An edge-based scheme on polyhedral meshes for vector advection-reaction equations

Pierre Cantin[1,2] & Alexandre Ern[2]

[1]Pontificia Universidad Catolica (PUC), Chile
[2]Université Paris Est - CERMICS (ENPC), France

École des Ponts ParisTech

Vector advection-reaction

➞ Find $\boldsymbol{u} \in \boldsymbol{V}_{\boldsymbol{\beta};2}(\Omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \,|\, \nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} \in \boldsymbol{L}^2(\Omega)\}$ s.t.

$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{u}) + (\nabla\times\boldsymbol{u})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{u} = \boldsymbol{s} \quad \text{in } \Omega,$$
$$\boldsymbol{u} = \boldsymbol{u}_D \text{ on } \partial\Omega^-.$$

Setting

Physical parameters $\boldsymbol{\beta} \in \mathbf{Lip}(\Omega)$ and $\boldsymbol{\mu} \in \boldsymbol{L}^\infty(\Omega)$

Data $\boldsymbol{s} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{u}_D \in \boldsymbol{L}^2(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$

Outflow/inflow boundary $\partial\Omega^\pm := \{\boldsymbol{x} \in \partial\Omega \,|\, \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$

Objectives

1) Low-order approximation using scalar degrees of freedom at mesh edges

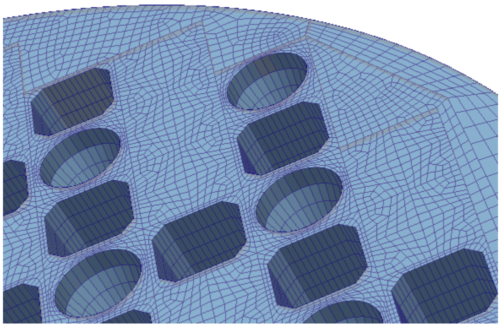
2) Approximation on 3D general meshes (polyhedral/non-conforming)

3) Extend the discrete stability under new assumptions

Why polyhedral meshes?

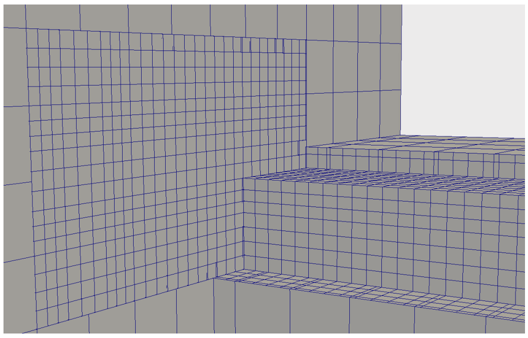
→ Complex industrial geometries

- Multi-element mesh

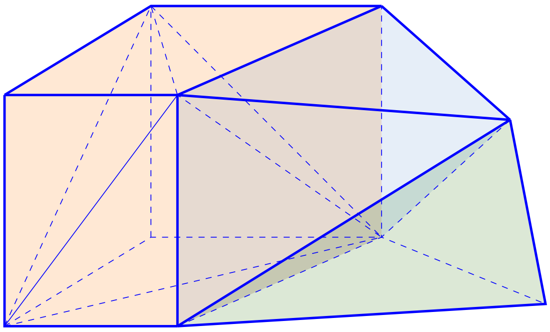


→ Non-conforming interfaces

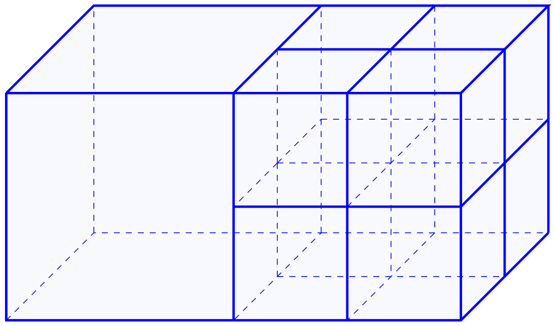
- Mesh agglomeration



- Reduced mesh cardinalities



- Locally refined mesh



Numerous formulations

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\boldsymbol{v}) = \nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta} + \boldsymbol{\mu}\boldsymbol{v}$$

→ Four advection operators up to the 0-th order operator $\boldsymbol{\mu}$

$\boldsymbol{\mu}$	$\boldsymbol{\mathcal{L}}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\boldsymbol{v})$

- $\mathbb{R}^3$-proxies of the Lie derivative of a 1-form and a 2-form

$\mathbf{0}$	$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta}$
$(\nabla\cdot\boldsymbol{\beta})\mathbf{Id} - (\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}})$	$\boldsymbol{\beta}(\nabla\cdot\boldsymbol{v}) + \nabla\times(\boldsymbol{v}\times\boldsymbol{\beta})$

- Non-conservative and conservative Oseen-like operators

$-\nabla\boldsymbol{\beta}^{\mathrm{T}}$	$(\boldsymbol{\beta}\cdot\nabla)\boldsymbol{v}$
$(\nabla\cdot\boldsymbol{\beta})\mathbf{Id} - \nabla\boldsymbol{\beta}^{\mathrm{T}}$	$\nabla\cdot(\boldsymbol{v}\otimes\boldsymbol{\beta})$

Well-posedness

$\mathbb{R}^{3\times3}$-valued Friedrichs tensor $\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}} := \dfrac{\nabla\boldsymbol{\beta} + \nabla\boldsymbol{\beta}^{\mathrm{T}}}{2} + \dfrac{\boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}}}{2} - \dfrac{1}{2}(\nabla\cdot\boldsymbol{\beta})\mathbf{Id}$

Minimal eigenvalue $\aleph := \min\{\langle \boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}}\boldsymbol{y}, \boldsymbol{y}\rangle \mid \boldsymbol{y} \in \mathbb{R}^3 \text{ s.t. } |\boldsymbol{y}| = 1\}$

The problem is well-posed in $\boldsymbol{V}_{\beta;2}(\Omega)$ if $(\mathcal{H})$ **or** $(\mathcal{H}')$ holds:

$(\mathcal{H})$ (*Heumann & Hiptmair* '15) $\tau^{-1} := \operatorname{ess\,inf}_\Omega \aleph > 0$.

$(\mathcal{H}')$ (*Cantin* '16) $\operatorname{ess\,inf}_\Omega \aleph \leq 0$ and $\exists \zeta \in \mathrm{Lip}(\Omega)$ s.t.

$$\zeta > 0 \text{ and } \tau^{-1} := \operatorname{ess\,inf}_\Omega \left(\zeta\aleph - \frac{1}{2}\boldsymbol{\beta}\cdot\nabla\zeta\right) > 0$$

- Let $\boldsymbol{a}(\boldsymbol{v}, \boldsymbol{w}) = \displaystyle\int_\Omega (\nabla(\boldsymbol{\beta}\cdot\boldsymbol{v}) + (\nabla\times\boldsymbol{v})\times\boldsymbol{\beta})\cdot\boldsymbol{w} + \int_\Omega \boldsymbol{\mu}\boldsymbol{v}\cdot\boldsymbol{w}$

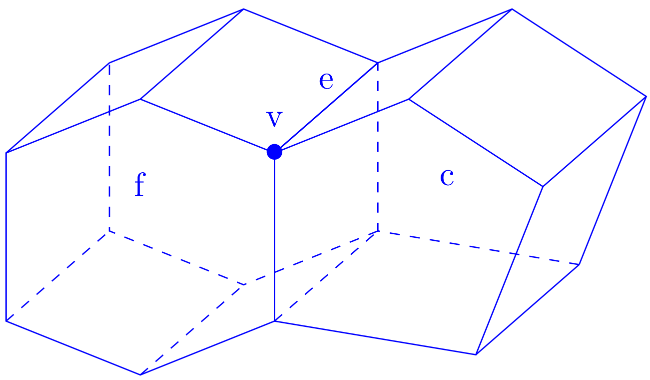
Uniqueness For all $\boldsymbol{v} \in \boldsymbol{V}_{\beta;2}(\Omega)$ s.t. $\boldsymbol{v}_{|\partial\Omega^-} = \mathbf{0}$,

$$\boldsymbol{a}(\boldsymbol{v}, \boldsymbol{v}) \geq \int_\Omega \boldsymbol{v}\cdot\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}}\cdot\boldsymbol{v} \xrightarrow{(\mathcal{H})} a(\boldsymbol{v}, \boldsymbol{v}) \geq \tau^{-1}\|\boldsymbol{v}\|^2_{L^2(\Omega)}$$

$$\boldsymbol{a}(\boldsymbol{v}, \zeta\boldsymbol{v}) \geq \int_\Omega \boldsymbol{v}\cdot\left(\zeta\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}} - \frac{1}{2}(\boldsymbol{\beta}\cdot\nabla\zeta)\mathbf{Id}\right)\cdot\boldsymbol{v} \xrightarrow{(\mathcal{H}')} \sup_{\boldsymbol{w}\in \boldsymbol{L}^2(\Omega)} \frac{a(\boldsymbol{v}, \boldsymbol{w})}{\|\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}} \geq \|\zeta\|^{-1}_{L^\infty(\Omega)}\tau^{-1}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}$$

Polyhedral meshes

→ Let M a polyhedral mesh of $\Omega \subset \mathbb{R}^3$ composed of



Cells	c ∈ C	
Faces	f ∈ F	
Edges	e ∈ E	$\mathcal{E}$
Vertices	v ∈ V	

Non-conforming edge reconstruction

➙ Edge-based scheme with **one dof** per mesh edge

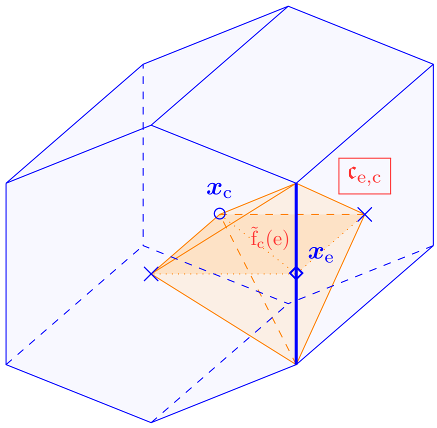
Piece-wise constant reconstruction

$$\mathbf{L}_{\mathcal{E}_c} : \mathcal{E}_c \to \mathbb{P}_0\left(\{\mathfrak{c}_{e,c}\}_{e\in E_c}; \mathbb{R}^3\right)$$

For all $w \in \mathcal{E}_c$, (*Codecasa & al.* '09)

$$\mathbf{C}_{\mathcal{E}_c}(w) := \frac{1}{|c|}\sum_{e\in E_c} w_e \tilde{\mathbf{f}}_c(e)$$

$$\mathbf{L}_{\mathcal{E}_c}(w)_{|\mathfrak{c}_{e,c}} := \mathbf{C}_{\mathcal{E}_c}(w) + \frac{\tilde{\mathbf{f}}_c(e)}{3|\mathfrak{c}_{e,c}|}\left(w_e - \mathbf{e}\cdot\mathbf{C}_{\mathcal{E}_c}(w)\right)$$

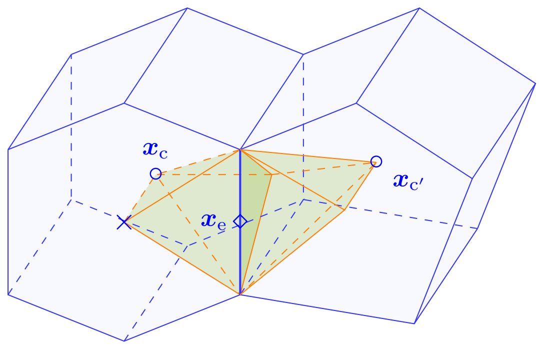


with $\mathbf{e} = \int_e \boldsymbol{t}_e$ and $\tilde{\mathbf{f}}_c(e) = \int_{\tilde{f}_c(e)} \boldsymbol{n}_{\tilde{f}_c(e)}$

➙ Polyhedral edge reconstruction $\mathbf{L}_{\mathcal{E}_c}(w) = \sum_{e\in E_c} w_e \boldsymbol{\ell}_{e,c}$ with shape functions

Properties

- Edge diamonds $\mathfrak{c}_e = \bigcup_{c\in C_e} \mathfrak{c}_{e,c}$
- Patch-cell $\hat{c} = \bigcup_{e\in E_c} \mathfrak{c}_e$



Quasi-local consistency. For all $\boldsymbol{v} \in \mathbb{P}_0(\hat{c};\mathbb{R}^3)$, $\mathbf{L}_{\mathcal{E}_c} \circ \widehat{R}_{\mathcal{E}_c}(\boldsymbol{v}) = \boldsymbol{v}_{|c}$ where

$$\forall e \in E_c, \quad \widehat{R}_{\mathcal{E}_c}(\boldsymbol{v})_{|e} := \frac{1}{|\mathfrak{c}_e|}\int_{\mathfrak{c}_e} \boldsymbol{v}\cdot\mathbf{e}$$

➙ Stability of $\widehat{R}_{\mathcal{E}_c}$ in $\boldsymbol{L}^1(\hat{c})$

Local consistency. For all $\boldsymbol{v} \in \mathbb{P}_0(c;\mathbb{R}^3)$, $\mathbf{L}_{\mathcal{E}_c} \circ R_{\mathcal{E}_c}(\boldsymbol{v}) = \boldsymbol{v}$ where

$$\forall e \in E_c, \quad R_{\mathcal{E}_c}(\boldsymbol{v})_{|e} := \frac{1}{|e|}\int_e \boldsymbol{v}\cdot\mathbf{e},$$

➙ Stability of $R_{\mathcal{E}_c}$ in $\boldsymbol{W}^{s,p}(c)$ is $sp > 2$

Stability. For all $c \in C$, $\boldsymbol{L}^p(c)$-stability of $\mathbf{L}_{\mathcal{E}_c}$ for all $p \in [1,\infty]$

Bilinear forms

➞ Bilinear form $A^{\mathcal{E}}_{\beta,\mu}$ on $\mathcal{E}\times\mathcal{E}$

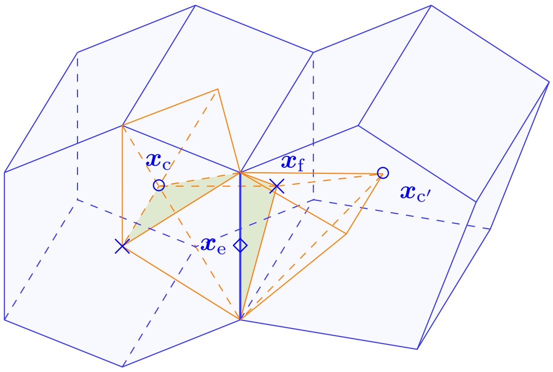
$$A^{\mathcal{E}}_{\beta,\mu}(u,v) = \sum_{c\in C} g_{\beta,\mu;c}(u,v) + \sum_{x\in\{F^\circ,C\}} n_{\beta,\mu;x}(u,v) + s_{\beta,\mu;x}(u,v)$$

- Galerkin formulation

$$g_{\beta,\mu;c}(u,v) = \int_c \Big(\nabla\cdot(\mathbf{L}_{\mathcal{E}_c}(u)\cdot\beta) + (\nabla\times\mathbf{L}_{\mathcal{E}_c}(u))\times\beta\Big)\cdot\mathbf{L}_{\mathcal{E}_c}(v) + \int_c \boldsymbol{\mu}\mathbf{L}_{\mathcal{E}_c}(u)\cdot\mathbf{L}_{\mathcal{E}_c}(v)$$

- Consistency with $x = f$ or c:

$$n_{\beta;x}(u,v) = -\sum_{f\in\mathfrak{F}_{E,x}} \int_f (\beta\cdot\boldsymbol{n})[\![\mathbf{L}_{\mathcal{E}}(u)]\!]\cdot\{\!\{\mathbf{L}_{\mathcal{E}}(v)\}\!\}$$



- Stabilization with $x = f$ or c:

$$s_{\beta;x}(u,v) = \sum_{f\in\mathfrak{F}_{E,x}} \int_f |\beta\cdot\boldsymbol{n}|\,[\![\mathbf{L}_{\mathcal{E}}(u)]\!]\cdot[\![\mathbf{L}_{\mathcal{E}}(v)]\!]$$

➞ Jumps across inter-cell sub-faces are also penalized

Scheme and stability

Find $u \in \mathcal{E}$ s.t., for all $v \in \mathcal{E}$,

(AR) $\mathbb{A}^{\mathcal{E}}_{\beta,\mu}(u,v) = \mathbb{S}(\boldsymbol{s},\boldsymbol{u}_D;v)$

$$\mathbb{A}^{\mathcal{E}}_{\beta,\mu}(u,v) = A^{\mathcal{E}}_{\beta,\mu}(u,v) + \text{BCs}$$

$$\mathbb{S}(\boldsymbol{s},\boldsymbol{u}_D;v) = \int_\Omega \boldsymbol{s}\cdot\mathbf{L}_{\mathcal{E}}(v) + \text{BCs}$$

Stability norm $\|w\|^2_{\mathcal{E},a} := \tau^{-1}\sum_{e\in E}\frac{|\mathfrak{c}_e|}{|e|^2}|w_e|^2 + \text{Stab.} + \text{BCs}$

$(\boldsymbol{\mathcal{H}})$ $\tau^{-1} := \operatorname{ess\,inf}_\Omega \aleph > 0$ ($\aleph$ the minimal eigenvalue of $\boldsymbol{\sigma}_{\beta,\mu}$)

Coercivity under $(\boldsymbol{\mathcal{H}})$

$$\mathbb{A}^{\mathcal{E}}_{\beta,\mu}(v,v) \gtrsim \|v\|^2_{\mathcal{E},a} \text{ for all } v\in\mathcal{E}$$

Inf-sup stability

$(\mathcal{H}_0')$ $-\mathsf{C}_\aleph < \text{ess inf}_\Omega\ \aleph \leq 0$ and there exists $\zeta \in \mathrm{Lip}(\Omega)$ such that $\zeta > 0$ and

$$\tau^{-1} = \text{ess inf}_\Omega \left(-\frac{1}{2}\boldsymbol{\beta}\cdot\nabla\zeta\right) > -\text{ess inf}_\Omega\ (\zeta\aleph)$$

Reference length $h_0 \simeq \left(|\zeta|_{W^{1,\infty}(\Omega)}\, \tau \|\nabla\boldsymbol{\beta}^{\mathrm{T}} + \boldsymbol{\mu}\|_{L^\infty(\Omega)}\right)^{-1}$

- $h_0 = +\infty$ if $\boldsymbol{\mu} = -\nabla\boldsymbol{\beta}^{\mathrm{T}}$ (Non-conservative Oseen operator)

Inf-sup stability under $(\mathcal{H}_0')$

Assume $h \lesssim h_0(1 + \mathsf{C}_\aleph^{-1}\text{ess inf}_\Omega\ \aleph)$. Then

$$\sup_{\mathsf{w}\in\mathcal{E}} \frac{\mathbb{A}^{\mathcal{E}}_{\boldsymbol{\beta},\boldsymbol{\mu}}(\mathsf{v},\mathsf{w})}{\|\mathsf{w}\|_{\mathcal{E},\mathsf{a}}} \gtrsim \|\mathsf{v}\|_{\mathcal{E},\mathsf{a}},\ \forall \mathsf{v} \in \mathcal{E}$$

Error estimates

Lebesgue exponent $p \in (\frac{3}{2}, 2]$

➙ Let $\mathsf{u} \in \mathcal{E}$ the **discrete** solution and let $\boldsymbol{u} : \Omega \to \mathbb{R}^3$ the **exact** solution

Quasi-local *a priori* estimate

Assume that $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$. Then,

$$\|\mathsf{u} - \widehat{\mathsf{R}}_{\mathcal{E}}(\boldsymbol{u})\|_{\mathcal{E},\mathsf{a}} \lesssim \left(\sum_{\mathrm{c}\in\mathrm{C}} \omega_{\mathrm{c}}^{p/2}\, h_{\mathrm{c}}^{2p-3}\, |\boldsymbol{u}|^p_{\boldsymbol{W}^{1,p}(\hat{\mathrm{c}})}\right)^{\frac{1}{p}}$$

Reduction map for all $\mathrm{e} \in \mathrm{E}$, $\widehat{\mathsf{R}}_{\mathcal{E}}(\boldsymbol{u})_{|\mathrm{e}} = \frac{1}{|\mathfrak{c}_\mathrm{e}|}\int_{\mathfrak{c}_\mathrm{e}} \boldsymbol{u}\cdot\mathbf{e}$ and $\mathsf{R}_{\mathcal{E}}(\boldsymbol{u})_{|\mathrm{e}} = \frac{1}{|\mathrm{e}|}\int_{\mathrm{e}} \boldsymbol{u}\cdot\mathbf{e}$

Quasi-optimal local estimate

Assume $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\mathrm{C})$ and $\nabla\times\boldsymbol{u} \in \boldsymbol{L}^{\tilde{p}}(\Omega)$ with $\tilde{p} = \frac{2p}{p-3} \in [2,4)$. Then,

$$\|\mathsf{u} - \mathsf{R}_{\mathcal{E}}(\boldsymbol{u})\|_{\mathcal{E},\mathsf{a}} \lesssim \left(\sum_{\mathrm{c}\in\mathrm{C}} \omega_{\mathrm{c}}^{p/2}\, h_{\mathrm{c}}^{2p-3}\left(|\boldsymbol{u}|^p_{\boldsymbol{W}^{1,p}(\mathrm{c})} + h_{\mathrm{c}}^{\frac{3(p-1)}{2}}\|\nabla\times\boldsymbol{u}\|^p_{\boldsymbol{L}^{\tilde{p}}(\mathrm{c})}\right)\right)^{\frac{1}{p}}$$

➙ See also *Amrouche & al.* '98

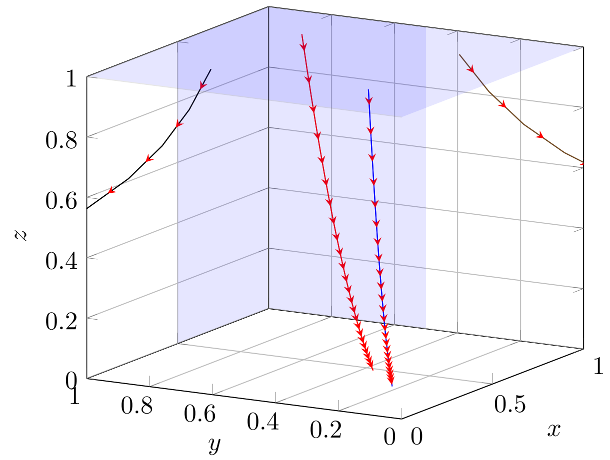
Reference velocity $\omega_{\mathrm{c}} = \left(\|\nabla\boldsymbol{\beta} + \boldsymbol{\mu}^{\mathrm{T}} - \nabla\cdot\boldsymbol{\beta}\|^p_{\boldsymbol{L}^\infty(\mathrm{c})}\tau^{\frac{p}{2}} h_{\mathrm{c}}^{\frac{p}{2}} + \|\boldsymbol{\beta}\|^{\frac{p}{2}}_{\boldsymbol{L}^\infty(\mathrm{c})}\right)^{\frac{2}{p}}$

Test case

Physical parameters

$$\boldsymbol{\beta} = \frac{1}{4}\begin{pmatrix} x-2y \\ y-2x \\ -2(z+1) \end{pmatrix} \text{ and } \boldsymbol{\mu} = \mu\mathbf{Id}$$

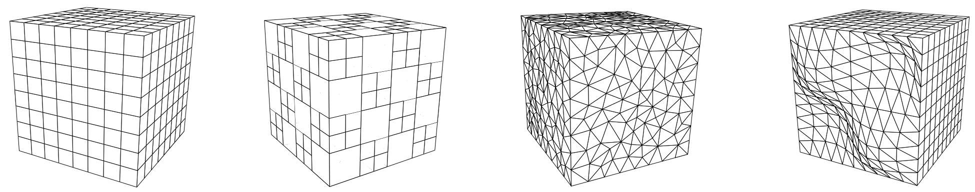
$$\boldsymbol{\sigma}_{\boldsymbol{\beta},\boldsymbol{\mu}} \sim \begin{pmatrix} \mu-\frac{1}{2} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu+2 \end{pmatrix}$$



➞ Well-posedness under $(\mathcal{H}_0')$ if $\mu = \frac{1}{2}$ with (e.g.) $\zeta(\boldsymbol{x}) = (z+1)^2$

Smooth solutions $\quad \boldsymbol{u}(x,y,z) = \begin{pmatrix} \sin(\pi x)\ \cos(\pi y/2)\ \cos(\pi z/2) \\ \cos(\pi x/2)\ \sin(\pi y)\ \cos(\pi z/2) \\ \cos(\pi x/2)\ \cos(\pi y/2)\ \sin(\pi z) \end{pmatrix}$

Computational setting



$\#\mathrm{E} \sim 810k$ $\quad \#\mathrm{E} \sim 700k$ $\quad \#\mathrm{E} \sim 1450k$ $\quad \#\mathrm{E} \sim 270k$

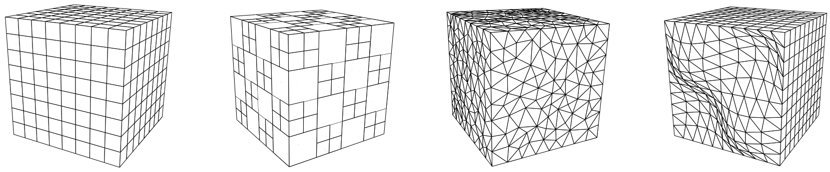
Computed quantities

Discrete relative $\boldsymbol{L}^2$-error w.r.t. the exact solution $\boldsymbol{u} : \Omega \to \mathbb{R}^3$

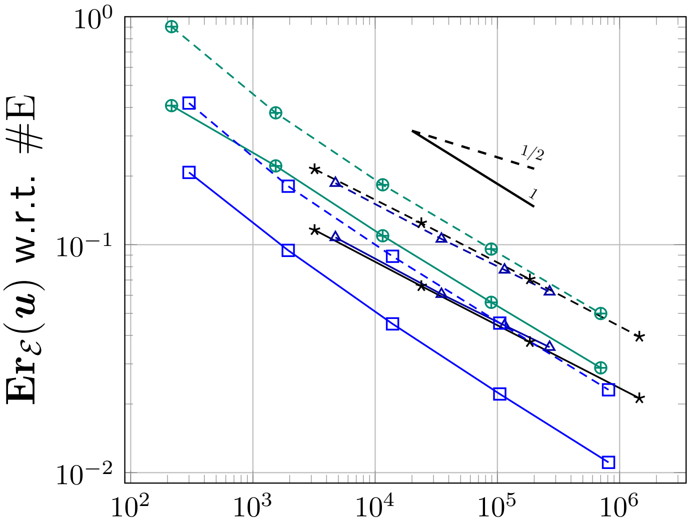
$$\mathbf{Er}_{\mathcal{E}}(\boldsymbol{u}) := \left(\frac{\sum_{\mathrm{v}\in\mathrm{V}} |\mathfrak{c}_{\mathrm{e}}|\,|\mathrm{e}|^{-2}\,\left|\mathrm{u}_{\mathrm{e}} - \mathrm{R}_{\mathcal{E}}(\boldsymbol{u})_{|\mathrm{e}}\right|^2}{\sum_{\mathrm{v}\in\mathrm{V}} |\mathfrak{c}_{\mathrm{e}}|\,|\mathrm{e}|^{-2}\,\left|\mathrm{R}_{\mathcal{E}}(\boldsymbol{u})_{|\mathrm{e}}\right|^2} \right)^{1/2} \quad \text{with} \quad \mathrm{R}_{\mathcal{E}}(\boldsymbol{u})_{|\mathrm{e}} := \int_{\mathrm{e}} \boldsymbol{u}\cdot\boldsymbol{t}_{\mathrm{e}}.$$

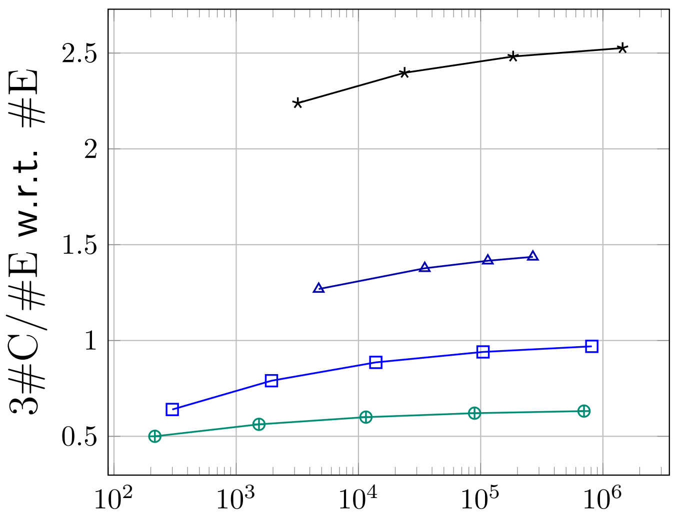
Mean stencil $\overline{\mathtt{St}} = \mathrm{NNZ}/\#\mathrm{E}$ and Max stencil `St.max`

Edge-based CDO scheme



$\mu = 5$	**??**	**??**	**??**	**??**
$\mu = 0.5$	**??**	**??**	**??**	**??**
$\overline{\texttt{St}}/\texttt{St.max}$	30/39	180/276	29/48	50/70





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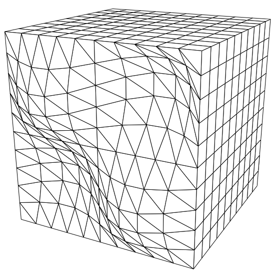

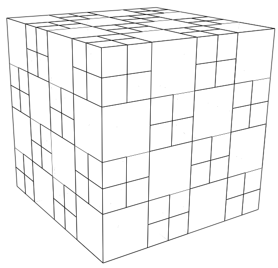

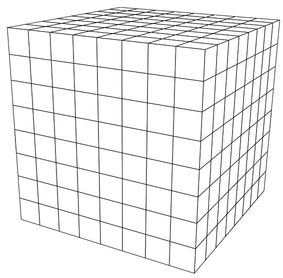

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