An hp-adaptive refinement strategy for hypersingular operators on surfaces

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Abstract

An adaptive refinement strategy for the hp-version of the boundary element method with hypersingular operators on surfaces is presented. The error indicators are based on local projections provided by two-level decompositions of ansatz spaces with additional bubble functions. Assuming a saturation property and locally quasi-uniform meshes, efficiency and reliability of the resulting error estimator is proved. A second error estimator based on mesh refinement and overlapping decompositions that better fulfills the saturation property is presented. The performance of the algorithm and the estimators is demonstrated for a model problem.

Key words: hp-version of the BEM, a-posteriori error estimate, two-level decomposition
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1 Introduction

Adaptive strategies based on subspace decompositions have become popular in recent years, see e.g. [6, 4, 17, 19, 18]. Here we exploit this idea to define error indicators and estimators for the hp-version of the boundary element method with hypersingular operators on surfaces. We especially consider meshes consisting of triangles and quadrilaterals. Our error indicators are based on solving local problems on individual elements with increased polynomial degrees where, on quadrilaterals, also direction control is provided. Essential ingredient in the analysis of the corresponding error estimator is the stability of the underlying subspace decomposition. Decoupling of standard basis functions in the p-version leads to unstable decompositions, see e.g. [3]. Therefore, when one does not want to switch to special basis functions, e.g. discrete harmonic functions which are not hierarchical, then one has to perform a Schur complement step. This step amounts to a basis transformation and is also referred to as orthogonalization. In the finite element method (FEM) this orthogonalization can be performed locally. In the boundary element method (BEM) however, we have to deal with a dense matrix which makes this step too expensive. We therefore propose to perform a partial orthogonalization that neglects the coupling of basis functions associated with different elements.

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To prove reliability of our error estimators we have to assume a saturation property. This property is not uniformly satisfied when introducing an improved approximation by increasing polynomial degrees by one or two. Reliability of such an estimator therefore will eventually decrease for increasing polynomial degrees. However, our numerical results show that the indicators then still lead to good adaption. To obtain an error estimator that is reliable also for higher polynomial degrees we propose a second strategy that is based on mesh refinement and an overlapping decomposition. But for the adaption process we always use the indicators based on increasing polynomial degrees since they directly provide local informations and direction control.

The methods proposed and analyzed in this paper are used in [13] where only numerical results for the p-version are given. In that paper, a parameter κ was introduced that characterizes the positive definiteness of the Schur complement. This unknown parameter is not present anymore in the theory of this paper.

We also note that a strategy for direction control analogous to ours was proposed in [19] for a weakly singular operator. In that paper performance of the direction control is demonstrated by numerical results but no analysis for the p-version is provided. For weakly singular operators continuous basis functions are not required and tensor products of Legendre polynomials can be used. Here we need to deal with continuous ansatz functions and a different norm and therefore, our results cannot be applied to the BEM with weakly singular operators, and vice versa. We give an analysis under the restriction of locally quasi-uniform meshes and demonstrate the performance by presenting numerical results for the hp-version.

For the adaptive hp-version it is not only necessary to provide error indicators to decide where to adapt. One also has to decide whether to refine the mesh at a selected element or to increase polynomial degrees. For hp-FEM a decision based on local regularity estimates is proposed in [1]. Here we simply follow the three-fold algorithm given in [12] that refines where indicators are very large and that increases polynomial degrees where indicators are only of medium size. This algorithm likely produces sub-optimal adaption when different singularities of different intensities are present. A criterion for the BEM that leads to optimal hp-refinement is not studied yet. However, for our model problem with different singularities this hp-adaptive algorithm still performs better than the other algorithms, as there are the uniform and adaptive h- and p-versions.

As a model problem we consider the weak form of the hypersingular integral equation

$$\langle Du, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on an open, piecewise plane Lipschitz surface $\Gamma \subset \mathbf{R}^3$. $f \in H^{-1/2}(\Gamma)$ is a given function and D is the hypersingular integral operator

$$Du(x) = \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad x \in \Gamma$$

which is a continuous and positive definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [5]. Hence, there holds the equivalence

$$\langle Dv, v\rangle_{L^2(\Gamma)} \simeq \|v\|^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all} \quad v \in \tilde{H}^{1/2}(\Gamma).$$

The Sobolev space $\tilde{H}^{1/2}(\Gamma)$ is an interpolation space between $L^2(\Gamma)$ and $H_0^1(\Gamma)$, and $H^{-1/2}(\Gamma)$ is the dual space of $\tilde{H}^{1/2}(\Gamma)$. The solution u of (1) is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathbf{R}^3 \setminus \bar{\Gamma}$, cf. [24].

The Galerkin scheme for the approximate solution of (1) reads as follows: *Given a finite dimensional subspace $S_{h,p}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$ find $u_N \in S_{h,p}(\Gamma)$ such that*

$$\langle Du_N, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \textit{for all} \quad v \in S_{h,p}(\Gamma). \tag{2}$$

Here the index N refers to the dimension of $S_{h,p}(\Gamma)$. Note that the conformity condition $S_{h,p}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$ for an open surface requires continuous functions which are zero on the boundary of Γ. For the h-version one uses piecewise polynomials of low degree ($p = 1, 2$) on a sequence of iteratively refined meshes. The p-version keeps a fixed mesh and uses piecewise polynomials of increasing degree p. Here we consider the hp-version that combines mesh refinement and appropriate polynomial degree distribution.

The outline of this paper is as follows. In Section 2 we introduce error indicators and estimator in an abstract way based on decomposing an enriched ansatz space. For efficiency and reliability of the estimator we assume a saturation property and stability of the decomposition. We then introduce the concept of partial orthogonalization to guarantee stability. In Section 3 we propose two strategies to define error indicators and estimators. The main results stating reliability and efficiency under certain restrictions are given. Proofs are provided in Section 4. Several numerical results given in Section 5 demonstrate performance of our algorithms.

2 Abstract formulation and partial orthogonalization

Let us consider an abstract ansatz space H_N of dimension N and an enriched space $\tilde{H}_N$ with $H_N \subset \tilde{H}_N$ that is decomposed like

$$\tilde{H}_N = Z_0 + Z_1 + \cdots + Z_J \tag{3}$$

where in general $Z_j \not\subset H_N$ and, in practice, functions of Z_j for $j > 0$ have a small support. The subspace Z_0 may consist of functions with non-local support, i.e. they may be non-zero on the whole domain Γ. In standard domain decomposition methods Z_0 serves to share global information and usually consists of piecewise polynomials of low degrees on a relatively coarse mesh. For our adaptive method one also may think of $Z_0 = H_N$ and of local spaces Z_j ($j > 0$) that consist of additional functions not included in H_N. The local subspaces form a space Z,

$$Z := Z_1 + \cdots + Z_J$$

that we call a second level of the decomposition (3). Later the index J will refer to the number of elements. Except when we further decompose the subspaces Z_j associated with elements, the decomposition (3) will be direct, and this is assumed in the following.

To introduce the error estimator we define projection operators P_j, $j = 0, \ldots, J$, by

$$P_j : \tilde{H}_N \to Z_j \quad a(P_j\phi, \psi) = a(\phi, \psi) \quad \text{for any } \psi \in Z_j.$$

Here we use the notation $a(\cdot,\cdot)$ for the bilinear form introduced by the hypersingular integral operator.

Now let $u_N \in H_N$ be the Galerkin approximation to the true solution u of (1), and let $\tilde{u}_N \in \tilde{H}_N$ be the improved approximation (obtained by projecting u with respect to $a(\cdot,\cdot)$ onto $\tilde{H}_N$). Then our *a-posteriori error estimator* is defined by

$$\Theta := \left(\sum_{j=0}^{J} \theta_j^2\right)^{1/2} \quad \text{with} \quad \theta_j := \|P_j(\tilde{u}_N - u_N)\|_D, \quad j = 0, \dots, J.$$

The norm $\|\cdot\|_D$ is the energy norm defined by the integral operator D. We note that, once the local stiffness matrices A_j for the projections P_j are assembled, the error indicators are simply given by $\theta_j = (c_j^T A_j c_j)^{1/2}$. Here, c_j is the vector of coefficients belonging to the projection $P_j(\tilde{u}_N - u_N)$.

Efficiency and reliability of the error estimator depend on the stability of the two-level decomposition and on a saturation property. More precisely we make the following assumptions:

(A1) **Stability of the two-level decomposition**:
(i) there exists $c_1 > 0$ such that for any $\phi \in \tilde{H}_N$ there exist $\phi_j \in Z_j$ $(j \geq 0)$ such that

$$\phi = \sum_{j=0}^{J} \phi_j \quad \text{and} \quad c_1 \sum_{j=0}^{J} a(\phi_j, \phi_j) \leq a(\phi, \phi),$$

(ii) there exists $c_2 > 0$ such that for any $\phi \in \tilde{H}_N$ and any $\phi_j \in Z_j$ $(j \geq 0)$ with $\phi = \sum_{j=0}^{J} \phi_j$ there holds

$$a(\phi, \phi) \leq c_2 \sum_{j=0}^{J} a(\phi_j, \phi_j).$$

(A2) **Saturation assumption**: there exists $\sigma \in [0, 1)$ such that

$$\|u - \tilde{u}_N\|_D \leq \sigma \|u - u_N\|_D.$$

Using the above assumptions the following abstract result proving efficiency and reliability of Θ can be shown, see, e.g., [4, 19].

Lemma 1 *Let the assumptions* (A1) *and* (A2) *be fulfilled. Then there holds*

$$\frac{1}{c_2}\Theta^2 \leq \|u - u_N\|_D^2 \leq \frac{1}{c_1(1-\sigma^2)}\Theta^2.$$

In concrete cases it is left to study the behavior of the parameters c_1, c_2 and σ in (A1) and (A2). Then Lemma 1 gives bounds for the efficiency and reliability of the a-posteriori error estimator under consideration.

There are different possibilities to find efficient decompositions (3). For standard high order subspaces stable decompositions can be obtained by taking overlapping subspaces (see [20, 22] for the FEM and [10] for the BEM). One may also consider direct decompositions when using sophisticated basis functions which are not hierarchical, see [9, 14] for the BEM and [23, 7] for the FEM. However, overlapping subspaces do not give error indicators which are associated with distinct elements. On the other hand, non-hierarchical basis functions are not appropriate for locally increasing polynomial degrees, what is necessary for *hp*-adaptivity. Therefore, to formulate a practical *hp*-adaptive algorithm, we here propose to use both standard hierarchical basis functions and direct decompositions of the second level Z of $\tilde{H}_N$. Then stability of the decomposition of $\tilde{H}_N$ cannot a-priorily be assumed. On the contrary an orthogonalization procedure is necessary. In Section 3.1 below we choose $Z_0 = H_N$ and define local subspaces Z_j only by some additional bubble (i.e. internal basis) functions. As we will see, the decomposition of the space Z of additional bubble functions with respect to the elements is almost uniformly stable. However, to ensure stability of the overall decomposition (3) one needs to decouple the two levels Z_0 and Z. This decoupling is nothing but a static condensation of the functions belonging to Z and is also known as a Schur complement step. This decoupling is described in the following.

Let us assume that the basis functions of $\tilde{H}_N$ are numbered in such a way that the space Z_0 is spanned first and $Z = Z_1 + \cdots + Z_J$ is spanned second. Then the stiffness matrix is of the form

$$A = \begin{pmatrix} A_1 & B^T \\ B & A_2 \end{pmatrix}$$

where A_1 and A_2 belong to Z_0 and Z, respectively. Since we are dealing with integral operators which are non-local the matrix A_2 is not block-diagonal even if functions of different subspaces $Z_j \neq Z_i$ have disjoint supports. In the FEM this is different. For instance if the subspaces Z_j belong to different elements then the static condensation in the FEM can be done independently, i.e. A_2 can be inverted by inverting its diagonal blocks (if basis functions are numbered appropriately). Here, for the BEM, a full Schur complement step requires the inversion of the full matrix A_2. We propose an approximate static condensation by only inverting the block-diagonal part C_2 of A_2 that belongs to the decomposition

$$Z = Z_1 + \ldots + Z_J.$$

We assume that this decomposition is direct such that the blocks belonging to different subspaces do not overlap. Later we will also decompose subspaces belonging to elements where some of the decompositions may overlap. We call this step *partial Schur complement* or *partial orthogonalization* and it is performed by the matrix transformation

$$T_{C_2} := \begin{pmatrix} I & -B^T C_2^{-1} \\ 0 & I \end{pmatrix}. \tag{4}$$

Note that the usual Schur complement step is defined by the transformation

$$T_{A_2} := \begin{pmatrix} I & -B^T A_2^{-1} \\ 0 & I \end{pmatrix}.$$

The partial Schur complement step leads to the transformed stiffness matrix

$$\tilde{A} := T_{C_2} A T_{C_2}^T = \begin{pmatrix} A_1 - B^T(2C_2^{-1} - C_2^{-1}A_2C_2^{-1})B & B^T(I - C_2^{-1}A_2) \\ (I - A_2C_2^{-1})B & A_2 \end{pmatrix} =: \begin{pmatrix} \tilde{S} & \tilde{B}^T \\ \tilde{B} & A_2 \end{pmatrix}.$$

In the case $C_2 = A_2$ there holds $\tilde{B} = 0$ and $\tilde{S} = A_1 - B^T A_2^{-1} B$ is the usual Schur complement. Note also that the matrix transformations T_{C_2} and T_{A_2} implicitly define basis transformations within $\tilde{H}_N$. From the form of $\tilde{A}$ it is obvious that basis functions in Z remain unchanged whereas basis functions in Z_0 are modified with functions of Z. The old subspace Z_0 is transformed into a new subspace that we call $T_{C_2}(Z_0) \subset \tilde{H}_N$ in the case of the partial Schur complement and $T_{A_2}(Z_0) \subset \tilde{H}_N$ when the full orthogonalization is performed. When considering both cases synonymously we use the generic symbol T. The basis transformation leads to the new decomposition

$$\tilde{H}_N = T(Z_0) + Z_1 + \ldots + Z_J \tag{5}$$

where in general $T(Z_0) \neq Z_0$. The basis transformation in the first level needs to be taken into account for the a-posteriori error estimate. In the case $Z_0 = H_N$ we have $P_0(\tilde{u}_N - u_N) = 0$. But performing a partial or full Schur complement step this yields a transformed space $T(Z_0) = T(H_N)$ such that $P_0(\tilde{u}_N - u_N) \neq 0$ in general.

In conclusion, when considering a decomposition (3) with $Z_0 = H_N$ then $\theta_0 = 0$ and $\Theta = (\sum_{j=1}^J \theta_j^2)^{1/2}$ is the corresponding error estimator. On the other hand, if $T(Z_0) = T(H_N)$ in (5) comes from a Schur complement step we have a global contribution θ_0 to the a-posteriori error estimator, i.e. $\Theta = (\sum_{j=0}^J \theta_j^2)^{1/2}$. Since θ_0 provides global information it is not used to steer the refinement procedure. Here, only the local terms θ_j $(j > 0)$ are of interest. For estimating the overall error as a stopping criterion it may be necessary to calculate the global term θ_0 from time to time.

3 Error indicators and estimators for the *hp*-version

In this section we describe the ansatz space H_N for the Galerkin system (2). We propose different strategies to enrich the ansatz space and we study different decompositions that lead to error indicators and estimators. The cheapest procedure is to locally enrich the ansatz space H_N in a hierarchical manner. For the p-version that means to add some piecewise polynomials of higher degrees. This strategy is studied in Section 3.1. In Section 3.2 we use an overlapping decomposition of an enriched space defined by mesh refinement. This procedure is more expensive than the strategy by local enrichment but leads to a more effective error estimator when the polynomial degrees are large.

To define our standard hp-ansatz space let $\mathcal{T}_h = \{\Gamma_1, \ldots, \Gamma_J\}$ be a regular mesh of triangles and quadrilaterals with $\cup_j \bar{\Gamma}_j = \bar{\Gamma}$. We consider the piecewise polynomial space

$$H_N := S_{h,p}(\Gamma) := \{\phi \in C^0(\Gamma);\ \phi|_{\partial\Gamma} = 0,\ \phi|_{\Gamma_j} \in \mathcal{P}_{p_j,q_j}(\Gamma_j), j = 1, \ldots, J\}.$$

Here, p stands for the vector $((p_1, q_1), \ldots, (p_J, q_J))$ specifying the polynomial degrees on the elements and h only symbolically refers to a mesh size that may vary. On a quadrilateral Γ_j (p_j, q_j) denotes the degrees in the two local coordinate directions and for a triangle Γ_j, $p_j = q_j$. The spaces $\mathcal{P}_{p_j,q_j}(\Gamma_j)$ are standard polynomial spaces defined by affine transformations from the reference square $\Gamma_Q := (-1, 1)^2$ or the reference triangle $\Gamma_T := \{(x_1, x_2);\ 0 < x_1 < 1, 0 < x_2 < 1 - x_1\}$ onto the actual element:

$$\mathcal{P}_{p_j,q_j}(\Gamma_Q) := \mathrm{span}\{x_1^k x_2^l;\ k = 0, \ldots, p_j,\ l = 0, \ldots, q_j\},$$

$$\mathcal{P}_{p_j,p_j}(\Gamma_T) := \mathrm{span}\{x_1^k x_2^l;\ k + l = 0, \ldots, p_j\}.$$

In the following we consider enriched ansatz spaces $\tilde{H}_N$ and different decompositions thereof to define error indicators and estimators. Proofs of theoretical results are given in Section 4.

3.1 Local enrichment with direction control

In principle we consider locally enriched ansatz spaces of the form

$$\tilde{H}_N = \tilde{S}_{h,p}(\Gamma) = Z_0 + Z_1 + \cdots + Z_J$$

where the spaces Z_j consist of bubble functions which are not present in $S_{h,p}(\Gamma)$. Using such a decomposition with one subspace Z_j for each element this would result in isotropic error indicators. In other words, no criterion would be given to decide in which direction to refine elements or increase polynomial degrees. Since in general the true solution u of (1) exhibit corner, corner-edge and edge singularities (see, e.g., [27, 26]) non-isotropic hp-refinement is important to provide efficient approximations. For three-dimensional problems with edge singularities exponentially fast convergence of the Galerkin method can be realized by using stretched quadrilaterals close to edge singularities (plus appropriate polynomial degrees), see [11]. Triangles are not considered in [11] but are convenient to match quadrilaterals with different orientations (e.g., to discretize general polyhedral surfaces) and to realize local mesh refinements without using hanging nodes. Therefore, on quadrilaterals we split the spaces of additional bubble functions to define error indicators for both local directions. On triangles we do not decompose the space of bubble functions and just take isotropic indicators, i.e. one indicator per element. To ensure existence of bubble functions on triangles the polynomial degree has to be at least three. On quadrilaterals one has one bubble function if the polynomial degree is two. To be able to split the spaces Z_j on quadrilaterals we therefore need increased degrees (p'_j, q'_j) where one of the degrees p'_j or q'_j is at least three and the other at least two. In case of $(p_j, q_j) = (1, 1)$ we choose new degrees $(p'_j, q'_j) = (3, 3)$ since no direction is distinguished and $(p'_j, q'_j) = (2, 2)$ would not be enough.

More precisely, on a triangle Γ_j we take

$$Z_j := \operatorname{span}\left\{\mathcal{P}_{p'_j,p'_j}(\Gamma_j) \setminus \mathcal{P}_{p_j,p_j}(\Gamma_j)\right\} \cap H^1_0(\Gamma_j), \quad p'_j := \max\{p_j+1, 3\}$$

and on a quadrilateral Γ_j we define

$$Z_{j1} := \begin{cases} \operatorname{span}\left\{\mathcal{P}_{p_j+1,q_j}(\Gamma_j) \setminus \mathcal{P}_{p_j,q_j}(\Gamma_j)\right\} \cap H^1_0(\Gamma_j) & \text{if } p_j > 1, q_j > 1 \quad \text{(a)} \\ \operatorname{span}\left\{\mathcal{P}_{p_j+1,2}(\Gamma_j) \setminus \mathcal{P}_{p_j,2}(\Gamma_j)\right\} \cap H^1_0(\Gamma_j) & \text{if } p_j > 1, q_j = 1 \quad \text{(b)} \\ \mathcal{P}_{2,q_j}(\Gamma_j) \cap H^1_0(\Gamma_j) & \text{if } p_j = 1, q_j > 1 \quad \text{(c)} \\ \mathcal{P}_{3,2}(\Gamma_j) \cap H^1_0(\Gamma_j) & \text{if } p_j = 1, q_j = 1 \quad \text{(d)} \end{cases}$$

which is the local enrichment on Γ_j in x_1-direction and

$$Z_{j2} := \begin{cases} \operatorname{span}\left\{\mathcal{P}_{p_j,q_j+1}(\Gamma_j) \setminus \mathcal{P}_{p_j,q_j}(\Gamma_j)\right\} \cap H^1_0(\Gamma_j) & \text{if } p_j > 1, q_j > 1 \quad \text{(a)} \\ \mathcal{P}_{p_j,2}(\Gamma_j) \cap H^1_0(\Gamma_j) & \text{if } p_j > 1, q_j = 1 \quad \text{(b)} \\ \operatorname{span}\left\{\mathcal{P}_{2,q_j+1}(\Gamma_j) \setminus \mathcal{P}_{2,q_j}(\Gamma_j)\right\} \cap H^1_0(\Gamma_j) & \text{if } p_j = 1, q_j > 1 \quad \text{(c)} \\ \mathcal{P}_{2,3}(\Gamma_j) \cap H^1_0(\Gamma_j) & \text{if } p_j = 1, q_j = 1 \quad \text{(d)} \end{cases}$$

is the local enrichment on Γ_j in x_2-direction. As basis functions for the subspaces Z_{j1}, Z_{j2} we take affine mappings of the tensor products

$$\psi_p(x_1)\psi_q(x_2) \quad \text{with} \quad \psi_p(x) := \int_{-1}^{x} L_{p-1}(t)\,dt \quad (p, q \geq 2)$$

defined on $(-1,1)^2$ where L_{p-1} is the Legendre polynomials of degree $p-1$.

In Figure 1 a sketch of the possible decompositions and corresponding p-increases is given. The shaded regions indicate pairs of polynomial degrees for the individual spaces. Only degrees larger than one are represented, i.e. only bubble functions are referred to. Case (a) is the general case if the local polynomial degrees (denoted by p and q) are bigger than one. We increase the degrees by one in each direction and take the local subspaces Z_{j1} and Z_{j2} as indicated. Cases (b), (c) and (d) deal with the situation that p or q is just one. These situations are not covered by (a) and we need a special definition of the p-enrichment to obtain subspaces that measure the error in different directions. Case (c) which is analogous to the case (b) when exchanging p and q is omitted. Only in the case (a) (where $p \geq 2$ and $q \geq 2$) bubble functions of the previous ansatz space are present. This is the diagonally shaded region. Note also that (d) is the only situation when the decomposition of the subspace Z_j is not direct.

Using the additional subspaces defined above we obtain an enriched ansatz space with two-level decomposition of the form

$$\tilde{S}_{h,p}(\Gamma) = T(S_{h,p}(\Gamma)) + Z_{11} + Z_{12} + \cdots + Z_{J1} + J_{J2}. \tag{6}$$

The decomposition without direction control is given by

$$\tilde{S}_{h,p}(\Gamma) = T(S_{h,p}(\Gamma)) + Z_1 + \cdots + Z_J \tag{7}$$

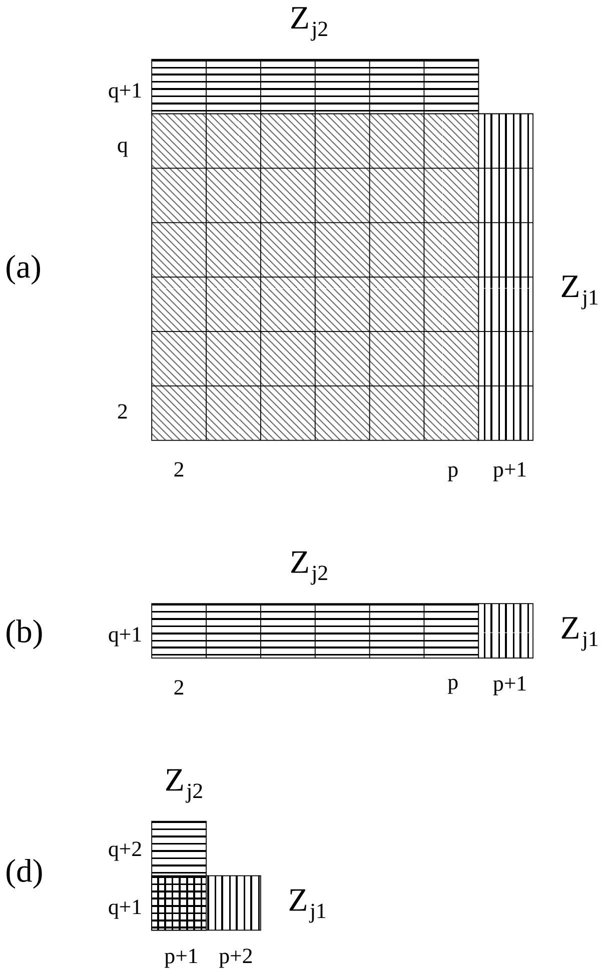

Figure 1: Local p-increase and decompositions for direction control.

with $Z_j := \mathrm{span}\{Z_{j1} \cup Z_{j2}\}$, $j = 1, \ldots, J$. On triangles the notations "$Z_{j1} + Z_{j2}$" and "$Z_{j1} \cup Z_{j2}$" are only symbolic. Here we always deal with a single space Z_j. As described in Section 2, the notation $T(S_{h,p}(\Gamma))$ refers to a basis transformation. Performing a complete Schur complement step with transformation T_{A_2} is expensive. We propose to perform a partial Schur complement step with transformation T_{C_2}. Instead of inverting the matrix A_2 that belongs to the second level space Z, we only invert the matrix C_2 that is the block-diagonal part of A_2. The blocks belong to individual elements, i.e. to the subspaces $Z_1, \ldots, Z_J$.

The following theorem prove reliability and efficiency of the error estimators based upon the above decompositions. The bounds depend only polylogarithmically on the maximum polynomial degree and on the saturation parameter. Asymptotically all the estimators with full or partial orthogonalization are equivalent.

Theorem 1 *Assume that the mesh $\mathcal{T}_h$ is locally quasi-uniform, i.e. the angles formed by adjacent sides are uniformly bounded from below by a positive number and the ratio of largest side length and smallest side length on each element is bounded from above by a global positive constant. Then there exist constants c_1, $c_2 > 0$ which are independent of h and p such that for the error estimator Θ based upon one of the decompositions* (6) *or* (7) *with $T = T_{A_2}$ or $T = T_{C_2}$ there holds*

$$c_1\Theta^2 \le \|u - u_N\|_D^2 \le \frac{c_2}{1-\sigma^2}(1 + \log p_{\max})^2\Theta^2.$$

Here, $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the true solution u of (1) *and $p_{\max} = \max\{p_1, q_1, \ldots, p_J, q_J\}$.*

From the above result it is clear that one expects reduced reliability of the error estimator when σ approaches one. This very likely will be the case for higher polynomial degrees. Our numerical results show that the indicators still lead to efficient adaptions in such cases.

To our knowledge there are no theoretical results available that prove saturation for hp-methods with non-uniform meshes or non-uniform degree distributions. Based on the algebraic convergence $O(h^\alpha p^{-2\alpha})$ (for an $\alpha > 0$) of quasi-uniform hp-methods for problems with singularities, the saturation assumption can be likely satisfied by the steps $h \to h/2$ and $p \to 2p$. However, a step $p \to 2p$ is very expensive for degrees larger than, e.g., $p = 3$. Moreover, by just adding bubble functions we do not span standard hp-ansatz spaces since additional edge basis functions are omitted.

To get an effective error estimator also for spaces with higher polynomial degrees one may split the strategy as follows. We consider different spaces $\tilde{H}_N$ and corresponding decompositions for error indicators and the error estimator. The error indicators are calculated and used to steer the adaption as before. But for an error estimator we define $\tilde{H}_N$ by mesh refinement $h \to h/2$. To avoid calculating the projection of the error $u - u_N$ onto $\tilde{H}_N$ we also perform a decomposition and calculate local projections instead. These local projections are not used as indicators and, therefore, we consider overlapping decompositions for this step. Such decompositions are known to be uniformly stable for hp-methods with quasi-uniform rectangular meshes and uniform polynomial degrees, see [20, 10]. Numerical experiments indicate that this is true for more general spaces and triangular meshes as well, see [22]. The error estimator based on mesh halving and overlapping decomposition is considered next.

3.2 Mesh refinement and overlapping decomposition

We define an enriched ansatz space by halving the mesh size, $\tilde{S}_{h,p}(\Gamma) := S_{h/2,q}(\Gamma)$. To compute local error indicators we decompose the enriched space in an overlapping manner like

$$\tilde{S}_{h,p}(\Gamma) := S_{h/2,q}(\Gamma) = S_{h/2,1}(\Gamma) \cup S_{h/2,q_1}(\Gamma'_1) \cup \cdots \cup S_{h/2,q_J}(\Gamma'_J). \tag{8}$$

Here, $S_{h/2,1}(\Gamma)$ is the space of piecewise linear/bilinear functions and J denotes the number of nodes of the mesh with halved size that are not on the boundary of Γ. The space $S_{h/2,q_j}(\Gamma'_j)$ ($j = 1, \ldots, J$) is a local subspace of piecewise polynomials on the union Γ'_j of elements that are adjacent to the node with number j. One needs to define the polynomial degrees q of the space $S_{h/2,q}(\Gamma)$. To prove a theoretical result we consider uniform polynomial degrees and therefore, take the maximum degree $p_{\max}$ from the previous space everywhere. Moreover, we assume quasi-uniform rectangular meshes.

Theorem 2 *Assume that the ansatz space $S_{h,p}(\Gamma)$ is given by a quasi-uniform rectangular mesh $\mathcal{T}_h$. Then, for the error estimator based upon the decomposition* (8) *with $q = p_{\max}$ uniformly, there holds*

$$c_1\Theta^2 \leq \|u - u_N\|_D^2 \leq \frac{c_2}{1-\sigma^2}\Theta^2.$$

Here, c_1 and c_2 are positive constants which are independent of h and p, σ is the saturation parameter and $u_N \in S_{h,p}(\Gamma)$ is the Galerkin approximation of the true solution u of (1)*.*

Proof. By Lemma 1 we only have to prove the stability of decomposition (8). For a uniform degree distribution $q = p_{\max}$ as considered here uniform stability has been proved by [10, Lemma 2]. □

In practice it is expensive to use the maximum polynomial degree everywhere on the refined mesh. Instead, for computations, we keep the dimension of the enriched space relatively small by inheriting polynomial degrees from the father elements. The degrees for the side functions then are defined as the minimum degrees of the degrees in side directions of the neighboring elements. This is also the standard procedure to define the degrees for the space $S_{h,p}(\Gamma)$. However, stability of an overlapping subspace decomposition for the non-uniform p-version has not been proved yet. Some results with local p-refinements for the FEM are given in [21]. Theoretical analysis of the general situation for both the FEM and the BEM is an open problem. Numerical studies of overlapping decompositions for the FEM with uniform polynomial degrees on general meshes in two and three dimensions are reported in [22].

4 Technical results and proof of the main theorem

To prove Theorem 1 we apply the general Lemma 1. Therefore, we only need to study stability of the appearing decompositions. First we give two preliminary results dealing with the partial orthogonalization and with the decomposition of local subspaces with respect to local coordinate directions as described in Section 3.1.

Lemma 2 *Assume that the decomposition of the second level Z is stable: there exist positive numbers c_4, c_5 such that*

$$c_4 \sum_{j=1}^{J} \|\phi_j\|_D^2 \leq \|\phi_1'\|_D^2 \leq c_5 \sum_{j=1}^{J} \|\phi_j\|_D^2 \tag{9}$$

for all $\phi_1' = \sum_{j=1}^{J} \phi_j \in Z$ with $\phi_j \in Z_j$, $j = 1, \ldots, J$. Without loss of generality we assume that $c_4 \leq 1 \leq c_5$. The direct decomposition of the second level defines a matrix C_2 that consists of a *block-diagonal part of A_2. This defines the transformation T_{C_2} (4) and also a transformed first level subspace $T_{C_2}(Z_0)$.*

The obtained decomposition (5) *is stable, i.e. there holds* (A1)*: there exist two numbers c_1, $c_2 > 0$ such that*

$$c_1 \sum_{j=0}^{J} a(\phi_j, \phi_j) \leq a(\phi, \phi) \leq c_2 \sum_{j=0}^{J} a(\phi_j, \phi_j)$$

for any $\phi = \sum_{j=0}^{J} \phi_j \in \tilde{H}_N$ with $\phi_0 \in T_{C_2}(Z_0)$ and $\phi_j \in Z_j$ $(j > 0)$. The number c_2 is bounded from above by $2c_5$ and c_1 is bounded from below by $1/O(c_5^2/c_4)$.

Proof. Let $\phi = \phi_0 + \phi_1' \in \tilde{H}_N$ be given with $\phi_0 \in T_{C_2}(Z_0)$ and $\phi_1' \in Z = Z_1 \cup \cdots \cup Z_J$. We also represent $\phi_1' = \sum_{j=1}^{J} \phi_j$ with $\phi_j \in Z_j$, $j = 1, \ldots, J$. Using the triangle inequality and the right estimate in (9) we obtain

$$\|\phi\|_D^2 \leq 2(\|\phi_0\|_D^2 + \|\phi_1'\|_D^2) \leq 2\|\phi_0\|_D^2 + 2c_5 \sum_{j=1}^{J} \|\phi_j\|_D^2$$

which proves the right estimate in (A1) with $c_2 \leq \max\{2, 2c_5\} = 2c_5$. To prove the left inequality in (A1) we only have to bound the number c in

$$\|\phi_0\|_D^2 + \|\phi_1'\|_D^2 \leq c\|\phi\|_D^2. \tag{10}$$

The assertion then follows by making use of the left inequality in (9).

For the chosen bases in $T_{C_2}(Z_0)$ and Z let x and y denote the vectors of coefficients that represent ϕ_0 and ϕ_1', respectively. Then (10) is equivalent to

$$\begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} \tilde{S} & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq c \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} \tilde{S} & \tilde{B}^T \\ \tilde{B} & A_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for all vectors x, y of appropriate dimensions. We bound the number c by bounding the largest eigenvalue of the generalized eigenvalue problem

$$\begin{aligned} \tilde{S}x &= \lambda(\tilde{S}x + \tilde{B}^T y), \\ A_2 y &= \lambda(\tilde{B}x + A_2 y). \end{aligned}$$

Let us assume without loss of generality that $\lambda > 1$ (otherwise c is bounded by $\lambda \leq 1$ and the proof is finished). Then, solving the second equation above for y and inserting this solution into the first equation, we obtain

$$\tilde{S}x = \lambda\tilde{S}x + \frac{\lambda^2}{1-\lambda}\tilde{B}^T A_2^{-1}\tilde{B}x.$$

We now assume that $\lambda > 1$ is large enough such that this equation is a small perturbation of the generalized eigenvalue problem

$$\tilde{S}x = \lambda\tilde{S}x - \lambda\tilde{B}^T A_2^{-1}\tilde{B}x \tag{11}$$

(otherwise λ is bounded from above). We bound the maximum eigenvalue of this problem which gives an asymptotic upper bound for the number c in (10).

The matrix $\tilde{S} - \tilde{B}^T A_2^{-1} \tilde{B}$ is the Schur complement of the transformed stiffness matrix $\tilde{A}$ which is obtained by eliminating the unknowns of the second level. This Schur complement is unique and therefore

$$\tilde{S} - \tilde{B}^T A_2^{-1} \tilde{B} = A_1 - B^T A_2^{-1} B = S$$

is the usual Schur complement. Hence, it is left to bound the eigenvalues of the generalized eigenvalue problem

$$\tilde{S}x = \lambda S x. \tag{12}$$

By definition of the Schur complement there holds

$$x^T S x = \begin{pmatrix} x \\ y \end{pmatrix}^T A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for} \quad y = -A_2^{-1} B x.$$

Accordingly, we find

$$x^T \tilde{S} x = \begin{pmatrix} x \\ y \end{pmatrix}^T A \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{for} \quad y = -C_2^{-1} B x.$$

We use the notations

$$\mathcal{A}_2^{-1} \mathcal{B} x := \begin{pmatrix} I & 0 \\ 0 & -A_2^{-1} \end{pmatrix} \begin{pmatrix} I \\ B \end{pmatrix} x = \begin{pmatrix} x \\ -A_2^{-1} B x \end{pmatrix}$$

and similarly

$$\mathcal{C}_2^{-1} \mathcal{B} x := \begin{pmatrix} x \\ -C_2^{-1} B x \end{pmatrix}.$$

Then, the eigenvalues of the problem (12) can be bounded as follows:

$$\begin{aligned}
\lambda &\leq \sup_{x \neq 0} \frac{x^T \tilde{S} x}{x^T S x} = \sup_{x \neq 0} \frac{(\mathcal{C}_2^{-1} \mathcal{B} x)^T A \mathcal{C}_2^{-1} \mathcal{B} x}{(\mathcal{A}_2^{-1} \mathcal{B} x)^T A \mathcal{A}_2^{-1} \mathcal{B} x} = \sup_{x \neq 0} \frac{(\mathcal{B} x)^T \mathcal{C}_2^{-T} A \mathcal{C}_2^{-1} \mathcal{B} x}{(\mathcal{B} x)^T \mathcal{A}_2^{-T} A \mathcal{A}_2^{-1} \mathcal{B} x} \\
&\leq \sup_{x \neq 0} \frac{x^T \mathcal{C}_2^{-T} A \mathcal{C}_2^{-1} x}{x^T \mathcal{A}_2^{-T} A \mathcal{A}_2^{-1} x} = \lambda_{\max}(A^{-1/2} \mathcal{A}_2^T \mathcal{C}_2^{-T} A \mathcal{C}_2^{-1} \mathcal{A}_2 A^{-1/2}) \\
&= \lambda_{\max}(A^{-1} \mathcal{A}_2^T \mathcal{C}_2^{-T} A \mathcal{C}_2^{-1} \mathcal{A}_2) \leq \lambda_{\max}(A^{-1} \mathcal{A}_2^T \mathcal{C}_2^{-T} A) \lambda_{\max}(\mathcal{C}_2^{-1} \mathcal{A}_2) \\
&= \lambda_{\max}(\mathcal{C}_2^{-1} \mathcal{A}_2)^2.
\end{aligned}$$

Here, the vector x is of generic dimension and $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue of a matrix. Using the assumption (9) we conclude that $\lambda_{\max}(\mathcal{C}_2^{-1} \mathcal{A}_2)$ is bounded by c_5 and, therefore, the number c in (10) is bounded like $O(c_5^2)$. Again making use of assumption (9) we obtain

$$\|\phi_0\|_D^2 + \sum_{j=1}^{J} \|\phi_j\|_D^2 \leq \|\phi_0\|_D^2 + \frac{1}{c_4} \|\phi_1'\|_D^2 \leq \frac{O(c_5^2)}{c_4} \|\phi\|_D^2.$$

Therefore, c_1 in (A1) is asymptotically bounded from below by c_4/c_5^2. This finishes the proof of the lemma. $\square$

Now we prove stability of the decomposition of local subspaces with respect to the directions, cf. Figure 1. This is done on the reference element.

Lemma 3 *On the reference element $(-1,1)^2$ the decomposition with respect to the directions $Z = Z_1 + Z_2$ with*

$$
\begin{array}{lll}
\text{(a)} & \begin{array}{l} Z_1 = \operatorname{span}\{\psi_{p+1}(x_1)\psi_j(x_2);\ j = 2,\dots,q\} \\ Z_2 = \operatorname{span}\{\psi_j(x_1)\psi_{q+1}(x_2);\ j = 2,\dots,p\} \end{array} & \textit{if } p > 1, q > 1, \\[2ex]
\text{(b)} & \begin{array}{l} Z_1 = \operatorname{span}\{\psi_{p+1}(x_1)\psi_2(x_2)\} \\ Z_2 = \operatorname{span}\{\psi_j(x_1)\psi_2(x_2);\ j = 2,\dots,p\} \end{array} & \textit{if } p > 1, q = 1, \\[2ex]
\text{(c)} & \begin{array}{l} Z_1 = \operatorname{span}\{\psi_2(x_1)\psi_j(x_2);\ j = 2,\dots,q\} \\ Z_2 = \operatorname{span}\{\psi_2(x_1)\psi_{q+1}(x_2)\} \end{array} & \textit{if } p = 1, q > 1, \\[2ex]
\text{(d)} & \begin{array}{l} Z_1 = \operatorname{span}\{\psi_2(x_1)\psi_2(x_2), \psi_3(x_1)\psi_2(x_2)\} \\ Z_2 = \operatorname{span}\{\psi_2(x_1)\psi_2(x_2), \psi_2(x_1)\psi_3(x_2)\} \end{array} & \textit{if } p = 1, q = 1
\end{array}
$$

is uniformly stable in $\tilde{H}^{1/2}((-1,1)^2)$ for $p, q \geq 1$.

Proof. The decompositions (a), (b), (c) are all direct. Only in the case (d) the two subspaces Z_1 and Z_2 overlap. However, (d) occurs only once for all $p, q \geq 1$ and nothing has to be proved for this. We only prove stability for (a). The cases (b) and (c) are analogous.

For given $\phi = \phi_1 + \phi_2 \in Z$ with $\phi_1 \in Z_1$ and $\phi_2 \in Z_2$ (which are unique) there holds

$$\|\phi\|^2_{\tilde{H}^{1/2}((-1,1)^2)} \leq 2(\|\phi_1\|^2_{\tilde{H}^{1/2}((-1,1)^2)} + \|\phi_2\|^2_{\tilde{H}^{1/2}((-1,1)^2)})$$

by the triangle inequality. It therefore remains to prove the opposite estimate. This is done by proving the inequality within $L^2((-1,1)^2)$ and $H_0^1((-1,1)^2)$. The assertion then follows by interpolation.

First let us prove the estimate within $H_0^1((-1,1)^2)$:

$$|\phi_1|^2_{H^1((-1,1)^2)} + |\phi_2|^2_{H^1((-1,1)^2)} \leq c|\phi|^2_{H^1((-1,1)^2)}.$$

There holds

$$
\begin{aligned}
\frac{|\phi_1|^2_{H^1} + |\phi_2|^2_{H^1}}{|\phi|^2_{H^1}} &= \frac{\|\frac{\partial}{\partial x_1}\phi_1\|^2_{L^2} + \|\frac{\partial}{\partial x_2}\phi_1\|^2_{L^2} + \|\frac{\partial}{\partial x_1}\phi_2\|^2_{L^2} + \|\frac{\partial}{\partial x_2}\phi_2\|^2_{L^2}}{\|\frac{\partial}{\partial x_1}\phi\|^2_{L^2} + \|\frac{\partial}{\partial x_2}\phi\|^2_{L^2}} \\
&\leq \frac{\|\frac{\partial}{\partial x_1}\phi_1\|^2_{L^2} + \|\frac{\partial}{\partial x_1}\phi_2\|^2_{L^2}}{\|\frac{\partial}{\partial x_1}\phi\|^2_{L^2}} + \frac{\|\frac{\partial}{\partial x_2}\phi_1\|^2_{L^2} + \|\frac{\partial}{\partial x_2}\phi_2\|^2_{L^2}}{\|\frac{\partial}{\partial x_2}\phi\|^2_{L^2}}.
\end{aligned}
$$

It suffices to bound one of the terms on the right hand side and we choose the first one. In order to due so we represent

$$\begin{aligned}\phi(x_1, x_2) &= \sum_{j=2}^{q} u_j \psi_{p+1}(x_1)\psi_j(x_2) + \sum_{i=2}^{p} v_i \psi_i(x_1)\psi_{q+1}(x_2) \\ &= \phi_1(x_1, x_2) + \phi_2(x_1, x_2)\end{aligned}$$

and obtain

$$\frac{\partial}{\partial x_1}\phi(x_1, x_2) = \sum_{j=2}^{q} u_j L_p(x_1)\psi_j(x_2) + \sum_{i=2}^{p} v_i L_{i-1}(x_1)\psi_{q+1}(x_2).$$

Obviously every pair $L_p(x_1)\psi_j(x_2)$, $L_{i-1}(x_1)\psi_{q+1}(x_2)$ $(i-1 = 1, \ldots, p-1,\ j = 2, \ldots, q)$ is orthogonal in $L^2((-1,1)^2)$ and one directly deduces that

$$\|\frac{\partial}{\partial x_1}\phi\|_{L^2}^2 = \|\frac{\partial}{\partial x_1}\phi_1\|_{L^2}^2 + \|\frac{\partial}{\partial x_1}\phi_2\|_{L^2}^2.$$

It is left to prove

$$\|\phi_1\|_{L^2((-1,1)^2)}^2 + \|\phi_2\|_{L^2((-1,1)^2)}^2 \le c\|\phi\|_{L^2((-1,1)^2)}^2. \tag{13}$$

We normalize the basis functions by defining $\psi_p^* := \psi_p/\|\psi_p\|_{L^2(-1,1)}$ and represent

$$\begin{aligned}\phi(x_1, x_2) &= \sum_{j=2}^{q} u_j \psi_{p+1}^*(x_1)\psi_j^*(x_2) + \sum_{i=2}^{p} v_i \psi_i^*(x_1)\psi_{q+1}^*(x_2) \\ &= \phi_1(x_1, x_2) + \phi_2(x_1, x_2).\end{aligned}$$

Numbering the basis functions in the order given by this representation we obtain

$$\|\phi\|_{L^2}^2 = \begin{pmatrix} u \\ v \end{pmatrix}^T \begin{pmatrix} M_{11} & M_{12}^T \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\|\phi_1\|_{L^2}^2 = u^T M_{11} u, \quad \|\phi_2\|_{L^2}^2 = v^T M_{22} v$$

with $u = (u_2, \ldots, u_q)^T$, $v = (v_2, \ldots, v_p)^T$ and

$$M_{11} = (\langle\psi_{p+1}^*, \psi_{p+1}^*\rangle\ \langle\psi_i^*, \psi_j^*\rangle)_{i,j=2,\ldots,q} = (\langle\psi_i^*, \psi_j^*\rangle)_{i,j=2,\ldots,q} \in \mathbf{R}^{(q-1)\times(q-1)},$$

$$M_{22} = (\langle\psi_i^*, \psi_j^*\rangle\ \langle\psi_{q+1}^*, \psi_{q+1}^*\rangle)_{i,j=2,\ldots,p} = (\langle\psi_i^*, \psi_j^*\rangle)_{i,j=2,\ldots,p} \in \mathbf{R}^{(p-1)\times(p-1)},$$

$$M_{12}^T = (\langle\psi_{p+1}^*, \psi_i^*\rangle\ \langle\psi_j^*, \psi_{q+1}^*\rangle)_{j=2,\ldots,q,\ i=2,\ldots,p} \in \mathbf{R}^{(q-1)\times(p-1)}.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the $L^2(-1,1)$-inner product. It is well-known that ψ_p is a linear combination of L_p and L_{p-2} for $p \ge 2$. Therefore (by using the Kronecker symbol δ_{ij})

$$\langle\psi_{p+1}^*, \psi_i^*\rangle\ \langle\psi_j^*, \psi_{q+1}^*\rangle = \delta_{p-1,i}\langle\psi_{p+1}^*, \psi_{p-1}^*\rangle\ \delta_{q-1,j}\langle\psi_{q+1}^*, \psi_{q-1}^*\rangle$$

and M_{12}^T has the only possible non-zero entry at the position that belongs to the column of $\psi_{p-1}(x_1)\psi_{q+1}(x_2)$ and the row of $\psi_{p+1}(x_1)\psi_{q-1}(x_2)$. The inner products $b_j := \langle \psi_j^*, \psi_{j+2}^* \rangle$ can be explicitly calculated (see [20]):

$$b_j = -\frac{1}{2}\sqrt{\frac{(2j-3)(2j+5)}{(2j-1)(2j+3)}}.$$

To prove (13) we bound the eigenvalues of the generalized eigenvalue problem

$$\begin{aligned} M_{11}u &= \lambda(M_{11}u + M_{12}^T v), \\ M_{22}v &= \lambda(M_{12}u + M_{22}v). \end{aligned}$$

We proceed as in the proof of Lemma 2 by eliminating v

$$M_{11}u = \lambda M_{11}u + \frac{\lambda^2}{1-\lambda} M_{12}^T M_{22}^{-1} M_{12}u,$$

and by considering the perturbed eigenvalue problem (by assuming that λ is large enough)

$$M_{11}u = \lambda(M_{11} - M_{12}^T M_{22}^{-1} M_{12})u.$$

As mentioned above, the matrix M_{12}^T has only one non-zero entry and, therefore,

$$M_{12}^T M_{22}^{-1} M_{12} = cee^T$$

for a number $c > 0$ and the unit vector $e \in \mathbf{R}^N$ that is 1 at the position that belongs to the basis function $\psi_{p+1}(x_1)\psi_{q-1}(x_2)$ and is 0 otherwise. Therefore, we have to bound λ in

$$M_{11}u = \lambda(M_{11} - cee^T)u$$

which is exactly of the form (14) in [20]. From [20, (17)] we know that

$$\lambda \leq \frac{1}{1 - ce^T M_{11}^{-1} e}$$

and, thus, we have to bound the entry of M_{11}^{-1} at the row and column positions that belong to the basis function $\psi_{p+1}(x_1)\psi_{q-1}(x_2)$. Since M_{11} is part of the matrix A_{p+1} (actually A_{q+1} in this paper) which has been studied in [20] the ideas of that paper can be applied also here to prove the boundedness of λ. We omit the details. □

Proof of Theorem 1. We only need to prove stability of the decompositions (6) and (7). The efficiency and reliability estimates for the error estimator then follow by using Lemma 1.

When the decompositions (6) or (7) are based upon a full Schur complement step then there holds

$$a(\phi_0, \phi_0) + a(\phi_1', \phi_1') = a(\phi_0 + \phi_1', \phi_0 + \phi_1')$$

for all functions $\phi_0 + \phi_1' \in \tilde{S}_{h,p}(\Gamma)$ with $\phi_0 \in T_{A_2}(S_{h,p}(\Gamma))$ and $\phi_1' \in Z = Z_1 \cup \cdots \cup Z_J$. In these cases it therefore remains to prove stability of the decompositions of the second level.

In case of partial orthogonalization stability of (6) and (7) is obtained by Lemma 2. Again, we only have to prove stability of the decompositions of the second level. This is done next.

First we consider the decomposition of the second level defined by (7), i.e. with respect to the elements. We claim that there holds

$$c(1+\log p_{\max})^{-2} \sum_{j=1}^{J} a(\phi_j,\phi_j) \le a(\phi_1',\phi_1') \le C \sum_{j=1}^{J} a(\phi_j,\phi_j) \tag{14}$$

for arbitrary $\phi_1' = \sum_{j=1}^{J} \phi_j \in Z$ with $\phi_j \in Z_j$ and constants c, $C > 0$. The right inequality is proved by

$$a(\phi_1',\phi_1') \simeq \| \sum_{j=1}^{J} \phi_j \|^2_{\tilde{H}^{1/2}(\Gamma)} \le C \sum_{j=1}^{J} \|\phi_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \simeq \sum_{j=1}^{J} a(\phi_j,\phi_j),$$

see [25, Lemma 3.2] or [8, Lemma 1].

Now we prove the left inequality. From [8, Lemma 5] we know that there holds

$$\|\phi_j\|_{\tilde{H}^{1/2-\epsilon}(\Gamma_j)} \le \frac{c}{\epsilon} \|\phi_j\|_{H^{1/2-\epsilon}(\Gamma_j)}$$

for a constant $c > 0$ that is independent of $\epsilon \in (0, 1/2)$ but depends on the diameter of Γ_j in general. However, since ϕ_j vanishes at the boundary of Γ_j its nodal interpolation (linear on triangles and bilinear on rectangles) vanishes. Therefore,

$$\|\phi_j\|_{H^{1/2-\epsilon}(\Gamma_j)} \le c|\phi_j|_{H^{1/2-\epsilon}(\Gamma_j)}$$

with a constant that is independent of the diameter of Γ_j if the L^2-contribution to the H^1-norm (that is used to define $H^{1/2-\epsilon}(\Gamma_j)$ via interpolation) is properly scaled (by h^{-2}). Here, the semi-norm $|\cdot|_{H^s(\Gamma_j)}$ is obtained by interpolating between $L^2(\Gamma_j)$ and $H^1(\Gamma_j)$ with semi-norm in $H^1(\Gamma_j)$, or by neglecting the L^2-contribution to the norm in the definition by Sobolev-Slobodeckij, cf. [8]. Now, the $\tilde{H}^{1/2-\epsilon}$-norm and the $H^{1/2-\epsilon}$-semi-norm scale with the same power of the diameter such that, finally,

$$\|\phi_j\|_{\tilde{H}^{1/2-\epsilon}(\Gamma_j)} \le \frac{c}{\epsilon} |\phi_j|_{H^{1/2-\epsilon}(\Gamma_j)}$$

with a constant c that is independent of ϵ, the diameter h_j of Γ_j and the polynomial degree of ϕ_j. Using the inverse property of the basis functions [8, Lemmas 4, 6] (that also holds on triangles), we obtain

$$\|\phi_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le c \frac{p_{j,\max}^{4\epsilon}}{h_j^{2\epsilon}\epsilon^2} |\phi_j|^2_{H^{1/2-\epsilon}(\Gamma_j)} \le c \frac{p_{j,\max}^{4\epsilon}}{\epsilon^2} |\phi_j|^2_{H^{1/2}(\Gamma_j)} \le c(1+\log p_{j,\max})^2 |\phi_j|^2_{H^{1/2}(\Gamma_j)} \tag{15}$$

with $p_{j,\max} := \max\{p_j, q_j\}$ and a constant $C > 0$ that is independent of p_j, q_j and the diameter of Γ_j. For the last step we chose $\epsilon := 1/\log(p_{j,\max}^2)$. Thus, we conclude that there holds

$$\begin{aligned}\sum_{j=1}^J a(\phi_j, \phi_j) \;&\simeq\; \sum_{j=1}^J \|\phi_j\|^2_{\tilde{H}^{1/2}(\Gamma_j)} \le \sum_{j=1}^J C(1 + \log p_{j,\max})^2 |\phi_j|^2_{H^{1/2}(\Gamma_j)} \\ &\le\; C(1 + \log p_{\max})^2 \|\sum_{j=1}^J \phi_j\|^2_{\tilde{H}^{1/2}(\Gamma)} \simeq C(1 + \log p_{\max})^2 a(\phi_1', \phi_1')\end{aligned}$$

which proves the left estimate in (14).

Now the stability of the decomposition of the second level defined by (6), i.e. with respect to elements and directions, follows from the previous observations and Lemma 3. Indeed, this lemma shows that the stability of the local decompositions is uniform such that the decomposition (7) has the same asymptotic properties as the decomposition (6). □

5 Numerical results

We solve the model problem (1) with $\Gamma = (-1/2, 1/2)^2 \times \{0\}$. For smooth right hand side functions f the solution u of (1) in general behaves singularly only at the boundary of Γ. In such a case (and with Γ as chosen above) non-uniform rectangular meshes suffice to obtain an exponential rate of convergence, see [11]. To study adaptively refined meshes that also comprise triangles we choose the right hand side function $f = 10 + 1/(\mathrm{dist}\,(x, (-0.1, -0.1)) \in H^{-\epsilon}(\Gamma)$ ($\epsilon > 0$). Then we expect that u still behaves singularly at the boundary of Γ and that it also exhibits a point singularity at $x = (-0.1, -0.1)$. The constant 10 in f is chosen after experiments such that our algorithm refines from the beginning at all the singularities. (A precise regularity analysis, e.g. within countably normed spaces like in two dimensions [15], is not available.)

For adaptive refinement we use the three-fold algorithm proposed in [12] for problems in two dimensions. On surfaces we add a direction control and modify the algorithm to avoid hanging nodes.

Three-fold algorithm: Define an initial ansatz space $S_{h,p}(\Gamma)$ with initial mesh $\mathcal{T}_h$ and low polynomial degrees. Choose an error tolerance $\epsilon > 0$ and steering parameters δ_1, δ_2, δ_3 with $0 < \delta_2 < \delta_1 < 1$ and $0 < \delta_3 < 1$. Then perform the following steps.

1. Galerkin solution. Compute the Galerkin solution $u_N \in S_{h,p}(\Gamma)$ of (2).

2. Error estimator. Calculate the terms $\theta_j = \|P_j(\tilde{u}_n - u_N)\|_D$ $(j \ge 0)$ and the estimator Θ, based on one of the decompositions (6) with partial orthogonalization or (8).

Stop if $\Theta \le \epsilon$.

3. Adaption steps.

(i) Indicators. (If necessary) compute the error indicators $\theta_{ji} = \|P_{ji}(\tilde{u}_n - u_N)\|_D$ ($j \geq 1$, $i = 1, 2$) based on the decomposition (6) with partial orthogonalization and determine $\theta_{\max} = \max_j (\theta_{j1}^2 + \theta_{j2}^2)^{0.5}$.

(ii) Classification of elements. Classify quadrilaterals Γ_j as follows (in pseudo Fortran90 language, directions are understood with respect to local coordinates):

```
if (θ_j > δ_1 θ_max) then    ! h-refinement
  if (θ_j1 < δ_3 θ_j2) then
     classify Γ_j for horizontal intersection
  elseif (θ_j2 < δ_3 θ_j1) then
     classify Γ_j for vertical intersection
  else
     classify Γ_j for intersections in both directions
  endif
elseif (θ_j > δ_2 θ_max) then    ! p-increase
  if (θ_j1 < δ_3 θ_j2) then
     classify Γ_j for increase of polynomial degree in vertical direction
  elseif (θ_j2 < δ_3 θ_j1) then
     classify Γ_j for increase of polynomial degree in horizontal direction
  else
     classify Γ_j for increase of polynomial degrees in both directions
  endif
endif
```

Triangles are classified without direction control, i.e. they are refined by halving all their edges if $\theta_j > \delta_1\theta_{max}$ and the polynomial degree is increased if $\delta_2\theta_{max} < \theta_j \leq \delta_1\theta_{max}$.

(iii) Adaption.

(a) Go through all the elements and refine as classified.

(b) Go through all the elements and refine as necessary to remove hanging nodes.

(c) Go through all the elements and increase polynomial degrees as classified if the corresponding element has not been refined in (b).

goto 1.

Remark 1 *When an element is refined then polynomial degrees for the new elements need to be given. To avoid high polynomial degrees on small elements we inherit the degrees reduced by one for the refinement direction whenever possible (i.e. when the degree is larger than one). Further we modify the refinements by bounding internal angles from below. Of course, there are more sophisticated remeshing algorithms. We mainly have in mind to test error indicators by allowing a certain hp-refinement.*

Remark 2 *An adaptive h-version can be realized by choosing $\delta_2 \geq \delta_1$. Pure p-adaptivity occurs when choosing $\delta_1 > 1$. Isotropic adaption (no direction control) can be performed by taking $\delta_3 = 0$ (this case is not reported here).*

Of course, with the above algorithm an exponential rate of convergence is not guaranteed. A regularity based decision for h-refinement or p-increase seems more appropriate in this context, see [1, 2] for hp-FEM.

In the following experiments with adaptivity we always classify elements for refinement or p-increase if $\theta_j > 0.1\,\theta_{\max}$. That means we take $\delta_1 = 0.1$ for the adaptive h-version and $\delta_2 = 0.1$ for the adaptive p-version. The adaptive hp-version is realized by $\delta_1 = 0.5$ and $\delta_2 = 0.1$. These choices are to make the individual versions comparable. However, we do not claim that this strategy (h- or p-adaption if $\theta_j > 0.1\,\theta_{\max}$) is optimal for any version.

In the following we first report on numerical results that underline that the proposed error indicators are appropriate for hp-adaptivity. Then we consider the effectivity of our error estimators.

For the integration of the stiffness entries of the Galerkin matrix we use subroutines of the maiprogs-library, see [16]. In order to compute the energy norm of the errors we use the relation $\|u - u_N\|_D^2 = \|u\|_D^2 - \|u_N\|_D^2$ and replace $\|u\|_D^2$ by an extrapolated value based on results of the uniform h-version.

Figure 2 presents the errors in energy norm for different versions of the boundary element method. These are the uniform and adaptive h-version, the uniform and adaptive p-version and the adaptive hp-version. Note that it is not obvious for our example how to design hp-ansatz spaces that guarantee optimal convergence. For our example the adaptive versions give better approximations than the uniform counterparts, and the adaptive hp-version is superior to all others. Corresponding to these results Figures 3, 4, 5 show the meshes and polynomial degrees after five steps of h-adaption, p-adaption and hp-adaption, respectively. The notation "pq" on an element means polynomial degrees p in x_1-direction and q in x_2-direction. The "." in the pairs $.q$ and $p.$ replaces 1. A missing pair represents lowest degrees, i.e. $p = q = 1$. All three adaptive methods adapt the ansatz spaces towards the singularities at the boundary of Γ and at $(-0.1, -0.1)$. Figure 6 shows (relatively crude approximations of) isolines of the piecewise constant field of error indicators $(\theta_{j1}^2 + \theta_{j2}^2)^{0.5}$ computed on a uniform mesh of 64 elements with lowest degree. The isolines indicate the singularities where expected for the given right hand side function.

Now we take a closer look at the error estimators. Tables 1–5 list the effectivity indices $\Theta/\|u - u_N\|_D$ and the saturation parameters σ for all versions under consideration. Results for both cases, Θ based on local enrichments by bubble functions with partial orthogonalization and Θ based on mesh refinement, are given. As expected, the effectivity deteriorates for increasing polynomial degrees when the estimator is based on local enrichments. However, effectivity is close to one for low polynomial degrees. This is even true for non-uniform meshes, cf. Table 2. The behavior of the effectivity index corresponds to that of σ which approaches one for increasing polynomial degrees and, therefore, confirms the assertion of Theorem 1. On the other hand, the results given in Tables 1–5 demonstrate better effectivity of the error estimator based on mesh refinement. But the first estimator is relatively cheap and therefore is attractive for low polynomial degrees.

The influence of the partial orthogonalization (for the method based on local enrichments) is studied in Figure 7 (uniform p-version) and Figure 8 (adaptive hp-version). When no orthogonalization is performed then the global contribution θ_0 to the error estimator vanishes and Θ consists only of local error indicators. The figures show that the global contribution is quite small for small number of unknowns (as considered here). However, for increasing number of unknowns this term becomes more important since it appears to decrease at a lower rate than the error, at least for the *hp*-version (within this range of unknowns). Figures 7 and 8 also underline that the global contributions θ_0 obtained by full and partial orthogonalizations do not differ much. This is expected due to Lemma 2 which proves asymptotically identical stability parameters of decomposition (6) for full and partial orthogonalization. Indeed, the parameter c_5 in (9) is bounded in our case, cf. the proof of Theorem 1.

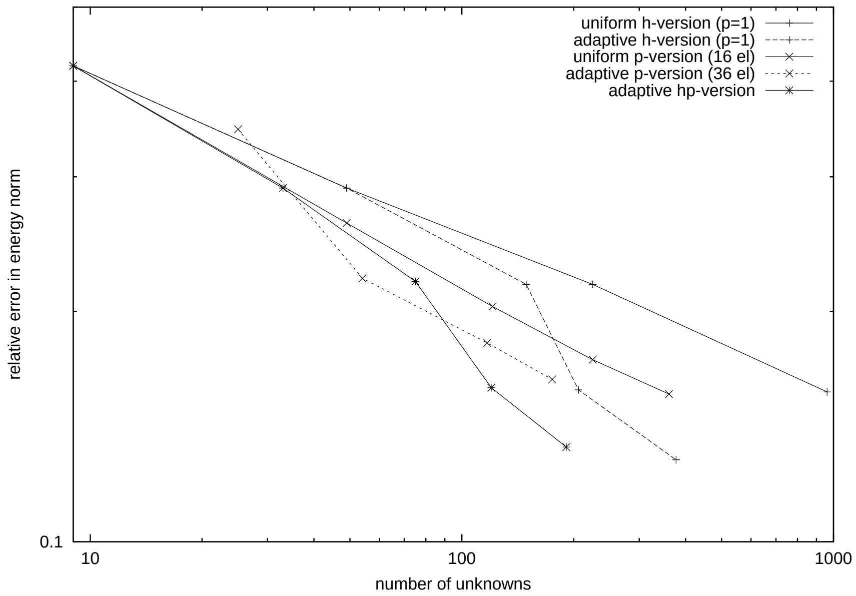

Figure 2: Relative errors in energy norm for different versions.

		local enrichment		mesh halving	
N	$p_{\max}$	Θ/error	σ	Θ/error	σ
1	1	1.02	0.61	1.00	0.73
9	1	0.93	0.67	1.02	0.69
49	1	0.88	0.70	0.99	0.75
225	1	0.84	0.74	0.96	0.72

Table 1: Effectivity indices and saturation parameters (uniform h-version).

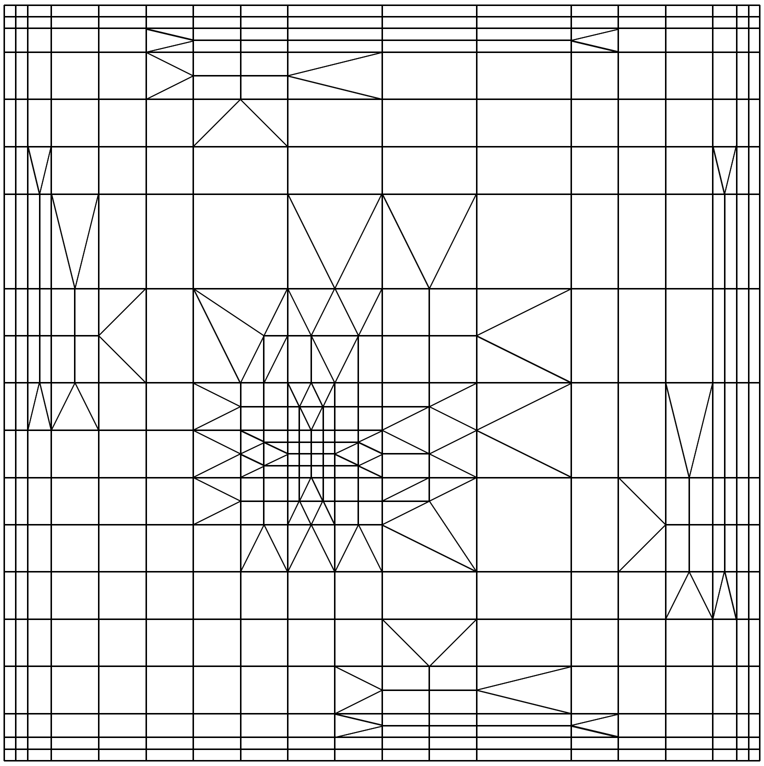

Figure 3: Mesh after five h-refinements.

N	$p_{\max}$	local enrichment Θ/error	local enrichment σ	mesh halving Θ/error	mesh halving σ
1	1	1.02	0.61	1.00	0.73
9	1	0.93	0.67	1.02	0.69
49	1	0.88	0.70	0.99	0.75
149	1	0.88	0.72	0.99	0.72
206	1	0.87	0.76	0.97	0.80

Table 2: Effectivity indices and saturation parameters (adaptive h-version).

66	.6	.6	.6	.6	66
6.	32	22	22	33	6.
6.	32	22	22	22	6.
6.	42	66	22	22	6.
6.	33	33	22	33	6.
66	.6	.6	.6	.6	66

Figure 4: Polynomial degrees after five p-adaptive steps.

		local enrichment		mesh halving	
N	$p_{\max}$	Θ/error	σ	Θ/error	σ
9	1	0.93	0.67	1.02	0.69
49	2	0.55	0.83	0.89	0.70
121	3	0.48	0.88	0.92	0.68
225	4	0.42	0.91	0.91	0.67

Table 3: Effectivity indices and saturation parameters (uniform p-version, 16 elements).

		local enrichment		mesh halving	
N	$p_{\max}$	Θ/error	σ	Θ/error	σ
9	1	0.93	0.67	1.02	0.69
33	2	0.54	0.84	0.86	0.70
67	3	0.46	0.88	0.87	0.68
110	4	0.41	0.91	0.90	0.66

Table 4: Effectivity indices and saturation parameters (adaptive p-version, 16 elements).

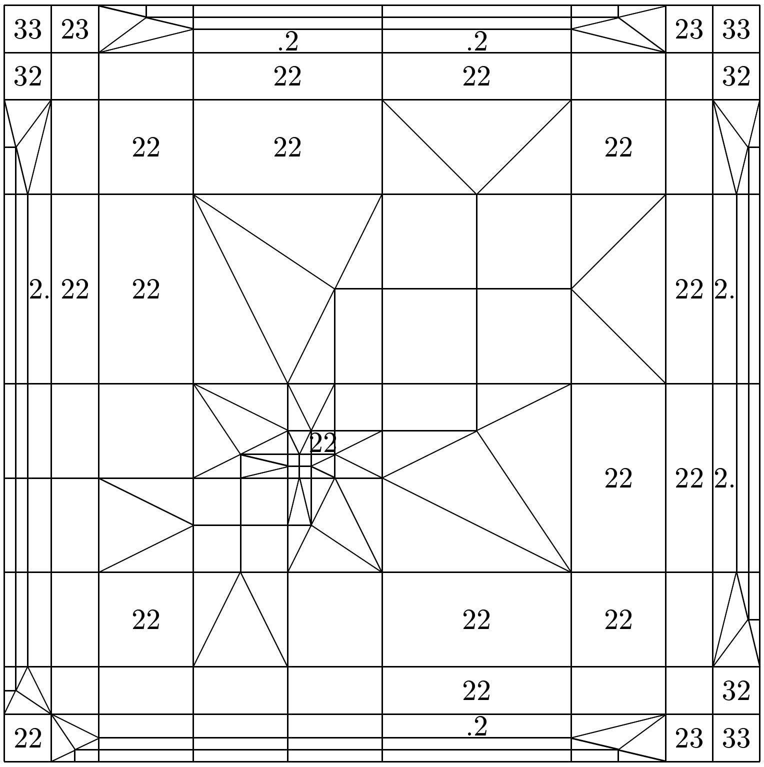

Figure 5: Mesh and polynomial degrees after five hp-adaptive steps.

		local enrichment		mesh halving	
N	$p_{\max}$	Θ/error	σ	Θ/error	σ
1	1	1.02	0.61	1.00	0.73
9	1	0.93	0.67	1.02	0.69
33	2	0.93	0.68	1.02	0.75
75	2	0.95	0.68	1.04	0.72
120	2	0.83	0.75	0.98	0.80
191	3	0.78	0.80	0.97	0.79

Table 5: Effectivity indices and saturation parameters (adaptive hp-version).

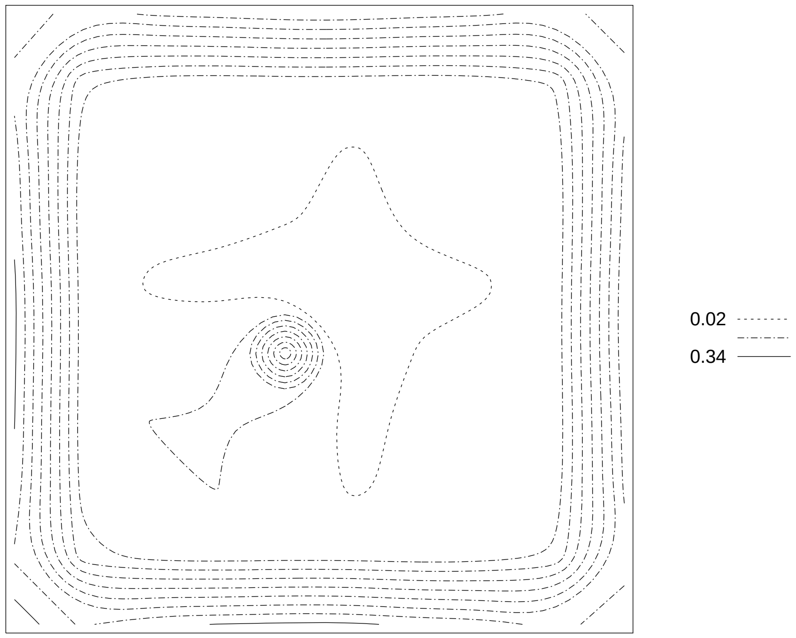

Figure 6: Approximated isolines of error estimator on uniform mesh of 64 elements with polynomial degree one (outer solid line is the boundary).

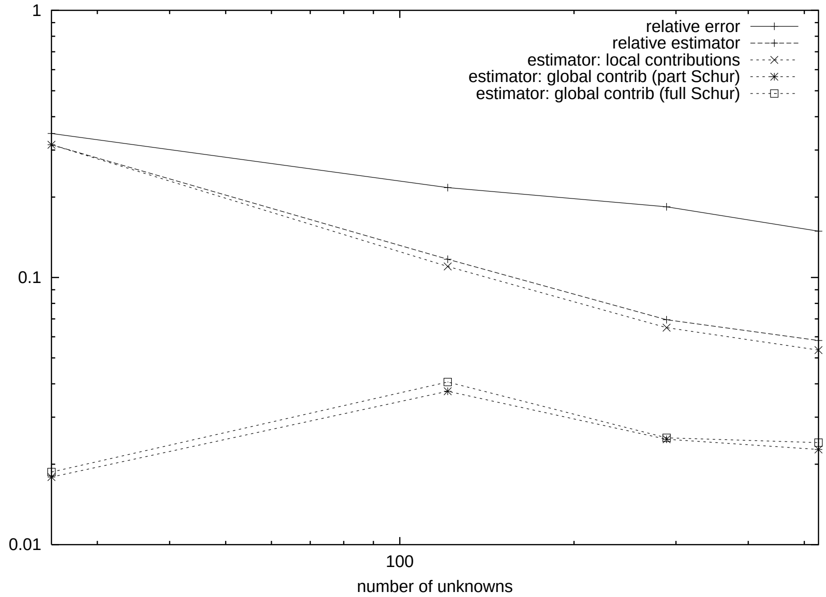

Figure 7: Error and estimator based on local enrichments for the uniform p-version (36 elements).

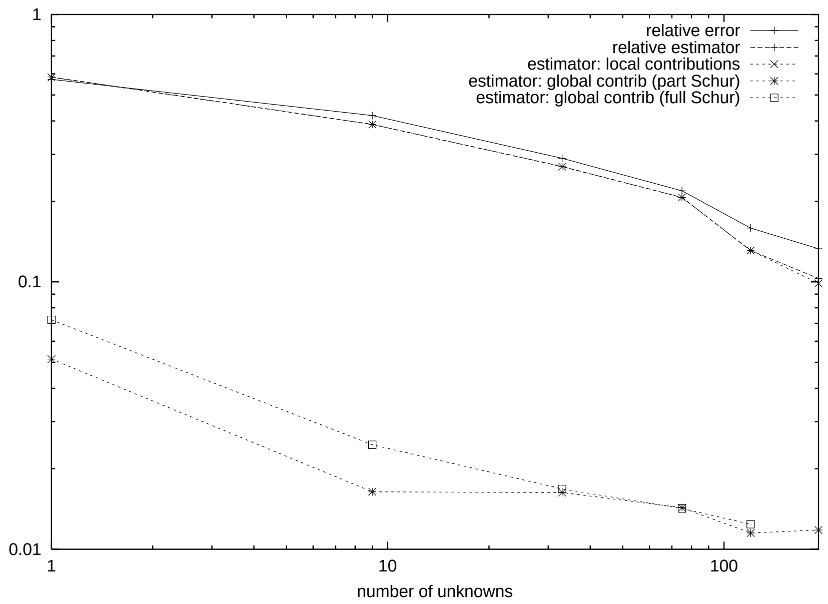

Figure 8: Error and estimator based on local enrichments for the adaptive hp-version.

References

[1] M. Ainsworth and W. Senior, *An adaptive refinement strategy for h-p finite element computations*, Appl. Numer. Math., 26 (1997), pp. 165–178.

[2] ——, *Aspects of an adaptive h-p finite element method: adaptive strategy, conforming approximation ands efficient solvers*, Comput. Methods Appl. Mech. Engrg., 150 (1997), pp. 65–87.

[3] I. Babuška, M. Griebel, and J. Pitkäranta, *The problem of selecting the shape functions for a p-type finite element*, Internat. J. Numer. Methods Engrg., 28 (1989), pp. 1891–1908.

[4] R. E. Bank and K. Smith, *A posteriori error estimates based on hierarchical bases*, SIAM J. Numer. Anal., 30 (1993), pp. 921–935.

[5] M. Costabel, *Boundary integral operators on Lipschitz domains: Elementary results*, SIAM J. Math. Anal., 19 (1988), pp. 613–626.

[6] P. Deuflhard, P. Leinen, and H. Yserentant, *Concepts of an adaptive hierarchical finite element code*, Impact Comput. Sci. Engrg., 1 (1989), pp. 3–35.

[7] B. Guo and W. Cao, *An additive Schwarz method for the h-p version of the finite element method in three dimensions*, SIAM J. Numer. Anal., 35 (1998), pp. 632–654.

[8] N. Heuer, *Additive Schwarz method for the p-version of the boundary element method for the single layer potential operator on a plane screen*, Numer. Math. Published online (2000) `http://dx.doi.org/10.1007/s002110000208`.

[9] ——, *An iterative substructuring method for the p-version of the boundary element method for hypersingular integral equations in three dimensions*, Numer. Math., 79 (1998), pp. 371–396. Erratum published online (2000) `http://dx.doi.org/10.1007/s002110000276`.

[10] ——, *Additive Schwarz methods for indefinite hypersingular integral equations in* $\mathbf{R}^3$ *– the p-version*, Appl. Anal., 72 (1999), pp. 411–437.

[11] N. Heuer, M. Maischak, and E. P. Stephan, *Exponential convergence of the hp-version for the boundary element method on open surfaces*, Numer. Math., 83 (1999), pp. 641–666.

[12] N. Heuer, M. E. Mellado, and E. P. Stephan, *hp-adaptive two-level methods for boundary integral equations on curves*, Tech. Rep. 00-09, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, 2000. `http://www.ing-mat.udec.cl/inf-loc-dim.html`

[13] ——, *A p-adaptive algorithm for the BEM with hypersingular operators on surfaces*, Tech. Rep. 00-12, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, 2000. `http://www.ing-mat.udec.cl/inf-loc-dim.html`

[14] N. Heuer and E. P. Stephan, *An additive Schwarz method for the h-p version of the boundary element method for hypersingular integral equations in* $\mathbf{R}^3$, IMA J. Numer. Anal. To appear.

[15] ——, *Boundary integral operators in countably normed spaces*, Math. Nachr., 191 (1998), pp. 123–151.

[16] M. Maischak, *Technical manual of the program system maiprogs*, Institut für Angewandte Mathematik, Universität Hannover, Hannover, Germany. `http://www.ifam.uni-hannover.de/~maischak/maiprogs.html`.

[17] M. Maischak, P. Mund, and E. P. Stephan, *Adaptive multilevel BEM for acoustic scattering*, Comput. Methods Appl. Mech. Engrg., 150 (1997), pp. 351–367.

[18] P. Mund and E. P. Stephan, *An adaptive two-level method for the coupling of nonlinear FEM-BEM equations*, SIAM J. Numer. Anal., 36 (1999), pp. 1001–1021.

[19] P. Mund, E. P. Stephan, and J. Weisse, *Two level methods for the single layer potential in* $\mathbf{R}^3$, Computing, 60 (1998), pp. 243–266.

[20] L. F. Pavarino, *Additive Schwarz methods for the p-version finite element method*, Numer. Math., 66 (1994), pp. 493–515.

[21] ——, *Schwarz methods with local refinement for the p-version finite element method*, Numer. Math., 69 (1994), pp. 185–211.

[22] L. F. Pavarino and T. Warburton, *Overlapping Schwarz methods for unstructured spectral elements*. Quaderni del Dipartimento di Matematica, Università di Milano, 1999, no. 15, to appear in Journal of Computational Physics.

[23] L. F. Pavarino and O. B. Widlund, *A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions*, SIAM J. Numer. Anal., 33 (1996), pp. 1303–1335.

[24] E. P. Stephan, *Boundary integral equations for screen problems in* $\mathbf{R}^3$, Integral Equations Operator Theory, 10 (1987), pp. 257–263.

[25] T. von Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder – Singularitäten und Approximation mit Randelementmethoden*, PhD thesis, Technische Hochschule Darmstadt, Germany, 1989.

[26] T. von Petersdorff and E. P. Stephan, *Decompositions in edge and corner singularities for the solution of the Dirichlet problem of the Laplacian in a polyhedron*, Math. Nachr., 149 (1990), pp. 71–104.

[27] ——, *Regularity of mixed boundary value problems in* $\mathbf{R}^3$ *and boundary element methods on graded meshes*, Math. Methods Appl. Sci., 12 (1990), pp. 229–249.