_j)} &\le \|\tilde{u}_j\|^2_{H^{-1}(\mathbb{R}^2)} = \|-\Delta E_S w_j\|^2_{H^{-1}(\mathbb{R}^2)} \\ &\simeq \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-1} |\mathcal{F}(-\Delta E_S w_j)(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^2} (1 + |\xi|^2)^{-1} (\xi_1^2 + \xi_2^2)^2 |\mathcal{F}(E_S w_j)(\xi)|^2 \, d\xi \\ &\le \int_{\mathbb{R}^2} (1 + |\xi|^2) |\mathcal{F}(E_S w_j)(\xi)|^2 \, d\xi \simeq \|E_S w_j\|^2_{H^1(\mathbb{R}^2)} \le c\|w_j\|^2_{H^1(\Gamma_j)}. \end{aligned}$$

Therefore, the estimate (12) with universal constant c being independent of j is proved and the proof of the lemma is finished. $\square$

The next corollary proves that, for negative s, the estimate (11) holds also for the dual norms when bounding s away from 0 and -1.

Corollary 2 (K-method, duality) *Let Γ be a Lipschitz domain in $\mathbb{R}^2$ and let $\{\Gamma_j;\ j=1,\ldots,J\}$ be a decomposition of Γ into Lipschitz domains, i.e.,*

$$\cup_{j=1}^{J}\bar{\Gamma}_j = \bar{\Gamma} \quad \textit{and} \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \textit{if} \quad i \neq j.$$

Moreover we assume that

(i) for each of the domains Γ_j, $j=1,\ldots,J$, a finite number (being independent of j and J) of Lipschitz mappings can be used to describe its boundary and that

(ii) the Lipschitz constants of all the boundary mappings are uniformly bounded.

Let $\delta \in (0,1/2)$ be given. Then, for each $s \in [-1+\delta, -\delta]$ and $u \in H^s(\Gamma)$ with $u_j := u|_{\Gamma_j}$, $j=1,\ldots,J$, there exists a constant $c>0$ which is independent of s, u and J such that

$$\sum_{j=1}^{J} \|u_j\|^2_{H^s(\Gamma_j)} \le c\|u\|^2_{H^s(\Gamma)} \quad \textit{(duality)} \tag{15}$$

Proof. By Lemma 1 we only need to check the uniform equivalence of the norms involved in (11) and (15). In the range of Sobolev spaces of negative order ($s<0$) the estimate (11) of Lemma 1 is proved by interpolating between the Sobolev spaces that, for integral s, are defined by duality or equivalent norms. More precisely, $H^s(\Gamma)$ for $s \in (-1,0)$ is defined by $[H^{-1}(\Gamma), L^2(\Gamma)]_{1+s}$, see the definition at the beginning of this section. On the other hand our standard Sobolev norm by duality uses the dual space of $[L^2(\Gamma), H^1_0(\Gamma)]_s$. Since $[H^{-1}(\Gamma), L^2(\Gamma)]_{1+s} = [L^2(\Gamma), H^{-1}(\Gamma)]_{-s}$ for $0 < -s < 1$ with equal norms (see [7, Theorem 3.4.1(a)]), we therefore need equivalence of norms in spaces of the form

$$[A_0', A_1']_s = [A_0, A_1]_s'$$

for Banach spaces A_0, A_1. This equality (equivalence of norms) is ensured by the Duality Theorem, see [7, Theorem 3.7.1]. The proof of Theorem 3.7.1 in [7] shows that this equivalence is uniform on intervals $s \in [\delta, 1-\delta]$ for $\delta \in (0,1/2)$. Indeed, this can be seen by checking the uniform equivalence of the norms for $s \in [\delta, 1-\delta]$ obtained by the K-method and the so-called J-method, see the proof of the Equivalence Theorem [7, Theorem 3.3.1]. This proves the corollary. $\square$

To estimate the norm of a function locally on the actual mesh Γ_h we use a transformation onto a reference element. We therefore need to know how Sobolev norms behave on regions of different sizes.

Lemma 2 (K-method, duality) *Let $I^n = (0,1)^n$, $I^n_h = (0,h)^n$ and let*

$$T^n_h : I^n_h \to I^n, \quad n = 1,2,3,$$

be an affine transformation. For functions u, $\tilde{u}$ such that $u = \tilde{u} \circ T_h^n$ on I_h^n there holds for $n = 1, 2, 3$

$$\|u\|^2_{\tilde{H}^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{\tilde{H}^s(I^n)} \quad (0 \le s \le 1)$$

and

$$\|u\|^2_{H^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{H^s(I^n)} \quad (-1 \le s \le 0).$$

Proof. For $s = 0, 1$ the first relation can be verified by substitution. For $0 < s < 1$ we use interpolation, see also [21, Lemma 3.2].

The second relation holds for $-1 \le s \le 0$ by duality:

$$\|u\|_{H^s(I_h^n)} = \sup_{\psi \in \tilde{H}^{-s}(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\|\psi\|_{\tilde{H}^{-s}(I_h^n)}} \simeq \sup_{\tilde{\psi} \in \tilde{H}^{-s}(I^n)} \frac{h^n \langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{h^{(n+2s)/2} \|\tilde{\psi}\|_{\tilde{H}^{-s}(I^n)}} = h^{(n-2s)/2} \|\tilde{u}\|_{H^s(I^n)}.$$

□

The scaling property of $\tilde{H}^s$-norms for $s \ge 0$ as given by the above lemma can be extended to negative values of s if one considers only functions with integral mean zero.

Lemma 3 (K-method, duality) *We use the same notations as in Lemma 2. For a function u with integral mean zero on I_h^n there holds for $n = 1, 2, 3$*

$$\|u\|^2_{H^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{H^s(I^n)} \quad (0 \le s \le 1)$$

and

$$\|u\|^2_{\tilde{H}^s(I_h^n)} \simeq h^{n-2s} \|\tilde{u}\|^2_{\tilde{H}^s(I^n)} \quad (-1 \le s \le 0),$$

provided one of the respective norms is finite.

Proof. The first assertion holds by Poincaré's inequality for $s = 1$ and for $0 < s < 1$ this follows by interpolation. Now let us prove the second assertion. First we consider the case $s = -1$. By duality we obtain, by using $\langle u, 1 \rangle_{L^2(I_h^n)} = 0$,

$$\begin{aligned}
\|u\|_{\tilde{H}^{-1}(I_h^n)} &= \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\|\psi\|_{H^1(I_h^n)}} = \sup_{\psi \in H^1(I_h^n)} \sup_{c \in \mathbb{R}} \frac{\langle u, \psi + c \rangle_{L^2(I_h^n)}}{\|\psi + c\|_{H^1(I_h^n)}} \\
&= \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{\inf_{c \in \mathbb{R}} \|\psi + c\|_{H^1(I_h^n)}} \simeq \sup_{\psi \in H^1(I_h^n)} \frac{\langle u, \psi \rangle_{L^2(I_h^n)}}{|\psi|_{H^1(I_h^n)}} \\
&\simeq \sup_{\tilde{\psi} \in H^1(I^n)} \frac{h^n \langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{h^{(n-2)/2} |\tilde{\psi}|_{H^1(I^n)}} = h^{(n+2)/2} \sup_{\tilde{\psi} \in H^1(I^n)} \frac{\langle \tilde{u}, \tilde{\psi} \rangle_{L^2(I^n)}}{|\tilde{\psi}|_{H^1(I^n)}} \\
&\simeq h^{(n+2)/2} \|\tilde{u}\|_{\tilde{H}^{-1}(I^n)}.
\end{aligned}$$

The last equivalence holds by the same arguments as used in the steps before. For the range $-1 < s < 0$ the assertion then follows by interpolation. □

The next lemma deals with the inverse property of piecewise polynomials. This property is well known for standard continuous finite element functions. Since we need this property also for higher degree polynomials and for discontinuous functions we include a proof for these cases. For simplicity we only consider rectangular meshes. The case of piecewise polynomials on curves is implicitly covered, see also [38, Remark 3.4].

Lemma 4 (K-method, K-method) *Let Γ be a polygonal domain and let $\{\Gamma_j;\ j = 1, \ldots, J\}$ be a rectangular mesh on Γ which is assumed to be quasi-uniform, the lengths of the elements being proportional to h. Further let v be a piecewise polynomial of degree p, i.e. $v|_{\Gamma_j}$ is a polynomial of degree p for $j = 1, \ldots, J$. If $v \in \tilde{H}^r(\Gamma)$ for a real number $r \le 1$, then for $s \le r$ there holds*

$$\|v\|_{\tilde{H}^r(\Gamma)} \le c h^{s-r} p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)}.$$

Accordingly, for $v \in H^r(\Gamma)$, there holds

$$\|v\|_{H^r(\Gamma)} \le c h^{s-r} p^{2(r-s)} \|v\|_{H^s(\Gamma)}.$$

Here, c is a constant that is independent of h and p and, for given r^*, s^* ($s^* \le r^*$), is also independent of r and s for $s^* \le s \le r \le r^*$.

Proof. First we deal with continuous piecewise polynomials. Let R be a Lipschitz domain such as one of the subdomains Γ_j. Locally for polynomials v we have by Schmidt's inequality (see [4])

$$|v|_{H^1(R)} \le c(h) p^2 \|v\|_{L^2(R)}.$$

Therefore, we conclude with the help of Lemma 2 that there holds

$$\begin{aligned} \|v\|^2_{H^1_0(\Gamma)} \simeq |v|^2_{H^1(\Gamma)} &= \sum_{j=1}^{J} |v|_{\Gamma_j}|^2_{H^1(\Gamma_j)} \simeq \sum_{j=1}^{J} |\widetilde{v|_{\Gamma_j}}|^2_{H^1(I^2)} \\ &\le c \sum_{j=1}^{J} p^4 \|\widetilde{v|_{\Gamma_j}}\|^2_{L^2(I^2)} \simeq \sum_{j=1}^{J} h^{-2} p^4 \|v|_{\Gamma_j}\|^2_{L^2(\Gamma_j)} = h^{-2} p^4 \|v\|^2_{L^2(\Gamma)}. \end{aligned} \tag{16}$$

Here, $\widetilde{v|_{\Gamma_j}}$ means the linearly transformed function $v|_{\Gamma_j}$ as defined in Lemma 2. By interpolation we now obtain

$$\|v\|_{\tilde{H}^r(\Gamma)} \le c h^{-(r-s)} p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)} \quad \text{for} \quad 0 \le s \le r \le 1. \tag{17}$$

Here we make use of the fact that interpolating between polynomial spaces reproduces the polynomials, see [29] for fundamental results on intervals and [5, 6] for an extension to domains in $\mathbb{R}^2$. Note also that the constant c in (17) does not depend on s or r (for $0 \le s \le r \le 1$) since the K-method is exact.

We note that the use of local scaling properties of Sobolev norms is standard in proving the usual inverse property of continuous piecewise polynomials of low degree, see, e.g., [12, Theorem 3.2.6].

To prove (17) for negative values of r and s we use the induction method of the proof of Theorem 4.1.3 in [2]. More precisely, we proceed as follows. Let $\alpha \in [0, 1/2)$ and $\varepsilon > 0$ be given such that $\alpha + \varepsilon \le 1$. Then from (17) we obtain by interpolation

$$\begin{aligned} \|v\|^2_{\tilde{H}^\alpha(\Gamma)} &= \|v\|^2_{[\tilde{H}^{\alpha-\varepsilon}(\Gamma), \tilde{H}^{\alpha+\varepsilon}(\Gamma)]_{1/2}} \le \|v\|_{\tilde{H}^{\alpha-\varepsilon}(\Gamma)} \|v\|_{\tilde{H}^{\alpha+\varepsilon}(\Gamma)} \\ &\le \|v\|_{\tilde{H}^{\alpha-\varepsilon}(\Gamma)} c h^{-\varepsilon} p^{2\varepsilon} \|v\|_{\tilde{H}^\alpha(\Gamma)}, \end{aligned}$$

i.e.

$$\|v\|_{\tilde H^{\alpha}(\Gamma)} \le c h^{-\varepsilon} p^{2\varepsilon} \|v\|_{\tilde H^{\alpha-\varepsilon}(\Gamma)}.$$

Therefore, (17) holds for $r \in [0,1]$ and $s \in [-1,r]$ with constant c^2 where c is from (17) for $0 \le s \le r \le 1$. By induction it follows that (17) holds for all numbers $s \le r$ where the constant c then depends on s (but is uniformly bounded for bounded $s \ge s^* > -\infty$). The extension to negative numbers r can be performed again by interpolation. For real numbers $s < r < 0 \le t \le 1$ and $0 < \theta < 1$ with $(1-\theta)s + \theta t = r$ there holds

$$\begin{aligned}
\|v\|_{\tilde H^{r}(\Gamma)} &= \|v\|_{[\tilde H^{s}(\Gamma), \tilde H^{t}(\Gamma)]_\theta} \le \|v\|^{1-\theta}_{\tilde H^{s}(\Gamma)} \|v\|^{\theta}_{\tilde H^{t}(\Gamma)} \\
&\le \|v\|^{1-\theta}_{\tilde H^{s}(\Gamma)} c h^{-\theta(t-s)} p^{2\theta(t-s)} \|v\|^{\theta}_{\tilde H^{s}(\Gamma)} = c h^{-(r-s)} p^{2(r-s)} \|v\|_{\tilde H^{s}(R)}.
\end{aligned}$$

This proves the assertion of the lemma for continuous functions.

We now prove the assertion when the piecewise polynomials are not required to be continuous. In this case we only deal with Sobolev norms of order less than 1/2. We simply make use of the inclusion $v|_{\Gamma_j} \in \tilde H^r(\Gamma_j)$ ($r < 1/2$) for a piecewise polynomial v. We then obtain by Lemmas 1 and 2 for an arbitrary but fixed parameter $0 \le r < 1/2$

$$\begin{aligned}
\|v\|^2_{\tilde H^{r}(\Gamma)} &\le c \sum_{j=1}^{J} \|v|_{\Gamma_j}\|^2_{\tilde H^{r}(\Gamma_j)} \simeq \sum_{j=1}^{J} h^{2-2r} \|\widetilde{v|_{\Gamma_j}}\|^2_{\tilde H^{r}(I^2)} \\
&\le c \sum_{j=1}^{J} h^{2-2r} p^{4r} \|\widetilde{v|_{\Gamma_j}}\|^2_{L^2(I^2)} \simeq \sum_{j=1}^{J} h^{2-2r} h^{-2} p^{4r} \|v|_{\Gamma_j}\|^2_{L^2(\Gamma_j)} \\
&= h^{-2r} p^{4r} \|v\|^2_{L^2(\Gamma)}.
\end{aligned} \tag{18}$$

Here again, $\widetilde{v|_{\Gamma_j}}$ means the linearly transformed function $v|_{\Gamma_j}$ as defined in Lemma 2. Further, we made use of Schmidt's inequality and its extension to real numbers $0 \le r \le 1$, see also [14, Lemma 4.1]. From (18) we obtain by interpolation

$$\|v\|^2_{\tilde H^{r}(\Gamma)} \le c h^{2(s-r)} p^{4(r-s)} \|v\|^2_{\tilde H^{s}(\Gamma)} \quad (0 \le s \le r < 1/2)$$

where c is uniformly bounded for $r \le r^* < 1/2$.

The extension of the above inverse property to negative numbers s and r can be performed inductively by interpolation as shown in the case of continuous piecewise polynomials. This finishes the proof of the first assertion of the lemma which deals with $\tilde H^r(\Gamma)$-norms.

The assertion for the $H^r(\Gamma)$-norms can be obtained by slight modifications of the above proof. In the case of continuous functions we note that (16) holds also when adding the $L^2(\Gamma)$-norm at the left hand side which gives

$$\|v\|^2_{H^1(\Gamma)} \le c h^{-2} p^4 \|v\|^2_{L^2(\Gamma)}.$$

The rest of the proof then is the same by replacing $\tilde H^r$-norms with H^r-norms. In the case of not necessarily continuous functions we note that (18) holds also for non-tilde norms which can be seen by using norms with weighted L^2-term. This weighting is admissible by Poincaré's inequality since C_0^∞ is dense in H^r for $|r| < 1/2$. For the first inequality in (18) one also needs Corollary 4 below and the fact that r is bounded by $r^* < 1/2$ in (18). The remainder of the proof for functions which are not necessarily continuous is analogous to the case when dealing with $\tilde{H}^r$-norms. □

The previous lemma uses the interpolation norm for Sobolev spaces $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ for positive and negative s. For negative s we need the inverse property of piecewise polynomials only in a small range s close to $-1/2$, but for the dual norm. In this range we can use the uniform equivalence of the interpolation norm and the dual norm to prove what is needed in this paper.

Corollary 3 (K-method, duality) *Let Γ be a polygonal domain and let $\{\Gamma_j;\ j = 1, \dots, J\}$ be a rectangular mesh on Γ which is assumed to be quasi-uniform, the lengths of the elements being proportional to h. Further let v be a piecewise polynomial of degree p, i.e. $v|_{\Gamma_j}$ is a polynomial of degree p for $j = 1, \dots, J$. Then, for given $\delta \in (0, 1/2)$ there exists a constant $c > 0$ such that for $s, r \in [-1+\delta, -\delta]$ with $s \le r$ there holds*

$$\|v\|_{\tilde{H}^r(\Gamma)} \le c h^{s-r} p^{2(r-s)} \|v\|_{\tilde{H}^s(\Gamma)}$$

and

$$\|v\|_{H^r(\Gamma)} \le c h^{s-r} p^{2(r-s)} \|v\|_{H^s(\Gamma)}.$$

Here, c is independent of h and p.

Proof. By Lemma 4 we only need to check the uniform equivalence of the norms involved. As in Corollary 2 this leads to considering equivalences of the form

$$[A_0', A_1']_s = [A_0, A_1]_s'$$

for Banach spaces A_0, A_1. In fact, this uniform equivalence for $s \in [\delta, 1-\delta]$ and given fixed $\delta \in (0, 1/2)$ is checked in the proof of Corollary 2. We therefore omit the details. □

The next lemma gives a bound for the norm of the injection $H^s \to \tilde{H}^s$ on Lipschitz domains for $|s| < 1/2$. In fact $\tilde{H}^s = H^s$ on Lipschitz domains for $|s| < 1/2$, see [16, Theorems 1.4.2.4, 1.4.5.2 (c)].

Lemma 5 (Sobolev-Slobodeckij, duality) *Let $R \subset \mathbb{R}^2$ be a Lipschitz domain. There exists a constant $c > 0$ such that for any $s \in (-1/2, 1/2)$ and any $v \in H^s(R)$ there holds*

$$\|v\|_{\tilde{H}^s(R)} \le \frac{c}{1/2 - |s|} \|v\|_{H^s(R)}.$$

Proof. For $0 < s < 1/2$ we use the equivalent Sobolev-Slobodeckij characterization of the $\tilde{H}^s(R)$-norm,

$$\|v\|^2_{\tilde{H}^s(R)} = \|v\|^2_{H^s(R)} + \left\|\frac{v(x)}{\operatorname{dist}(x, \partial R)^s}\right\|^2_{L^2(R)} \tag{19}$$

(see, e.g., [28, Theorem 11.7] and [16, Lemma 1.3.2.6]) where $\|v\|_{H^s(R)}$ is given by (9). Following the proof of Theorem 1.4.4.4 in [16] we estimate the latter term in the above relation.

First we consider the case when $\Omega = \mathbb{R}_+$ the nonnegative real axis. We make use of the identity [16, (1,4,4,9)]

$$v(x) = -w(x) + \int_x^\infty \frac{w(y)}{y}\,dy \quad \text{where} \quad w(x) = \frac{1}{x}\int_0^x (v(t) - v(x))\,dt.$$

Due to Hardy's inequality [18, Theorem 330],

$$\int_0^\infty x^{-r}\left(\int_x^\infty f(t)\,dt\right)^2 dx \le \frac{4}{(r-1)^2}\int_0^\infty x^{2-r} f^2(x)\,dx$$

for $r < 1$, there holds

$$\int_0^\infty \left(x^{-s}\int_x^\infty \frac{w(y)}{y}\,dy\right)^2 dx \le \frac{1}{(1/2-s)^2}\int_0^\infty w^2(x)x^{-2s}\,dx \quad \text{for } s < 1/2.$$

We estimate the latter integral above by

$$\begin{aligned}
\int_0^\infty w^2(x)x^{-2s}\,dx &= \int_0^\infty x^{-2(s+1)}\left(\int_0^x (v(t)-v(x))\,dt\right)^2 dx\\
&\le \int_0^\infty x^{-2(s+1)}\int_0^x 1\,dt\int_0^x (v(t)-v(x))^2\,dt\,dx\\
&= \int_0^\infty x^{-2s-1}\int_0^x (v(t)-v(x))^2\,dt\,dx\\
&\le \int_0^\infty\int_0^\infty \frac{(v(t)-v(x))^2}{|x-t|^{2s+1}}\,dt\,dx \le \|v\|^2_{H^s(0,\infty)}
\end{aligned}$$

where the last inequality is due to the definition of the Sobolev-Slobodeckij norm

$$\|v\|^2_{H^s(0,\infty)} = \|v\|^2_{L^2(0,\infty)} + \int_0^\infty\int_0^\infty \frac{|v(x)-v(y)|^2}{|x-y|^{1+2s}}\,dx\,dy,$$

see (9) with $n = 1$. Then we obtain

$$\begin{aligned}
\int_0^\infty (x^{-s}v(x))^2\,dx &= \int_0^\infty \left(-x^{-s}w(x) + x^{-s}\int_x^\infty \frac{w(y)}{y}\,dy\right)^2 dx\\
&\le 2\left(\int_0^\infty (x^{-s}w(x))^2\,dx + \int_0^\infty (x^{-s}\int_x^\infty \frac{w(y)}{y}\,dy)^2\,dx\right)\\
&\le 2\left(\|x^{-s}w\|^2_{L^2(0,\infty)} + \frac{1}{(1/2-s)^2}\|x^{-s}w\|^2_{L^2(0,\infty)}\right)\\
&\le \frac{c}{(1/2-s)^2}\|v\|^2_{H^s(0,\infty)}.
\end{aligned}$$

By (19) this proves the assertion when $\Omega = \mathbb{R}_+$ for $0 < s < 1/2$. The constant c does not depend on v and s.

This result applies also to general Lipschitz domains which can be seen by using the technique described in the proof of Theorem 1.4.4.4 in [16]. More precisely the boundary of R is locally the graph of a Lipschitz continuous function φ and locally we can define

$$v_{x_1}(t) = v(x_1, \varphi(x_1) - t)$$

where $(x_1, \varphi(x_1))$ describes a boundary piece and $(x_1, \varphi(x_1) - t) \in R$ for $0 < t < t_0$ and a $t_0 > 0$. (Here we assume that the local boundary piece under consideration is a Lipschitz continuous function of the variable x_1.) By using a partition of unity we can assume that the support of v is contained in a local neighborhood U of the boundary. We obtain

$$\|t^{-s} v_{x_1}\|^2_{L^2(0,\infty)} \le \frac{c}{(1/2 - s)^2} \|v_{x_1}\|^2_{H^s(0,\infty)}$$

for almost every x_1 such that $(x_1, \varphi(x_1)) \in U \cap \partial R$ where the constant c does not depend on x_1 and s. Integrating with respect to x_1 this yields

$$\begin{aligned}
& \int_{\mathbb{R}} \int_0^\infty t^{-2s} v^2(x_1, \varphi(x_1) - t)\, dt\, dx_1 \\
\simeq\ & \int_{\mathbb{R}} \int_{R \cap \{(x_1, \varphi(x_1) - t);\, t > 0\}} (\varphi(x_1) - x_2)^{-2s} v^2(x_1, x_2)\, dx_2\, dx_1 \\
\simeq\ & \left\| \frac{v(x)}{\operatorname{dist}(x, \partial R)^s} \right\|^2_{L^2(R)} \le \frac{c}{(1/2 - s)^2} \|v\|^2_{H^s(R)}
\end{aligned}$$

and therefore

$$\|v\|^2_{\tilde{H}^s(R)} \le \frac{c}{|s - 1/2|^2} \|v\|^2_{H^s(R)} \quad \text{for } 0 < s < 1/2.$$

For $-1/2 < s < 0$ we obtain

$$\|v\|_{\tilde{H}^s(R)} = \sup_{\varphi \in C_0^\infty(R)} \frac{\langle v, \varphi \rangle_{L^2(R)}}{\|\varphi\|_{H^{-s}(R)}} \le \frac{c}{|-s - 1/2|} \sup_{\varphi \in C_0^\infty(R)} \frac{\langle v, \varphi \rangle_{L^2(R)}}{\|\varphi\|_{\tilde{H}^{-s}(R)}} = \frac{c}{|s + 1/2|} \|v\|_{H^s(R)}$$

and the proof is finished. □

Since the previous lemma deals (for $s > 0$) with the Sobolev-Slobodeckij norm we need to check its equivalence with the norm obtained by interpolation.

Corollary 4 (K-method, duality) *Let $R \subset \mathbb{R}^2$ be a Lipschitz domain. For given $\delta \in (0, 1/2)$ there exists a constant $c > 0$ such that for any $s \in \mathbb{R}$ with $\delta \le |s| < 1/2$ and for any $v \in H^s(R)$ there holds*

$$\|v\|_{\tilde{H}^s(R)} \le \frac{c}{1/2 - |s|} \|v\|_{H^s(R)}.$$

Proof. By Lemma 5 it suffices to check the uniform equivalence for $s \in [\delta, 1/2]$ of the Sobolev-Slobodeckij norm and the norm defined by the K-method of interpolation. The assertion then follows for $s \in [\delta, 1/2)$ and by duality, which is used both in Lemma 5 and this corollary, the assertion holds also for $s \in (-1/2, -\delta]$.

As demonstrated by Brenner and Scott in the proof of Theorem 12.2.3 in [10], the equivalence of both norms on the Lipschitz domain R can be proved by their equivalence on $\mathbb{R}^2$. Therefore, for the time being, let us assume that the Sobolev-Slobodeckij norm and the interpolation norm (K-method) are uniformly equivalent on $\mathbb{R}^2$ for $s \in [\delta, 1/2]$. Then, using the extension operator by Stein $E_S:\ H^k(R) \to H^k(\mathbb{R}^2)$ for integer k (already considered in the proof of Lemma 1), one proves

$$\|u\|_{H^s(R)} \le \|E_S u\|_{H^s(\mathbb{R}^2)} \le c\|E_S u\|_{[L^2(\mathbb{R}^2),H^1(\mathbb{R}^2)]_s} \tag{20}$$

$$\le cC\|u\|_{[L^2(R),H^1(R)]_s}. \tag{21}$$

Here, $\|u\|_{H^s(R)}$ is the Sobolev-Slobodeckij norm. For (20) the equivalence of the norms on $\mathbb{R}^2$ is used and (21) follows by interpolating the continuity of E_S. Since the K-method is exact the second constant C in (21) can be chosen independently of $s \in (0,1)$. To prove the other direction Brenner and Scott are using an extension operator by Grisvard $E_G:\ H^s(R) \to H^s(\mathbb{R}^2)$ that is continuous with respect to the Sobolev-Slobodeckij norm, see [16, Theorem 1.4.3.1]. One obtains

$$\|u\|_{[L^2(R),H^1(R)]_s} \le \|E_G u\|_{[L^2(\mathbb{R}^2),H^1(\mathbb{R}^2)]_s} \le c\|E_G u\|_{H^s(\mathbb{R}^2)} \tag{22}$$

$$\le c\|E_G\|\|u\|_{H^s(R)}.$$

Here, (22) uses the equivalence of the norms on $\mathbb{R}^2$. Grisvard notes that the extension operator E_G (which he denotes by P_s) can be chosen independently of s, see [34] and [1]. Therefore, $\|E_G\| = \|E_G\|_{H^s(R)\to H^s(\mathbb{R}^2)}$ depends continuously on s and we can bound $\|E_G\| \le C$ for $s \in [\delta, 1/2]$.

It remains to check the equivalence of the norms on $\mathbb{R}^2$ for $s \in [\delta, 1/2]$. Combination of Theorems 6.4.4 and 6.4.5(4) in [7] gives

$$[L^2(\mathbb{R}^2), H^1(\mathbb{R}^2)]_s = B^s_{22} = H^s(\mathbb{R}^2) \tag{23}$$

where B^s_{22} is a so-called Besov space and $H^s(\mathbb{R}^2)$ is endowed with the norm defined by the Bessel potential,

$$\|u\|_s^2 := \int_{\mathbb{R}^2} (1+|\xi|^2)^s |\mathcal{F}u(\xi)|^2\, d\xi$$

with $\mathcal{F}u$ being the Fourier transform of u. By checking the proofs in [7] one can see that the equivalence constants of the norms involved in (23) depend continuously on s and therefore, the interpolation norm by the K-method and the norm given by the Bessel potential are uniformly equivalent for $s \in [\delta, 1/2]$. On the other hand the Bessel potential norm and the Sobolev-Slobodeckij norm are equivalent on $\mathbb{R}^2$ and the equivalence constants depend continuously on s, see, e.g., [15, Lemma 4.1]. This proves the uniform equivalence of the Sobolev-Slobodeckij norm and the interpolation norm by the K-method for Sobolev spaces $H^s(\mathbb{R}^2)$ with $s \in [\delta, 1/2]$.

By the argumentation at the beginning of the proof (using the extension operators E_S and E_G) this uniform equivalence carries over to the Lipschitz domain R and, by duality, this uniform equivalence of the corresponding dual norms holds for $s \in [-1/2, -\delta]$ as well. Therefore, the proof is finished by Lemma 5. □

With the help of Corollary 4 we are now able to bound the $\tilde{H}^{-1/2}$-norm of polynomials by their $H^{-1/2}$-norm. This result might be of interest in itself and is given for completeness. In the proof of the main theorem this estimate is only implicitly used by applying the technique of its proof. See [3] for corresponding results regarding the $\tilde{H}^{1/2}$- and $H^{1/2}$-norms of polynomials on an interval.

Lemma 6 *Let v be a piecewise polynomial of degree p defined on a quasi-uniform rectangular mesh R_h on R with mesh size h. There exists a constant $c > 0$ which is independent of p but may depend on the diameter of R such that there holds*

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq c(1 + \log\frac{p+1}{h})\|v\|_{H^{-1/2}(R)} \quad (h \leq 1).$$

Analogously, if $v \in \tilde{H}^{1/2}(R)$ is a piecewise polynomial then there holds

$$\|v\|_{\tilde{H}^{1/2}(R)} \leq c(1 + \log\frac{p}{h})\|v\|_{H^{1/2}(R)} \quad (h \leq 1).$$

Proof. The first assertion is proved as follows. We have $v \in \tilde{H}^{-1/2}(R)$ since $v \in L^2(R) \subset \tilde{H}^{-1/2}(R)$. Now, for fixed $\delta^* \in (0, 1/2)$, let $\delta \in (0, 1/2 - \delta^*)$ be given. Using Corollary 4 we deduce

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq \|v\|_{\tilde{H}^{-1/2+\delta}(R)} \leq c/\delta\|v\|_{H^{-1/2+\delta}(R)},$$

and by the inverse property (Corollary 3) we conclude for $p > 0$

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq \frac{c}{\delta}h^{-\delta}p^{2\delta}\|v\|_{H^{-1/2}(R)}.$$

Here the constant c does not depend on p or h. Setting $\delta := 1/\log\frac{p^2}{h}$ we obtain

$$\|v\|_{\tilde{H}^{-1/2}(R)} \leq c\log\frac{p}{h}\|v\|_{H^{-1/2}(R)} \quad (p/h > 1)$$

which gives the first assertion.

Analogously, we obtain for continuous piecewise polynomials v by using the inverse property (Lemma 4) and Corollary 4

$$\begin{aligned}\|v\|_{\tilde{H}^{1/2}(R)} &\leq ch^{-\delta}p^{2\delta}\|v\|_{\tilde{H}^{1/2-\delta}(R)} \leq \frac{c}{\delta}h^{-\delta}p^{2\delta}\|v\|_{H^{1/2-\delta}(R)}\\ &\leq \frac{c}{\delta}h^{-\delta}p^{2\delta}\|v\|_{H^{1/2}(R)}\end{aligned}$$

($\delta \in (0, 1/2 - \delta^*)$). Choosing $\delta := 1/\log\frac{p^2}{h}$ the second assertion of the lemma follows. □

3.1 Proof of the main theorem.

We are now in the position to prove Theorem 1. Due to the equivalence of the $\tilde{H}^{-1/2}$-norm and the norm induced by the weakly singular operator V the boundedness of the maximum eigenvalue of P is a direct consequence of Lemma 1 (estimate (10)) and the triangle inequality. In order to bound the minimum eigenvalue we proceed as follows.

First we note that, for $s \geq 0$, the L^2-projection

$$Q_{0,j} : \left\{ \begin{array}{ccc} H^s(G_j) & \to & H^s(G_j) \\ \psi & \mapsto & \frac{\langle\psi,1\rangle_{L^2(G_j)}}{|G_j|} \end{array} \right. \tag{24}$$

is continuous. This follows by interpolating $\|Q_{0,j}\psi\|_{L^2(G_j)} \le \|\psi\|_{L^2(G_j)}$ and

$$\|Q_{0,j}\psi\|_{H^k(G_j)} = \|Q_{0,j}\psi\|_{L^2(G_j)} \le \|\psi\|_{H^k(G_j)}$$

where k is an integer greater than or equal to s. Note that $\|Q_{0,j}\|_{H^s(G_j)\to H^s(G_j)}$ is bounded independently of s since the K-method of interpolation is exact, cf. (8).

Now let $v = v_0 + \sum_{j=1}^n w_j$ be the representation of a function $v \in H_N$ which corresponds to the decomposition (6). We have to bound the constant C in

$$\|v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j=1}^n \|w_j\|^2_{\tilde{H}^{-1/2}(\Gamma_j)} \le C\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)}$$

which is an upper bound for the inverse of the minimum eigenvalue of P. First we bound the sum of the local terms $\sum_{j=1}^n \|w_j\|^2_{\tilde{H}^{-1/2}(\Gamma_j)}$. Using the above projection operator $Q_{0,j}$ and noting that

$$w_j = \big(v - v_0\big)|_{G_j} = v|_{G_j} - Q_{0,j}\big(v|_{G_j}\big)$$

we obtain, for any $0 < \varepsilon \le \varepsilon^* < 1/2$,

$$\begin{aligned}\|w_j\|_{\tilde{H}^{-1/2+\varepsilon}(G_j)} &= \sup_{\psi\in H^{1/2-\varepsilon}(G_j)} \frac{\langle v - Q_{0,j}v, \psi\rangle_{L^2(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} = \sup_{\psi\in H^{1/2-\varepsilon}(G_j)} \frac{\langle v, \psi - Q_{0,j}\psi\rangle_{L^2(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \\ &\le \|v\|_{H^{-1/2+\varepsilon}(G_j)} \sup_{\psi\in H^{1/2-\varepsilon}(G_j)} \frac{\|\psi - Q_{0,j}\psi\|_{\tilde{H}^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}}.\end{aligned}$$

With Corollary 4 we estimate the $\tilde{H}^{1/2-\varepsilon}(G_j)$-norm by the $H^{1/2-\varepsilon}(G_j)$-norm. Together with the continuity of $Q_{0,j}$ given by (24) this gives for the last term the bound

$$\frac{\|\psi - Q_{0,j}\psi\|_{\tilde{H}^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \le \frac{c}{\varepsilon}\frac{\|\psi - Q_{0,j}\psi\|_{H^{1/2-\varepsilon}(G_j)}}{\|\psi\|_{H^{1/2-\varepsilon}(G_j)}} \le \frac{c}{\varepsilon},$$

i.e.,

$$\|w_j\|_{\tilde{H}^{-1/2+\varepsilon}(G_j)} \le \frac{c}{\varepsilon}\|v\|_{H^{-1/2+\varepsilon}(G_j)}.$$

The constant c does not depend on the size H of G_j since the $\tilde{H}^{-1/2+\varepsilon}(G_j)$ and the $H^{-1/2+\varepsilon}(G_j)$-norms scale with the same order of H since w_j has integral mean zero, cf. Lemmas 2 and 3. Therefore, by (15), we obtain

$$\sum_{j=1}^n \|w_j\|^2_{\tilde{H}^{-1/2+\varepsilon}(G_j)} \le \frac{c}{\varepsilon^2}\sum_{j=1}^n \|v\|^2_{H^{-1/2+\varepsilon}(G_j)} \le \frac{c}{\varepsilon^2}\|v\|^2_{H^{-1/2+\varepsilon}(\Gamma)}. \tag{25}$$

By noting that $\langle w_j, 1\rangle_{L^2(G_j)} = 0$, we use the scaling property given by Lemma 3 and obtain

$$\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \le cH_j^{2\varepsilon}\|w_j\|^2_{\tilde{H}^{-1/2+\varepsilon}(G_j)}.$$

Together with the inverse property (Corollary 3) and (25) we therefore conclude

$$\sum_{j=1}^{n} \|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \le c\frac{H^{2\varepsilon}}{\varepsilon^2}\|v\|^2_{H^{-1/2+\varepsilon}(G)} \le c\Big(\frac{H}{h}\Big)^{2\varepsilon}\frac{p^{4\varepsilon}}{\varepsilon^2}\|v\|^2_{\tilde{H}^{-1/2}(G)}. \tag{26}$$

Setting

$$\varepsilon := \frac{1}{2\log(\frac{H}{h}p^2)} \quad \text{for } \frac{H}{h}p^2 > 1$$

this gives

$$\sum_{j=1}^{n} \|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \le c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)} \tag{27}$$

for $p \ge 0$ where the constant c does not depend on p, H, and h.

The bound for the component v_0 of v then follows by the triangle inequality and by (10):

$$\begin{aligned}\|v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} &\le 2\big(\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \|v-v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)}\big) \le c\big(\|v\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j=1}^{n}\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)}\big)\\ &\le c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)}.\end{aligned}$$

Adding (27) this yields

$$\|v_0\|^2_{\tilde{H}^{-1/2}(\Gamma)} + \sum_{j=1}^{n}\|w_j\|^2_{\tilde{H}^{-1/2}(G_j)} \le c(1+\log\frac{H}{h}(p+1))^2\|v\|^2_{\tilde{H}^{-1/2}(G)}$$

which proves that the minimum eigenvalue of P is bounded from below by a term which behaves like $(1+\log\frac{H}{h}(p+1))^{-2}$. This concludes the proof of Theorem 1.

4 Numerical results.

In this section we present numerical results for the Galerkin method for solving the problem (1). We choose the domain Γ to be the square plate $(-1/2,1/2)^2 \times \{0\}$. The finite-dimensional subspaces X_N of $\tilde{H}^{-1/2}(\Gamma)$ are constructed by discontinuous piecewise tensor products of scaled Legendre polynomials (3) on a partition Γ_h of Γ. We note that the scaling of the polynomials already reduces substantially the spectral condition number of the stiffness matrix, cf. [20].

The entries of the stiffness matrix of the system (2) are implemented by analytical recurrence formulas. They have been developed and implemented by M. Maischak at the University of Hannover, Germany. For details we refer to [30]. The condition numbers are computed by reducing the matrices to tridiagonal form and by calculating the minimum and maximum eigenvalues via the QR–algorithm. For this we use standard LAPACK-routines.

First we look at the pure p-version. We use a uniform mesh of 4 rectangles and increase successively the degree p of the polynomials uniformly on the elements. Table 1 shows the condition numbers of the stiffness matrices and the numbers of iterations of the conjugate gradient algorithm which are required to reduce the iteration error to 10% of the approximation error. (We choose the right hand side $g = 1$ and use an approximation for the energy norm of the true solution by extrapolating results obtained by the pure h-version.) The number N denotes the number of unknowns in the ansatz space X_N and $\kappa(A)$ is the condition number of the stiffness matrix which is constructed by using the scaled Legendre polynomials. To demonstrate the effect on the condition number which is due to the scaling we also show the condition number $\kappa(A_{\mathrm{u}})$ where we neglect the scaling. Finally, $\kappa(P)$ is the condition number of the preconditioned stiffness matrix where we choose $H = h$, i.e., the subdomains consist of a single element of the fine mesh. As proved by Theorem 1, $\kappa(P)$ increases just slowly with p. A comparison of the predicted bound $O(1 + \log(p + 1))^2$ (in the case $H = h$) for the condition numbers and the actual numerical results is given in Figure 1 in a log-log scale where almost coincidence can be observed.

p	N	$\kappa(A_{\mathrm{u}})$	$\kappa(A)$	#it	$\kappa(P)$	#it
1	16	96	11.32	2	2.95	2
2	36	445	21.52	3	4.42	2
3	64	1277	35.05	3	5.36	3
4	100	2875	52.13	4	6.51	3
5	144	5608	72.82	5	7.33	3
6	196	9904	97.15	6	8.26	3
7	256	16292	125.15	7	8.98	3
8	324	26425	154.13	8	9.77	3

Table 1: The condition and iteration numbers for the p-version with 4 elements.

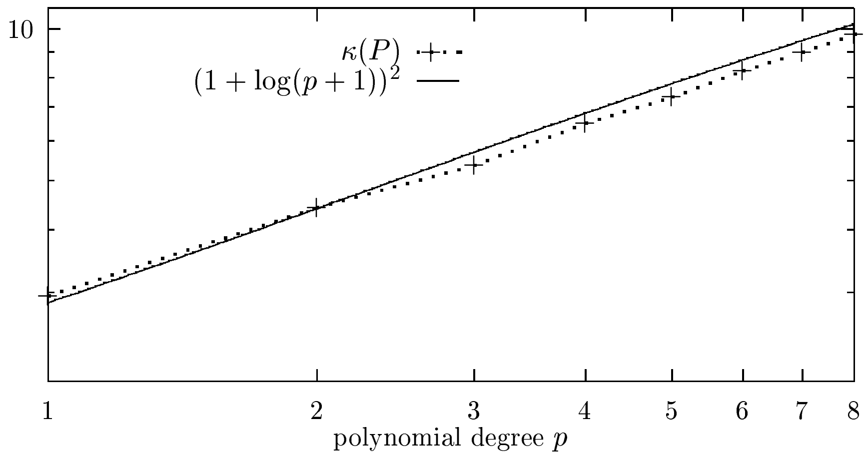

Figure 1: Condition numbers of the preconditioned stiffness matrix ($H = h$).

To check the boundedness of the condition number of the additive Schwarz operator when refining the mesh and choosing $H = h$ we also present results for the h-version where we fix the polynomial degree at $p = 1$, see Table 2. In that case each element is associated with four unknowns, cf. (3). Actually the condition numbers are very small and they seem to support boundedness.

$1/h$	N	$\kappa(A)$	#it	$\kappa(P)$	#it
2	16	11.32	2	2.95	2
4	64	26.47	3	3.67	3
6	144	41.23	4	3.91	3
8	256	55.78	5	4.08	3
10	400	70.23	5	4.14	3

Table 2: The condition and iteration numbers for the h-version with $p = 1$.

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