Analyse à la Friedrichs des problèmes d'advection réaction scalaire et vectoriel

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Scalar advection-reaction

- Ω be a domain of $\mathbb{R}^3$ with Lipschitz-continuous boundary $\partial\Omega$

Find $u : \Omega \to \mathbb{R}$ solving

$$\begin{aligned} \boldsymbol{\beta}\cdot\nabla u + \mu u &= s \text{ a.e. in } \Omega, \\ u &= 0 \text{ a.e. on } \partial\Omega^-. \end{aligned}$$

Inflow/Outflow Boundary $\partial\Omega^\pm := \{\boldsymbol{x} \in \partial\Omega \,|\, \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$

Settings

Advection field $\boldsymbol{\beta} : \Omega \to \mathbb{R}^3 \in \mathbf{Lip}(\Omega)$

Reaction field $\mu : \Omega \to \mathbb{R} \in L^\infty(\Omega)$

Data $s : \Omega \to \mathbb{R} \in L^p(\Omega)$ with $p \in (1, \infty)$

Hilbertian analysis *Bardos* (1970), *Beirão da Veiga* (1988), *DiPerna & Lions* (1989), *Ern & Guermond* (2006)

$W^{1,p}(\Omega)$-regularity *Girault & Tartar* (2015)

Vector advection-reaction

Find $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ solving

$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{u}) + (\nabla\times\boldsymbol{u})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{u} = \boldsymbol{s} \text{ a.e. in } \Omega, \qquad \boldsymbol{u} = \mathbf{0} \text{ a.e. on } \partial\Omega^-.$$

Inflow/Outflow Boundary $\partial\Omega^{\pm} := \{\boldsymbol{x} \in \partial\Omega \,|\, \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$

Settings

- **Advection field** $\boldsymbol{\beta} : \Omega \to \mathbb{R}^3 \in \mathbf{Lip}(\Omega)$
- **Conductivity field** $\boldsymbol{\mu} : \Omega \to \mathbb{R}^{3\times 3} \in \boldsymbol{L}^{\infty}(\Omega)$
- **Data** $\boldsymbol{s} : \Omega \to \mathbb{R}^3 \in \boldsymbol{L}^p(\Omega)$ with $p \in (1, \infty)$

Physical modelling Static advection of a magnetic field
e.g., *Heumann & Hiptmair* (2013)

Numerous formulations

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\boldsymbol{v}) = \nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{v}$$

→ Four advection operators up to the 0-th order operator $\boldsymbol{\mu}$

$\boldsymbol{\mu}$	$\boldsymbol{\mathcal{L}}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\boldsymbol{v})$

- Non-conservative and conservative Oseen-like operators

$-\nabla\boldsymbol{\beta}^{\mathrm{T}}$	$(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}$
$(\nabla\cdot\boldsymbol{\beta})\mathbf{Id} - \nabla\boldsymbol{\beta}^{\mathrm{T}}$	$\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{\beta})$

- $\mathbb{R}^3$-proxies of the Lie derivative of a 1-form and a 2-form

$\mathbf{0}$	$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta}$
$(\nabla\cdot\boldsymbol{\beta})\mathbf{Id} - (\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}})$	$\boldsymbol{\beta}(\nabla\cdot\boldsymbol{v}) + \nabla\times(\boldsymbol{v}\times\boldsymbol{\beta})$

Generalization

Lie derivative

- Ω be a d-dimensional manifold
- $\Lambda^k(\Omega)$ collecting the k-differential forms on Ω with $k \in [\![0, d]\!]$
- Let $\beta : \Omega \to T\Omega$ be a given vector field, isomorphic to $\Lambda^1(\Omega)$

Exterior derivative

$$\mathrm{d} : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$$

"Classical differential operators"

Interior product

$$\iota_{\boldsymbol{\beta}} : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$$

"Multiplication with $\boldsymbol{\beta}$"

➙ Lie derivative: Cartan homotopy formula

$$\mathcal{L}_{\boldsymbol{\beta}} : \Lambda^k(\Omega) \to \Lambda^k(\Omega), \quad \mathcal{L}_{\boldsymbol{\beta}}\omega^k = (\mathrm{d}\iota_{\boldsymbol{\beta}} + \iota_{\boldsymbol{\beta}}\mathrm{d})\omega^k, \quad \forall \omega^k \in \Lambda^k(\Omega)$$

Scalar graph space

Find $u : \Omega \to \mathbb{R}$ solving

$$\boldsymbol{\beta}\cdot\nabla u + \mu u = s \text{ a.e. in } \Omega, \quad u = 0 \text{ a.e. on } \partial\Omega^-.$$

Let $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$$V_{\boldsymbol{\beta};p}(\Omega) := \{\boldsymbol{v} \in L^p(\Omega) \mid \boldsymbol{\beta}\cdot\nabla v \in L^p(\Omega)\}$$

➙ Reflexive Banach space with $\|v\|_{V_{\boldsymbol{\beta};p}(\Omega)} = \left(\|v\|^p_{L^p(\Omega)} + \|\boldsymbol{\beta}\cdot\nabla v\|^p_{L^p(\Omega)}\right)^{1/p}$

- $\boldsymbol{\beta}\cdot\nabla v \in L^p(\Omega)$ means that the linear form

$$T^0_{\boldsymbol{\beta}} : \mathcal{C}^\infty_c(\Omega) \ni \varphi \mapsto -\int_\Omega v\nabla\cdot(\boldsymbol{\beta}\varphi)$$

is bounded in $L^{p'}(\Omega)$, and $\boldsymbol{\beta}\cdot\nabla v$ is the Riesz representation of $T^0_{\boldsymbol{\beta}}$ in $L^p(\Omega)$.

Existence of trace in $V_{\boldsymbol{\beta};p}(\Omega)$

$$L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega) := \left\{ v : \partial\Omega \to \mathbb{R} \mid v \text{ is measurable on } \partial\Omega \text{ and } \int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|v|^p < \infty \right\}$$

$$\|v\|_{L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|;\partial\Omega)} := \left(\int_{\partial\Omega} |\boldsymbol{\beta}\cdot\boldsymbol{n}|\,|v|^p \right)^{\frac{1}{p}}, \quad \forall v \in L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$$

Continuity of the trace

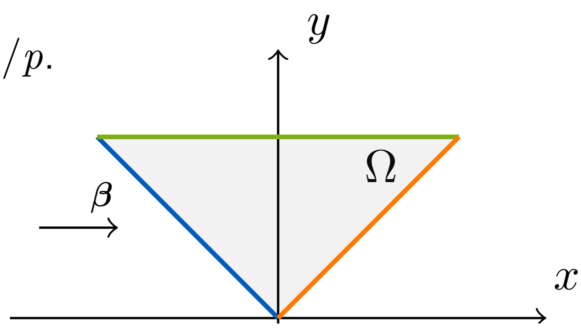
The trace map $\gamma : V_{\boldsymbol{\beta};p}(\Omega) \to L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$ is continuous if and only if

$$\mathrm{dist}(\partial\Omega^+, \partial\Omega^-) > 0.$$

CNS (From *Jensen* '04) $\boldsymbol{\beta} = (1,0)^{\mathrm{T}}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < 1 \text{ s.t. } |x| < y\}$

- Define $u(x,y) = y^\alpha$ with $-2/p < \alpha \leq -1/p$.

$u \in V_{\boldsymbol{\beta};p}(\Omega)$ **and** $u \notin L^p(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$



Integration by parts

Integration by parts

For all $u \in V_{\boldsymbol{\beta};p}(\Omega)$ and $v \in V_{\boldsymbol{\beta};p'}(\Omega)$, we have

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla u)v + \int_\Omega (\boldsymbol{\beta}\cdot\nabla v)u + \int_\Omega (\nabla\cdot\boldsymbol{\beta})u\,v = \int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})uv$$

It follows from the dense inclusion $C^\infty(\overline{\Omega}) \hookrightarrow V_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,\infty)$ and

$$\nabla |\varphi|^p = p\varphi\,|\varphi|^{p-2}\,\nabla\varphi, \quad \forall \varphi \in C^\infty(\overline{\Omega}).$$

Corollary

For all $u \in V_{\boldsymbol{\beta};p}(\Omega)$ and $z \in W^{1,\infty}(\Omega)$, we have

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla u)u\,|u|^{p-2}\,z + \frac{1}{p}\int_\Omega (\boldsymbol{\beta}\cdot\nabla z)\,|u|^p + \frac{1}{p}\int_\Omega (\nabla\cdot\boldsymbol{\beta})\,|u|^p\,z = \frac{1}{p}\int_{\partial\Omega} (\boldsymbol{\beta}\cdot\boldsymbol{n})\,|u|^p\,z$$

$$\boldsymbol{\beta}\cdot\nabla(\varphi\,|\varphi|^{p-2}\,z) = \varphi\,|\varphi|^{p-2}\,\boldsymbol{\beta}\cdot\nabla z + (p-1)\,|\varphi|^{p-2}\,z\boldsymbol{\beta}\cdot\nabla\varphi, \quad \forall \varphi \in C^\infty(\overline{\Omega}),\ z \in W^{1,\infty}(\Omega).$$

Weak-formulation

Scalar problem in Banach graph space

- Strong enforcement of BCs $V^0_{\boldsymbol{\beta};p}(\Omega) := \left\{ v \in V_{\boldsymbol{\beta};p}(\Omega) \mid v_{|\partial\Omega^-} = 0 \right\}$

Bilinear form $a_{\boldsymbol{\beta},\mu;p} \in \mathcal{L}(V^0_{\boldsymbol{\beta};p}(\Omega) \times L^{p'}(\Omega))$, $\quad a_{\boldsymbol{\beta},\mu;p}(u,v) = \int_\Omega (\boldsymbol{\beta}\cdot\nabla u)\, v + \int_\Omega \mu\, uv$

Find $u \in V^0_{\boldsymbol{\beta};p}(\Omega)$ s.t.

$$(P) \quad a_{\boldsymbol{\beta},\mu;p}(u,v) = \int_\Omega s\, v, \quad \forall v \in L^{p'}(\Omega)$$

- Weak enforcement of BCs $a^\sharp_{\boldsymbol{\beta},\mu;p}\mathcal{L}(V_{\boldsymbol{\beta};p}(\Omega) \times L^{p'}(\Omega))$

$$a^\sharp_{\boldsymbol{\beta},\mu;p}(u,v) = \int_\Omega (\boldsymbol{\beta}\cdot\nabla u)\, v + \int_\Omega \mu\, uv + \int_{\partial\Omega^-} (\boldsymbol{\beta}\cdot\boldsymbol{n})\, uv.$$

BNB theorem

- Let $a \in \mathcal{L}(U \times V; \mathbb{R})$ with U, V two reflexive Banach spaces
- Let $s \in V'$ and consider the abstract problem

$$\text{Find } u \in U \text{ s.t. } a(u,v) = \langle f, v \rangle_{V' \times V}, \quad \forall v \in V. \qquad (P_0)$$

Banach-Necas-Babuska Theorem

(P_0) is well-posed if, and only if

- There exists $\gamma > 0$ such that $\quad \inf_{u \in U} \sup_{v \in V} \frac{|a(u,v)|}{\|u\|_U \|v\|_V} \geq \gamma$
- $\{v \in V \mid a(u,v) = 0\ \forall u \in U\} \equiv \{0\}$.

Positive well-posedness

$\mathbb{R}$-valued Friedrichs tensor

$$\sigma_{\boldsymbol{\beta},\mu;p} := \mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}$$

The problem (P) is well-posed in $V^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1, +\infty)$ if

$$(\mathcal{H}_p) \quad \tau^{-1} := \text{ess inf}_\Omega\, \sigma_{\boldsymbol{\beta},\mu;p} > 0$$

Uniqueness Owing to integration by parts for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$

$$a_{\boldsymbol{\beta},\mu;p}(v, v\,|v|^{p-2}) = \int_\Omega \left(\mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}\right)|v|^p + \int_{\partial\Omega}(\boldsymbol{\beta}\cdot\boldsymbol{n})\,|u|^p \geq \int_\Omega \sigma_{\boldsymbol{\beta},\mu;p}\,|v|^p \overset{(\mathcal{H}_p)}{\geq} \tau^{-1}\|v\|^p_{L^p(\Omega)}$$

$$\|v\|_{L^p(\Omega)} \leq \tau \sup_{w\in L^{p'}(\Omega)} \frac{a_{\boldsymbol{\beta},\mu;p}(v,w)}{\|w\|_{L^{p'}(\Omega)}}$$

$$\|\boldsymbol{\beta}\cdot\nabla v\|_{L^p(\Omega)} \leq \sup_{w\in L^{p'}(\Omega)} \frac{a_{\boldsymbol{\beta},\mu;p}(v,w)}{\|w\|_{L^{p'}(\Omega)}}\left(1 + \tau\|\mu\|_{L^\infty(\Omega)}\right)$$

Extended well-posedness

$$\sigma_{\boldsymbol{\beta},\mu;p} := \mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}$$

➙ The analysis fails down if e.g. $\mu = 0$ and $\nabla\cdot\boldsymbol{\beta} = 0$.

The problem (P) is well-posed in $V^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1, +\infty)$ if there exists $\zeta \in \mathrm{Lip}(\Omega)$ such that

$$(\mathcal{H}'_p) \quad \tau^{-1} := \text{ess inf}_\Omega\, e^\zeta\left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) > 0$$

Existence of ζ Every solution $t \mapsto \boldsymbol{x}(t)$ solving $d_t\boldsymbol{x}(t) = \boldsymbol{\beta}(\boldsymbol{x}(t))$ with $\boldsymbol{x}_0 = \boldsymbol{x}(0) \in \Omega$ remains in Ω for a finite time.

Example If $\boldsymbol{\beta}$ has no stationnary point and closed curves.

Extended well-posedness

Proof

$$\sigma_{\boldsymbol{\beta},\mu;p} := \mu - \frac{1}{p}\nabla\cdot\boldsymbol{\beta}$$

The problem (P) is well-posed in $V^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,+\infty)$ if there exists $\zeta \in \mathrm{Lip}(\Omega)$ such that

$$(\mathcal{H}'_p) \quad \tau^{-1} := \text{ess inf}_\Omega\, e^\zeta \left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) > 0$$

IPP For all $v \in L^p(\Omega)$ and $z \in W^{1,\infty}(\Omega)$,

$$\int_\Omega (\boldsymbol{\beta}\cdot\nabla v)v\,|v|^{p-2}\,z = \frac{1}{p}\left(\int_{\partial\Omega}(\boldsymbol{\beta}\cdot\boldsymbol{n})\,|v|^p\,z - \int_\Omega(\nabla\cdot\boldsymbol{\beta})\,|v|^p\,z - \int_\Omega(\boldsymbol{\beta}\cdot\nabla z)\,|v|^p\right)$$

Owing to integration by parts for all $v \in V^0_{\boldsymbol{\beta};p}(\Omega)$

$$a_{\boldsymbol{\beta},\mu;p}(v, e^\zeta v\,|v|^{p-2}) = \int_\Omega e^\zeta\left(\sigma_{\boldsymbol{\beta},\mu;p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right)|v|^p + \frac{1}{p}\int_{\partial\Omega^+}(\boldsymbol{\beta}\cdot\boldsymbol{n})\,|u|^p \overset{(\mathcal{H}'_p)}{\geq} \tau^{-1}\|v\|^p_{L^p(\Omega)}$$

→ We conclude as in the positive case.

Vector advection-reaction

Find $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ solving

$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{u}) + (\nabla\times\boldsymbol{u})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{u} = \boldsymbol{s} \text{ a.e. in } \Omega,$$
$$\boldsymbol{u} = \boldsymbol{0} \text{ a.e. on } \partial\Omega^-.$$

Inflow/Outflow Boundary $\partial\Omega^\pm := \{\boldsymbol{x} \in \partial\Omega \mid \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$

Settings

- **Advection field** $\boldsymbol{\beta} : \Omega \to \mathbb{R}^3 \in \mathbf{Lip}(\Omega)$
- **Conductivity field** $\boldsymbol{\mu} : \Omega \to \mathbb{R}^{3\times3} \in \boldsymbol{L}^\infty(\Omega)$
- **Data** $\boldsymbol{s} \in \boldsymbol{L}^p(\Omega)$ with $p \in (1,\infty)$

Vector graph space $\boldsymbol{V}_{\boldsymbol{\beta};p}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega) \mid (\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)\}$

Weak-formulation

Vector problem in Banach graph space

➙ Strong enforcement of BCs $\boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega) := \left\{\boldsymbol{v} \in \boldsymbol{V}_{\boldsymbol{\beta};p}(\Omega) \mid \boldsymbol{v}_{|\partial\Omega^-} = 0\right\}$

Bilinear form $\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p} \in \mathcal{L}(\boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega) \times \boldsymbol{L}^{p'}(\Omega))$ defined as

$$\begin{aligned} \boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{u},\boldsymbol{v}) &= \int_\Omega \nabla(\boldsymbol{\beta}\cdot\boldsymbol{u})\cdot\boldsymbol{v} + \int_\Omega ((\nabla\times\boldsymbol{u})\times\boldsymbol{\beta})\cdot\boldsymbol{v} + \int_\Omega \boldsymbol{\mu}\boldsymbol{u}\cdot\boldsymbol{v} \\ &= \int_\Omega \boldsymbol{v}\cdot(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{u} + \int_\Omega \boldsymbol{v}\cdot(\nabla\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\mu})\,\boldsymbol{u} \end{aligned}$$

Find $\boldsymbol{u} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ s.t.

$$(\boldsymbol{P}) \quad \boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{s}\cdot\boldsymbol{v}, \quad \forall \boldsymbol{v} \in \boldsymbol{L}^{p'}(\Omega)$$

Positive well-posedness

$\mathbb{R}^{3\times 3}$-valued Friedrichs tensor

$$\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p} := \frac{\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}}}{2} + \frac{\boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}}}{2} - \frac{1}{p}(\nabla\cdot\boldsymbol{\beta})\mathbf{Id}$$

Minimal eigenvalue

$$\forall \boldsymbol{x} \in \Omega,\ \aleph_p(\boldsymbol{x}) := \min\left\{ \frac{\langle \boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{x})\boldsymbol{y}, \boldsymbol{y}\rangle}{|\boldsymbol{y}|^2_{\ell^2}} \;\middle|\; \boldsymbol{y} \in \mathbb{R}^3 \right\}$$

The problem $(\boldsymbol{P})$ is well-posed in $\boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1,+\infty)$ if

$$(\boldsymbol{\mathcal{H}}_p) \quad \tau^{-1} := \text{ess inf}_\Omega\, \aleph_p > 0$$

Uniqueness For all $\boldsymbol{v} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ and owing to integration by parts

$$\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) = \int_\Omega |\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p}\cdot\boldsymbol{v} + \frac{1}{p}\int_{\partial\Omega^+} (\boldsymbol{\beta}\cdot\boldsymbol{n})\,|\boldsymbol{v}|^p \overset{(\boldsymbol{\mathcal{H}}_p)}{\geq} \tau^{-1}\|\boldsymbol{v}\|^p_{\boldsymbol{L}^p(\Omega)}$$

Extended well-posedness

$$\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p} := \frac{\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}}}{2} + \frac{\boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}}}{2} - \frac{1}{p}(\nabla\cdot\boldsymbol{\beta})\mathbf{Id}$$

The problem $(\boldsymbol{P})$ is well-posed in $\boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ for all $p \in (1, +\infty)$ if there exists $\zeta \in \mathrm{Lip}(\Omega)$ such that

$$(\boldsymbol{\mathcal{H}}'_p) \quad \tau^{-1} := \text{ess inf}_\Omega\ e^\zeta \left(\aleph_p - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\right) > 0$$

Uniqueness For all $\boldsymbol{v} \in \boldsymbol{V}^0_{\boldsymbol{\beta};p}(\Omega)$ and owing to integration by parts

$$\begin{aligned}
\boldsymbol{a}_{\boldsymbol{\beta},\boldsymbol{\mu};p}(\boldsymbol{v}, e^\zeta \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) &= \boldsymbol{a}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p}(\boldsymbol{v}, \boldsymbol{v}\,|\boldsymbol{v}|^{p-2}) - \int_\Omega e^\zeta\,|\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\left(\frac{\nabla\zeta\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\nabla\zeta}{2}\right)\cdot\boldsymbol{v} \\
&\geq \int_\Omega |\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\boldsymbol{\sigma}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p}\cdot\boldsymbol{v} - \int_\Omega e^\zeta\,|\boldsymbol{v}|^{p-2}\,\boldsymbol{v}\cdot\left(\frac{\nabla\zeta\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\nabla\zeta}{2}\right)\cdot\boldsymbol{v}
\end{aligned}$$

with $\widetilde{\boldsymbol{\beta}} = e^\zeta\boldsymbol{\beta}$ and $\widetilde{\boldsymbol{\mu}} = e^\zeta\boldsymbol{\mu}$. Conclude with the identity

$$\boldsymbol{\sigma}_{\widetilde{\boldsymbol{\beta}},\widetilde{\boldsymbol{\mu}};p} = e^\zeta\left(\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu};p} - \frac{1}{p}\boldsymbol{\beta}\cdot\nabla\zeta\mathbf{Id}\right) + e^\zeta\left(\frac{\nabla\zeta\otimes\boldsymbol{\beta} + \boldsymbol{\beta}\otimes\nabla\zeta}{2}\right)$$

Conclusion et perspectives

- Extension de l'analyse des problèmes d'advection réaction scalaire et vectoriel
  - Dans les espaces du graphe de Banach
  - Sous des hypothèses sur les champs physiques plus générales

- Généralisation de cette analyse en géométrie différentielle (dérivée de Lie)
- Traitement des conditions à la limite non-homogènes, surjectivité de la trace

Merci pour votre attention

• P. Cantin, *"Well-posedness of the scalar and the vector advection-reaction problems in Banach graph space"*
Compte-rendu de l'Académie des Sciences, 2017.