A nonlinear least-square framework for strongly monotone operators

Pierre Cantin[1] & Norbert Heuer[1]

[1]Pontificia Universidad Catolica (PUC), Chile

Outline

Objective Extension of the dPG technology to solve nonlinear problems

- ➙ Analysis of a nonlinear mixed penalized problem
- ➙ Approximation using broken test space and *a priori* error analysis
- ➙ Application to a strongly monotone problem, numerical experiment

Least-square formulation

- $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ two reflexive Hilbert spaces
- $A : U \to V'$ a continuous operator and $R : V \to V'$ the Riesz map
- $F \in V'$ a source term

Abstract problem

$$u \in U \quad : \quad \langle Au, v \rangle_{V',V} = \langle F, v \rangle_{V',V} \quad \forall v \in V. \tag{1}$$

Least-square formulation

$$u \in U \quad : \quad \langle Au, R^{-1}Aw \rangle_{V',V} = \langle F, R^{-1}Aw \rangle_{V',V} \quad \forall w \in U. \tag{2}$$

➞ If A^* is injective (linear or not), then (1) ⇔ (2)

➞ If A is linear and A^* is surjective, (2) is well-posed

Practical Least-square approximation

The dPG method

Conforming approximation $U_h \subset U$ and $V_h \subset V$

Discrete operators $A_h : U_h \to V_h'$ and $R_h : V_h \to V_h'$

Discrete leasts-square problem

$$u_h \in U_h \quad : \quad \langle A_h u_h, R_h^{-1} A_h v_h \rangle = \langle F_h, R_h^{-1} A_h v_h \rangle \quad \forall v_h \in U_h \tag{3}$$

Discrete well-posedness

Assume that • A is linear and A^* is surjective

• There is $\Pi \in \mathcal{L}(V, V_h)$ such that

$$\langle A_h w_h, v - \Pi v \rangle = 0 \quad \text{for all} \quad (w_h, v) \in U_h \times V$$

Then, (3) is well-posed and $\|u_h\|_{U_h} \lesssim \|F_h\|_{V_h'}$.

♠ Computing $R_h^{-1} : V_h \to V_h'$ amounts to inverse a Gramm matrix....

dPG method : LS formulation (3) using discontinuous test spaces

$$V_h = \bigtimes_{K \in \Omega_h} V_h(K)$$

Penalized mixed formulation

♠ If A is nonlinear (and so as A_h), the LS/dPG technologies are not competitive

Let $B : U \to V'$ and $C : U \to U'$ such that

- B is linear continuous
- C is nonlinear continuous
- $A(u) = F$ in V' $\Leftrightarrow$ $Bu = F_V$ in V' and $C(u) = F_U$ in U'

Equivalent formulations

$$\begin{pmatrix} C & B^* \\ B & -R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_U \\ F_V \end{pmatrix} \Leftrightarrow \begin{cases} Bu - Rv & = F_V \\ B^* R^{-1} Bu + C(u) & = B^* R^{-1} F_V + F_U \end{cases}$$

Assume that • B is surjective and • $C(u) = F_U$ in $\mathcal{N}(B)' \Rightarrow C(u) = F_U$ in U'

$$\begin{pmatrix} C & B^* \\ B & -R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_U \\ F_V \end{pmatrix} \Leftrightarrow \begin{cases} Bu = F_V, \; C(u) = F_U \\ v = 0 \end{cases}$$

Penalized mixed formulation

- Let $F_U \in U'$, $F_V \in V'$ and $\kappa > 0$.

$$(u, v) \in U \times V \quad : \quad \begin{pmatrix} C & B^* \\ B & -\kappa^{-1} R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_U \\ F_V \end{pmatrix} \in U' \times V' \tag{4}$$

Well-posedness

Assume that • B is surjective

- for all $u_1, u_2 \in U$,

$$\|P_B(u_1 - u_2)\|_U^2 \lesssim \langle C(u_1) - C(u_2), u_1 - u_2 \rangle + \|B(u_1 - u_2)\|_{V'}^2, \tag{5}$$

Then, for all $\kappa > 0$ large enough, (4) is well-posed and $(u, v) \in U \times V$ satisfies

$$\|u\|_U \lesssim \|F_U + \kappa B^* R^{-1} F_V\|_{U'}, \quad \|v\|_V \leq \kappa \|Bu - F_V\|_{V'}$$

- Assumption (5) means in particular that C is strongly monotone on $\mathcal{N}(B)$
- If C is linear, auto-adjoint and (u, v) solves (4), then

$$u = \text{argmin}_{v \in U} \langle C(v), v \rangle - 2\langle F_U, v \rangle + \kappa \langle R^{-1}(Bv - F_V), Bv - F_V \rangle$$

Conforming approximation

Conforming approximation $U_h \subset U$ and $V_h \subset V$

Discrete operators $B_h : U_h \to V_h'$, $C_h : U_h \to U_h'$ and $R_h : V_h \to V_h'$

Discrete problem

$$(u_h, v_h) \in U_h \times V_h \quad : \quad \begin{pmatrix} C_h & B_h^* \\ B_h & -\kappa^{-1} R_h \end{pmatrix} \begin{pmatrix} u_h \\ v_h \end{pmatrix} = \begin{pmatrix} F_{U_h} \\ F_{V_h} \end{pmatrix} \in U_h' \times V_h' \qquad (6)$$

Discrete well-posedness

Assume that • B is surjective

• for all $u_1, u_2 \in U$,

$$\|P_B(u_1 - u_2)\|_U^2 \lesssim \langle C(u_1) - C(u_2), u_1 - u_2 \rangle + \|B(u_1 - u_2)\|_{V'}^2,$$

• There is $\Pi \in \mathcal{L}(V, V_h)$ such that

$$\langle B_h w_h, v - \Pi v \rangle = 0 \text{ for all } (w_h, v) \in U_h \times V$$

Then, for all $\kappa > 0$ large enough, (6) is well-posed and $(u_h, v_h) \in U_h \times V_h$ satisfies

$$\|u_h\|_{U_h} \lesssim \|F_{U_h} + \kappa B_h^* R_h^{-1} F_{V_h}\|_{U_h'}, \quad \|v_h\|_{V_h} \le \kappa \|B_h u_h - F_{V_h}\|_{V_h'}$$

A priori error estimate

Continuous and discrete problems $(u, v) \in U \times V$ and $(u_h, v_h) \in U_h \times V_h$ s.t.

$$\begin{pmatrix} C & B^* \\ B & -R \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} F_U \\ F_V \end{pmatrix}, \quad \begin{pmatrix} C_h & B_h^* \\ B_h & -\kappa^{-1} R_h \end{pmatrix} \begin{pmatrix} u_h \\ v_h \end{pmatrix} = \begin{pmatrix} F_{U_h} \\ F_{V_h} \end{pmatrix}$$

Assume that • B is surjective

• for all $u_1, u_2 \in U$,

$$\|P_B(u_1 - u_2)\|_U^2 \lesssim \langle C(u_1) - C(u_2), u_1 - u_2 \rangle + \|B(u_1 - u_2)\|_{V'}^2,$$

• There is $\Pi \in \mathcal{L}(V, V_h)$ such that

$$\langle B_h w_h, v - \Pi v \rangle = 0 \text{ for all } (w_h, v) \in U_h \times V$$

• C is Lipschitz-continuous

• $C(u) = F_U$ in $\mathcal{N}(B)' \Rightarrow C(u) = F_U$ in U'

Then, for all $\kappa > 0$ small enough,

$$\|v_h\|_{V_h} \lesssim \|u - u_h\|_U \lesssim (1 + \kappa \|B^* R_h^{-1} B\|_{\mathcal{L}(U, U')}) \inf_{w_h \in U_h} \|u - w_h\|_U.$$

with $(u, 0) \in U \times V$ and $(u_h, v_h) \in U_h \times V_h$ the exact and the discrete solution.

A model problem

Objectives Consider this penalized mixed formulation to approximate the solution of a strongly monotone problem using optimal broken test functions (dPG)

Advective field $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$ s.t. $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and ess $\inf_{\Omega} (-\nabla\cdot\boldsymbol{\beta}) \geq 0$

Diffusion tensor $\boldsymbol{\lambda} : \mathbb{R} \to \mathbb{R}^{d\times d}$ s.t. for all $\boldsymbol{\sigma}, \boldsymbol{\theta} \in \boldsymbol{L}^2(\Omega)$,

$$\begin{aligned} \lambda_\sharp^{-1} \|\boldsymbol{\lambda}(|\boldsymbol{\sigma}|) - \boldsymbol{\lambda}(|\boldsymbol{\theta}|)\|^2_{\boldsymbol{L}^2(\Omega)} &\leq \|\boldsymbol{\sigma} - \boldsymbol{\theta}\|^2_{\boldsymbol{L}^2(\Omega)} \\ &\leq \lambda_\flat^{-1} \left(\boldsymbol{\lambda}(|\boldsymbol{\sigma}| - \boldsymbol{\lambda}(|\boldsymbol{\theta}|), \boldsymbol{\sigma} - \boldsymbol{\theta}\right)_{\boldsymbol{L}^2(\Omega)} \end{aligned}$$

Well-posedness

Under the above assumptions, there exists an unique $u \in H^1_0(\Omega)$ such that

$$-\nabla\cdot(\boldsymbol{\lambda}(|\nabla u|)\nabla u + \boldsymbol{\beta} u) = f \quad \text{a.e. in } \Omega \quad \text{and} \quad u = 0 \quad \text{a.e. on } \partial\Omega.$$

Applications Nonlinear diffusion/filtration, image processing (Perona-Malik diffusion), power-law materials, ...

Linear very-weak formulation

- Formally, consider the linear/nonlinear decomposition

$$-\nabla\cdot(\boldsymbol{\lambda}(|\nabla u|)\nabla u+\boldsymbol{\beta}u)=f \quad \blacktriangleright \quad \underbrace{-\nabla\cdot(\boldsymbol{\rho}+\boldsymbol{\beta}u)=f,\ \ \boldsymbol{\sigma}=\nabla u,}_{:=B}\ \ \underbrace{\boldsymbol{\rho}=\boldsymbol{\lambda}(|\boldsymbol{\sigma}|)\boldsymbol{\sigma}}_{:=C}$$

- Linear surjective operator $B:\boldsymbol{U}\to\boldsymbol{V}'$ such that

$$\begin{aligned}\langle B\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{V}',\boldsymbol{V}}=(u,\boldsymbol{\beta}\cdot\nabla v+\nabla\cdot\boldsymbol{\tau})_{L^2(\Omega_h)}+(\boldsymbol{\rho},\nabla v)_{L^2(\Omega_h)}+(\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2(\Omega)}\\ -\langle\hat{\rho},\gamma(v)\rangle_{\partial\Omega_h}-\langle\gamma_n(\boldsymbol{\tau}),\hat{u}\rangle_{\partial\Omega_h}\end{aligned}$$

with $\boldsymbol{u}=(u,\boldsymbol{\sigma},\boldsymbol{\rho},\hat{u},\hat{\rho})\in\boldsymbol{U}:=L^2(\Omega)\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)\times H^{1/2}_{00}(\partial\Omega_h)\times H^{-1/2}(\partial\Omega_h)$

$\boldsymbol{v}=(v,\boldsymbol{\tau})\quad\in\boldsymbol{V}:=H^1(\Omega_h)\times\boldsymbol{H}(\mathrm{div};\Omega_h)$ (Broken spaces)

Equivalent formulations

$$B\boldsymbol{u}=\tilde{f}\quad\Leftrightarrow\quad\begin{cases}u\in H^1_0(\Omega),\ \boldsymbol{\rho}\in\boldsymbol{H}(\mathrm{div};\Omega)\\ -\nabla\cdot(\boldsymbol{\rho}+\boldsymbol{\beta}u)=f,\ \boldsymbol{\sigma}=\nabla u,\ \hat{u}=\gamma(u),\ \hat{\rho}=\gamma_n(\boldsymbol{\rho}+\boldsymbol{\beta}u)\end{cases}$$

Nonlinear penalty operator

- Linear surjective operator $B : \boldsymbol{U} \to \boldsymbol{V}'$ such that

$$\langle B\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{V}',\boldsymbol{V}} = (u, \boldsymbol{\beta}\cdot\nabla v + \nabla\cdot\boldsymbol{\tau})_{L^2(\Omega_h)} + (\boldsymbol{\rho}, \nabla v)_{L^2(\Omega_h)} + (\boldsymbol{\sigma}, \boldsymbol{\tau})_{L^2(\Omega)} - \langle \hat{\rho}, \gamma(v)\rangle_{\partial\Omega_h} - \langle \gamma_n(\boldsymbol{\tau}), \hat{u}\rangle_{\partial\Omega_h}$$

- Nonlinear penalty operator $C : \boldsymbol{U} \to \boldsymbol{U}'$ such that

$$\langle C(\boldsymbol{u}), \boldsymbol{w}\rangle_{\boldsymbol{U}',\boldsymbol{U}} = (\boldsymbol{\lambda}(|\boldsymbol{\sigma}|)\boldsymbol{\sigma} - \boldsymbol{\rho}, \alpha\lambda_\flat\boldsymbol{\theta} - \boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)}, \quad \alpha > 0$$

with $\boldsymbol{u} = (u, \boldsymbol{\sigma}, \boldsymbol{\rho}, \hat{u}, \hat{\rho}) \in \boldsymbol{U} := L^2(\Omega)\times\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)\times H^{1/2}_{00}(\partial\Omega_h)\times H^{-1/2}(\partial\Omega_h)$
$\boldsymbol{w} = (\,\cdot\,, \boldsymbol{\theta}, \boldsymbol{\eta}, \,\cdot\,, \,\cdot\,) \in \boldsymbol{U}$

Well-posedness properties (I)

(i) $C : \boldsymbol{U} \to \boldsymbol{U}'$ is Lipschitz-continuous

(ii) Assume that $\alpha > (\lambda_\sharp/\lambda_\flat)^2$, then for all $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{U}$,

$$\|P_B(\boldsymbol{u}-\boldsymbol{w})\|^2_{\boldsymbol{U}} \lesssim \langle C(\boldsymbol{u}) - C(\boldsymbol{w}), \boldsymbol{u}-\boldsymbol{w}\rangle + \|B(\boldsymbol{u}-\boldsymbol{w})\|^2_{V'}$$

Nonlinear penalty operator

Well-posedness properties (II)

(iii) For all $\boldsymbol{u} \in \boldsymbol{U}$, $C(\boldsymbol{u}) = 0$ in $\mathcal{N}(B)' \Rightarrow C(\boldsymbol{u}) = 0$ in U'

Sketch of the proof

- We have $\mathcal{N}(B) = E(H_0^1(\Omega) \times \boldsymbol{H})$ with $\boldsymbol{H} := \{\boldsymbol{\eta} \in \boldsymbol{H}(\text{div};\Omega) | \nabla\cdot\boldsymbol{\eta} = 0\}$ and

$$E(\psi, \boldsymbol{\eta}) = (\psi, \nabla\psi, \boldsymbol{\eta} - \boldsymbol{\beta}\psi, \gamma(\psi), \gamma_n(\boldsymbol{\eta}))$$

- Then, testing $C(\boldsymbol{u}) = 0$ with $E(0, \boldsymbol{\eta})$ and $E(\psi, 0)$ yield successively
  - Using the orthogonal decomposition $\boldsymbol{L}^2(\Omega) = \nabla H_0^1(\Omega) \oplus \boldsymbol{H}$

$$\langle C(\boldsymbol{u}), E(0, \boldsymbol{\eta})\rangle = (\boldsymbol{\lambda}(|\boldsymbol{\sigma}|)\boldsymbol{\sigma} - \boldsymbol{\rho}, \boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)} = 0 \;\Rightarrow\; \begin{cases} \text{there is } \varphi \in H_0^1(\Omega) \\ \nabla\varphi = \boldsymbol{\lambda}(|\boldsymbol{\sigma}|)\boldsymbol{\sigma} - \boldsymbol{\rho} \end{cases}$$

  - Using Lax-Milgram Th. and assumption $\operatorname{ess\,inf}_\Omega (-\nabla\cdot\boldsymbol{\beta}) \geq 0$

$$\langle C(\boldsymbol{u}), E(\psi, 0)\rangle = (\nabla\varphi, \alpha\lambda_\flat\nabla\psi + \boldsymbol{\beta}\psi)_{\boldsymbol{L}^2(\Omega)} = 0 \;\Rightarrow\; \varphi = 0$$

Penalized mixed formulation

$$\langle C(\boldsymbol{u}), \boldsymbol{w}\rangle_{\boldsymbol{U}',\boldsymbol{U}} = (\boldsymbol{\lambda}(|\boldsymbol{\sigma}|)\boldsymbol{\sigma} - \boldsymbol{\rho}, \alpha\lambda_\flat\theta - \boldsymbol{\eta})_{\boldsymbol{L}^2(\Omega)}$$

$$\langle B\boldsymbol{u}, (v,\boldsymbol{\tau})\rangle_{\boldsymbol{V}',\boldsymbol{V}} = (u, \boldsymbol{\beta}\cdot\nabla v + \nabla\cdot\boldsymbol{\tau})_{L^2(\Omega_h)} + (\boldsymbol{\sigma},\boldsymbol{\tau})_{L^2(\Omega)} + (\boldsymbol{\rho}, \nabla v)_{L^2(\Omega_h)} - \langle\gamma_n(\boldsymbol{\tau}), \hat{u}\rangle_{\partial\Omega_h} - \langle\hat{\rho}, \gamma(v)\rangle_{\partial\Omega_h}$$

Continuous problem Find $\boldsymbol{u} \in \boldsymbol{U}$ such that

$$\langle C(\boldsymbol{u}), \boldsymbol{w}\rangle + \kappa\langle B\boldsymbol{u}, R^{-1}B\boldsymbol{w}\rangle = \kappa\langle F, R^{-1}B\boldsymbol{w}\rangle, \quad \forall \boldsymbol{w}\in\boldsymbol{U}$$

- Well-posed in $\boldsymbol{U}$ for all $\kappa > 0$ large enough
- Problem equivalent to our model problem

Discrete problem Find $\boldsymbol{u}_h \in \boldsymbol{U}_h$ such that

$$\langle C_h(\boldsymbol{u}_h), \boldsymbol{w}_h\rangle + \kappa\langle B_h\boldsymbol{u}_h, R_h^{-1}B_h\boldsymbol{w}_h\rangle = \kappa\langle F_h, R_h^{-1}B_h\boldsymbol{w}_h\rangle, \quad \forall \boldsymbol{w}_h\in\boldsymbol{U}_h$$

- Well-posed in $\boldsymbol{U}_h$ for all $\kappa > 0$ large enough
- Satisfying the quasi-optimal a priori error estimate

$$\|u - u_h\|_U \lesssim (1+\kappa) \inf_{w_h \in U_h} \|u - w_h\|_U$$

Numerical experiment

Low-order scheme

- $\mathbb{P}_k^d(\Omega_h)$: Ω_h-piecewise d-variate polynomials of degree k
- $\mathbb{P}_{1,0}^1(\partial\Omega_h) \subset \mathbb{P}_1^1(\partial\Omega_h)$ the largest subspace s.t. $\gamma(\mathbb{P}_{1,0}^1(\partial\Omega_h)) \subset H_{00}^{1/2}(\partial\Omega_h)$

Discrete trial and test spaces

$$\boldsymbol{U}_h = \mathbb{P}_0^1(\Omega_h) \times \mathbb{P}_0^2(\Omega_h) \times \mathbb{P}_0^2(\Omega_h) \times \mathbb{P}_{1,0}^1(\partial\Omega_h) \times \mathbb{P}_0^1(\partial\Omega_h)$$
$$\boldsymbol{V}_h = \mathbb{P}_2^1(\Omega_h) \times \mathbb{P}_2^2(\Omega_h)$$

Conforming approximation

- $\boldsymbol{U}_h \subset \boldsymbol{U} = L^2(\Omega) \times \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega) \times H_{00}^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h)$
- $\boldsymbol{V}_h \subset \boldsymbol{V} = H^1(\Omega_h) \times \boldsymbol{H}(\mathrm{div};\Omega_h)$

Fortin operator

Assume that $\beta \in \mathbb{P}_0^2(\Omega_h)$. Then, there is $\Pi \in \mathcal{L}(\boldsymbol{V}, \boldsymbol{V}_h)$ such that

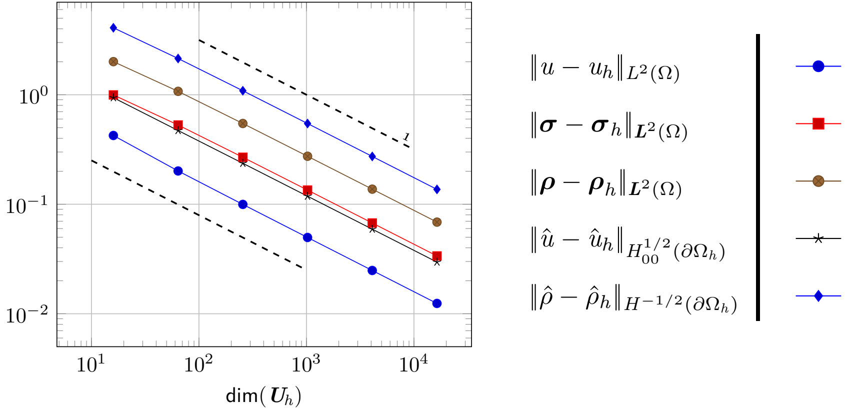
$$\langle B_h \boldsymbol{w}_h, \boldsymbol{v} - \Pi \boldsymbol{v} \rangle = 0 \quad \text{for all } (\boldsymbol{w}_h, \boldsymbol{v}) \in \boldsymbol{U}_h \times \boldsymbol{V}$$

Numerical experiment

Exact solution $u(x, y) = \cos(\pi x/2)\cos(\pi y/2)$ in $\Omega = [-1, 1]^2$

Physical parameters $\boldsymbol{\beta}(x, y) = (y, -x)$ and $\lambda(|\nabla u|) = 2 - (1 + |\nabla u|)^{-2}$

Mesh Uniform refinement of a triangular mesh



➙ Expected behavior $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}} = \mathcal{O}(h)$

Thank you for your attention

- P. Cantin & N. Heuer, 2018
  "A DPG framework for strongly monotone operators", available on HAL.