A compact-stencil scheme on polyhedral meshes for steady transport equations

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Scalar advection-reaction

➞ Find $u \in V_{\beta;2}(\Omega) := \{v \in L^2(\Omega) \,|\, \boldsymbol{\beta}\cdot\nabla v \in L^2(\Omega)\}$ such that

$$\begin{aligned} \boldsymbol{\beta}\cdot\nabla u + \mu u = s \quad &\text{in } \Omega, \\ u = u_D \ &\text{on } \partial\Omega^-. \end{aligned}$$

Physical parameters $\boldsymbol{\beta} \in \mathbf{Lip}(\Omega)$ and $\mu \in L^\infty(\Omega)$

Data $s \in L^2(\Omega)$ and $u_D \in L^2(|\boldsymbol{\beta}\cdot\boldsymbol{n}|\,;\partial\Omega)$

Outflow/inflow boundary $\partial\Omega^{\pm} := \{\boldsymbol{x} \in \partial\Omega \,|\, \pm\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x}) > 0\}$

The problem is well-posed in $V_{\beta;2}(\Omega)$ if $\tau^{-1} := \text{ess inf}_\Omega \left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) > 0$

Objectives

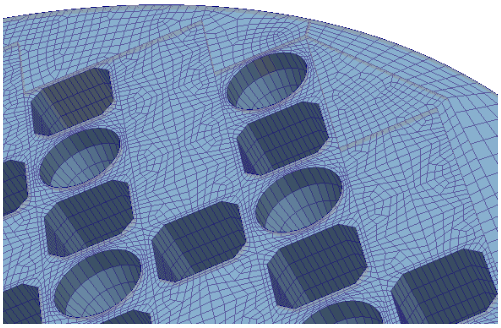
1) Low-order approximation using degrees of freedom at mesh vertices

2) Approximation on 3D general meshes (polyhedral/non-conforming)

Why polyhedral meshes?

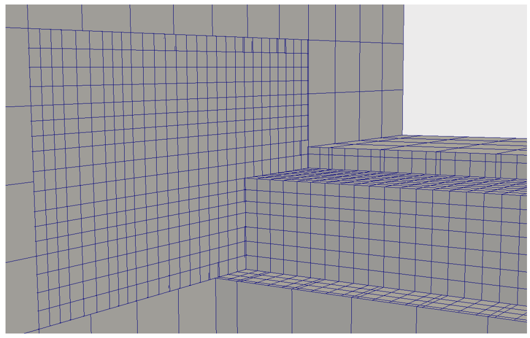
➙ Complex industrial geometries

- Multi-element mesh

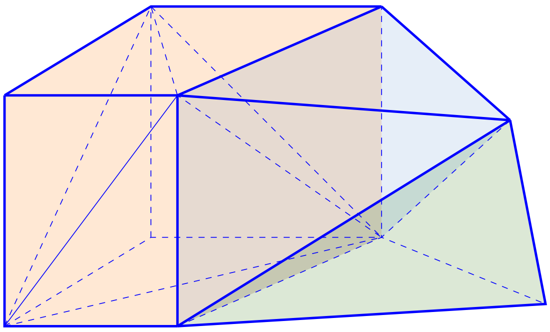


➙ Non-conforming interfaces

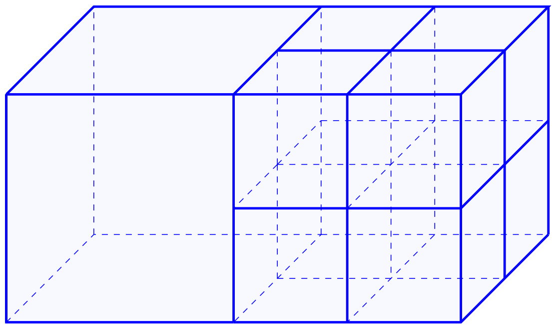
- Mesh agglomeration



- Reduced mesh cardinalities

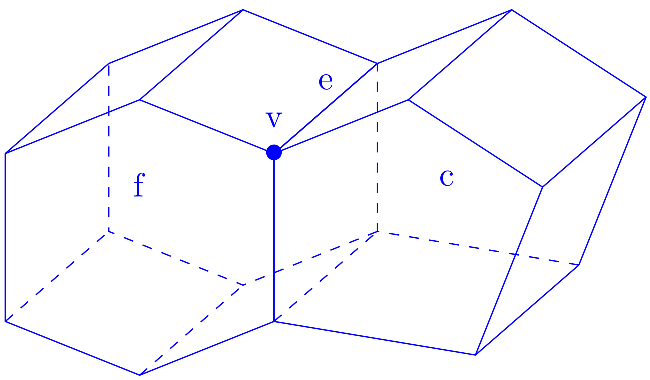


- Locally refined mesh



Polyhedral meshes and CDO tools

➙ Let M a polyhedral mesh of $\Omega \subset \mathbb{R}^3$ composed of



Cells	$\mathrm{c} \in \mathrm{C}$	$\mathcal{C}$
Faces	$\mathrm{f} \in \mathrm{F}$	
Edges	$\mathrm{e} \in \mathrm{E}$	
Vertices	$\mathrm{v} \in \mathrm{V}$	$\mathcal{V}$

Comparison of low-order approaches

Stability norm

$\tau^{-\frac{1}{2}}\|\cdot\|_{L^2(\Omega)}$ + Stab. + BCs

Stronger graph norm

Stability norm $+h^{\frac{1}{2}}\|\boldsymbol{\beta}\cdot\nabla(\cdot)\|_{L^2(\Omega)}$

$\mathbb{P}_1$-dG scheme	$\mathcal{O}(h^{\frac{3}{2}})$ in stronger norm	Polyhedral meshes	$4\#C$
CDO scheme*	$\mathcal{O}(h^{\frac{1}{2}})$ in stability norm	Polyhedral meshes	$\#V$
$\mathbb{P}_1$-stabilized FE scheme	$\mathcal{O}(h^{\frac{3}{2}})$ in stronger norm	Simplicial meshes	$\#V$

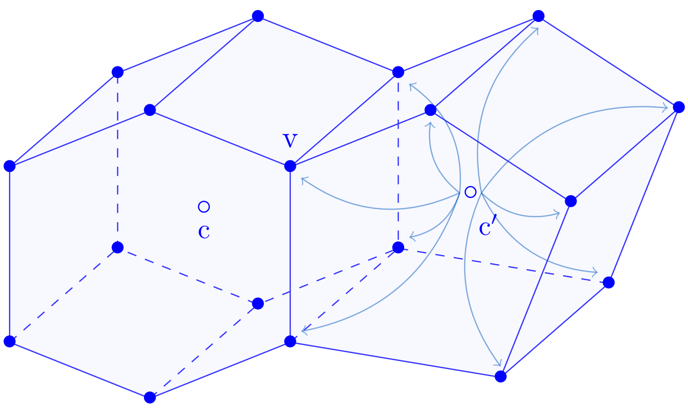
$\mathbb{P}_1$-polyhedral FE scheme	$\mathcal{O}(h^{\frac{3}{2}})$ in stronger norm	Polyhedral meshes	$\#V$

In a nutshell, $\mathbb{P}_1$-polyhedral FE scheme consists of

- Introduction of condensable dofs attached to mesh cells
- Gradient jump penalty across cell sub-faces
- Quasi-optimal convergence rate in L^2-norm of order $\frac{3}{2}$

* Cantin & Ern CMAM 2016, "Vertex-Based Compatible Discrete Operators Schemes on Polyhedral Meshes for Advection-Diffusion Equations".

$\mathbb{P}_1$-polyhedral finite element: guideline



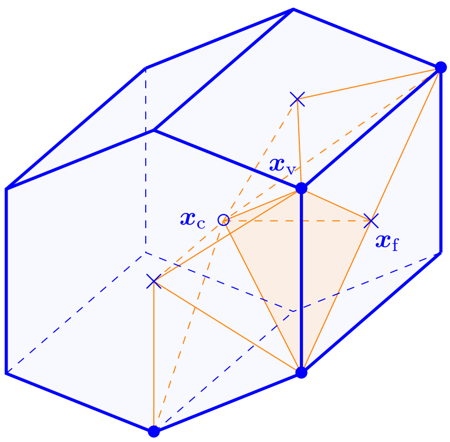
❶ Additional unknowns attached to mesh cells: Dofs space $\mathcal{P} = \mathcal{V} \times \mathcal{C}$
Similar to VAG schemes for elliptic PDEs (*Eymard & al.* '12 & '14)

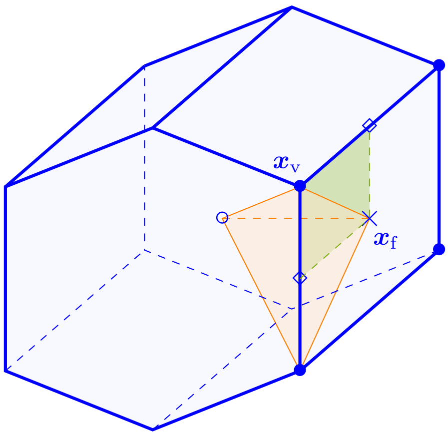
❷ $\mathbb{P}_1$-polyhedral finite element based on a simplicial sub-division

❸ Gradient jump penalty across internal sub-faces for each cell
➙ Cell-based dofs remain uncoupled

❹ Static condensation of $\mathcal{C}$ at modest marginal cost

Geometric simplicial sub-division





① Mesh cells c divided into $2\#\mathrm{E}_{\mathrm{c}}$ tetrahedra $(\mathfrak{C}_{\mathrm{EF,c}})$

→ Nodal Courant shape functions

$$\{\theta_{\mathrm{v}}, \theta_{\mathrm{f}}, \theta_{\mathrm{c}}\} \text{ for all } \mathrm{v} \in \mathrm{V}_{\mathrm{c}},\ \mathrm{f} \in \mathrm{F}_{\mathrm{c}}$$

② Classical $\mathbb{P}_1$-reconstruction: $1 + \#\mathrm{V}_{\mathrm{c}} + \#\mathrm{F}_{\mathrm{c}}$ dofs

③ Geometric $\mathbb{P}_1$-consistent elimination of face dofs

$$\forall v \in \mathbb{P}_1(\mathrm{c};\mathbb{R}),\ v(\boldsymbol{x}_{\mathrm{f}}) = \sum_{\mathrm{v}\in\mathrm{V}_{\mathrm{f}}} \frac{|\mathrm{f} \cap \tilde{\mathrm{c}}(\mathrm{v})|}{|\mathrm{f}|} v(\boldsymbol{x}_{\mathrm{v}})$$

$$v = \sum_{\mathrm{v}\in\mathrm{V}_{\mathrm{c}}} v(\boldsymbol{x}_{\mathrm{v}}) \underbrace{\left(\theta_{\mathrm{v}} + \sum_{\mathrm{f}\in\mathrm{F}_{\mathrm{v}}} \frac{|\mathrm{f} \cap \tilde{\mathrm{c}}(\mathrm{v})|}{|\mathrm{f}|}\theta_{\mathrm{f}}\right)}_{\ell_{\mathrm{v,c}}} + v(\boldsymbol{x}_{\mathrm{c}})\theta_{\mathrm{c}}$$

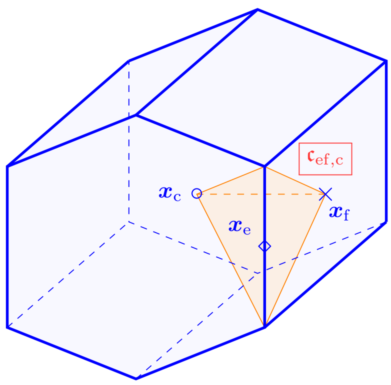
④ Local dofs space $\mathcal{P}_{\mathrm{c}}$ attached to $\mathrm{V}_{\mathrm{c}} \times \{\mathrm{c}\}$

$\mathbb{P}_1$-polyhedral FE $\{\mathcal{P}, \mathsf{L}_{\mathcal{P}}, \mathsf{R}_{\mathcal{P}}\}$

H^1-conforming reconstruction

$$\mathcal{P}_{\mathrm{c}}\ (= \mathcal{V}_{\mathrm{c}} \times \mathcal{C}_{\mathrm{c}}) \to \mathbb{P}_1(\mathfrak{C}_{\mathrm{EF,c}};\mathbb{R}) \cap \mathcal{C}^0(\mathrm{c})$$

$$\mathsf{w} \mapsto \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{w}) = \sum_{\mathrm{v}\in\mathrm{V}_{\mathrm{c}}} \mathsf{w}_{\mathrm{v}}\ell_{\mathrm{v,c}} + \mathsf{w}_{\mathrm{c}}\ell_{\mathrm{c}}$$



Consistency. For all $v \in \mathbb{P}_1(\mathrm{c},\mathbb{R})$, $\mathsf{L}_{\mathcal{P}_{\mathrm{c}}} \circ \mathsf{R}_{\mathcal{P}_{\mathrm{c}}}(v) = v$

$\mathsf{R}_{\mathcal{P}_{\mathrm{c}}}(\cdot)$ point-wise evaluation at mesh cells and vertices

Stability. For all $\mathsf{v} \in \mathcal{P}_{\mathrm{c}}$ and $p \in [1,\infty]$,

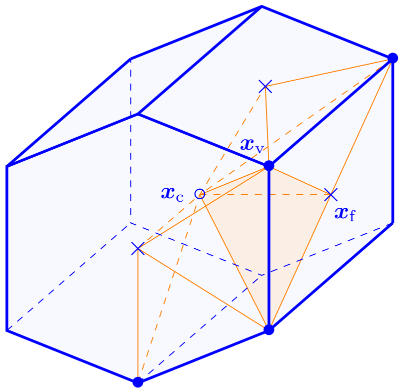
$$\|\mathsf{v}\|_{\mathcal{P}_{\mathrm{c}},p} \lesssim \|\mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{v})\|_{L^p(\mathrm{c})} \lesssim \|\mathsf{v}\|_{\mathcal{P}_{\mathrm{c}},p}, \text{ with } \|\mathsf{v}\|_{\mathcal{P}_{\mathrm{c}},p} := h_{\mathrm{c}}^{\frac{3}{p}} \left(|\mathsf{v}_{\mathrm{c}}|^p + \sum_{\mathrm{v}\in\mathrm{V}_{\mathrm{c}}} |\mathsf{v}_{\mathrm{v}}|^p\right)^{\frac{1}{p}}.$$

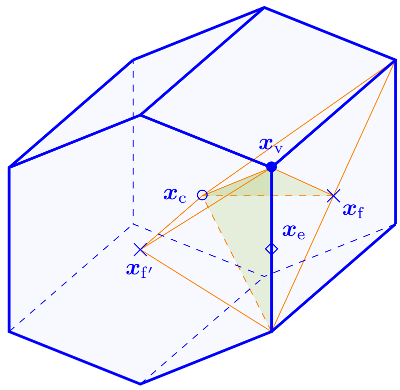
Interpolation. For all $p \in (\frac{3}{2},\infty]$ and for all $v \in W^{2,p}(\mathrm{c})$,

$$\|v - \mathsf{L}_{\mathcal{P}_{\mathrm{c}}} \circ \mathsf{R}_{\mathcal{P}_{\mathrm{c}}}(v)\|_{L^p(\mathrm{c})} + h_{\mathrm{c}}\,|v - \mathsf{L}_{\mathcal{P}_{\mathrm{c}}} \circ \mathsf{R}_{\mathcal{P}_{\mathrm{c}}}(v)|_{W^{1,p}(\mathrm{c})} \lesssim h_{\mathrm{c}}^2\,|v|_{W^{2,p}(\mathrm{c})}$$

Bilinear forms

➞ Bilinear form on $\mathcal{P}_{\mathrm{c}} \times \mathcal{P}_{\mathrm{c}}$ $\quad$ $\mathsf{A}^{\mathcal{P}}_{\boldsymbol{\beta},\mu;\mathrm{c}}(\mathsf{u},\mathsf{v}) = \mathsf{g}_{\boldsymbol{\beta},\mu;\mathrm{c}}(\mathsf{u},\mathsf{v}) + \mathsf{s}_{\boldsymbol{\beta};\mathrm{c}}(\mathsf{u},\mathsf{v})$





- Galerkin formulation

$$\mathsf{g}_{\boldsymbol{\beta},\mu;\mathrm{c}}(\mathsf{u},\mathsf{v}) = \int_{\mathrm{c}} \boldsymbol{\beta}\cdot\nabla \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{u})\, \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{v}) + \int_{\mathrm{c}} \mu \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{u})\, \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{v})$$

- Gradient jump penalty (*Burman & Hansbo* '04, *Burman* '05)

$$\mathsf{s}_{\boldsymbol{\beta};\mathrm{c}}(\mathsf{u},\mathsf{v}) = h_{\mathrm{c}}^2\, |\boldsymbol{\beta}_{\mathrm{c}}|^{-1} \sum_{\mathfrak{f}\in\mathfrak{F}_{\mathrm{EF},\mathrm{c}}} \int_{\mathfrak{f}} (\boldsymbol{\beta}_{\mathrm{c}}\cdot[\![\nabla \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{u})]\!])(\boldsymbol{\beta}_{\mathrm{c}}\cdot[\![\nabla \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{v})]\!])$$

➞ We **only** penalize jumps across inter-cell sub-faces and **not** across faces

$\mathbb{P}_1$-polyhedral finite element scheme

Find $\mathsf{u} \in \mathcal{P}$ s.t., for all $\mathsf{v} \in \mathcal{P}$,

$$\mathbb{A}^{\mathcal{P}}_{\boldsymbol{\beta},\mu}(\mathsf{u},\mathsf{v}) = \mathbb{S}(s, u_D; \mathsf{v})$$

$$\mathbb{A}^{\mathcal{P}}_{\boldsymbol{\beta},\mu}(\mathsf{u},\mathsf{v}) = \sum_{\mathrm{c}\in\mathrm{C}} \mathsf{A}^{\mathcal{P}}_{\boldsymbol{\beta},\mu;\mathrm{c}}(\mathsf{u},\mathsf{v}) + \text{BCs}$$

$$\mathbb{S}(s, u_D; \mathsf{v}) = \int_{\Omega} s\, \mathsf{L}_{\mathcal{P}}(\mathsf{v}) + \text{BCs}$$

Stronger graph norm $\quad$ For all $\mathsf{w} \in \mathcal{P}$

$$\|\mathsf{w}\|^2_{\mathcal{P},\sharp\mathsf{a}} := \sum_{\mathrm{c}\in\mathrm{C}} \tau^{-1} \|\mathsf{w}\|^2_{\mathcal{P}_{\mathrm{c}},2} + h_{\mathrm{c}}\, |\boldsymbol{\beta}_{\mathrm{c}}|^{-1}\, \|\boldsymbol{\beta}\cdot\nabla \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{w})\|^2_{L^2(\mathrm{c})} + \text{Stab.} + \text{BCs}$$

Quasi-optimal local estimate

Let $\mathsf{u} \in \mathcal{P}$ the **discrete** solution and let $u : \Omega \to \mathbb{R}$ the **exact** solution. Assume that $u \in W^{2,p}(\mathrm{C})$ with $p \in (\frac{3}{2}, 2]$. Then

$$\|\mathsf{u} - \mathsf{R}_{\mathcal{P}}(u)\|_{\mathcal{P},\sharp\mathsf{a}} \lesssim \left(\sum_{\mathrm{c}\in\mathrm{C}} \omega_{\mathrm{c}}^{\frac{p}{2}}\, h_{\mathrm{c}}^{3(p-1)}\, |u|^p_{W^{2,p}(\mathrm{c})} \right)^{\frac{1}{p}}$$

Reference velocity $\quad \omega_{\mathrm{c}} := \left(|\boldsymbol{\beta}_{\mathrm{c}}|^{\frac{p}{2}} + h_{\mathrm{c}}^{\frac{p}{2}}\, |\boldsymbol{\beta}|^{\frac{p}{2}}_{\boldsymbol{W}^{1,\infty}(\mathrm{c})} \right)^{\frac{2}{p}}$

Stability analysis

Inf-sup stability under $(\mathcal{H})$

Assume $\tau^{-1} := \text{ess inf}_{\Omega} \left(\mu - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\right) > 0$. Then, for all $\mathsf{v} \in \mathcal{P}$

$$\sup_{\mathsf{w}\in\mathcal{P}} \frac{\mathbb{A}^{\mathcal{P}}_{\boldsymbol{\beta},\mu}(\mathsf{v},\mathsf{w})}{\|\mathsf{w}\|_{\mathcal{P},\sharp\mathsf{a}}} \gtrsim \|\mathsf{v}\|_{\mathcal{P},\sharp\mathsf{a}}$$

- Test function: discrete bubble $\mathsf{w} \in \mathcal{P}$ attached to mesh cells :

$$\mathsf{w}_{\mathrm{v}} = 0 \text{ for all } \mathrm{v} \in \mathrm{V}_{\mathrm{c}}$$

- Bubble intensity: average advective derivative in mesh sub-cells

$$\mathsf{w}_{\mathrm{c}} := h_{\mathrm{c}}\, |\boldsymbol{\beta}_{\mathrm{c}}|^{-1} \frac{1}{\#\mathfrak{C}_{\mathrm{EF,c}}} \sum_{\mathfrak{c}\in\mathfrak{C}_{\mathrm{EF,c}}} \boldsymbol{\beta}_{\mathrm{c}}\cdot\nabla \mathsf{L}_{\mathcal{P}_{\mathrm{c}}}(\mathsf{v})_{|\mathfrak{c}} \text{ for all } \mathrm{c} \in \mathrm{C}$$

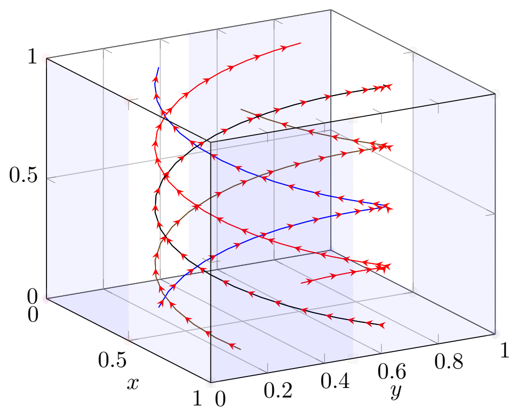
(*Burman & Schieweck* '15)

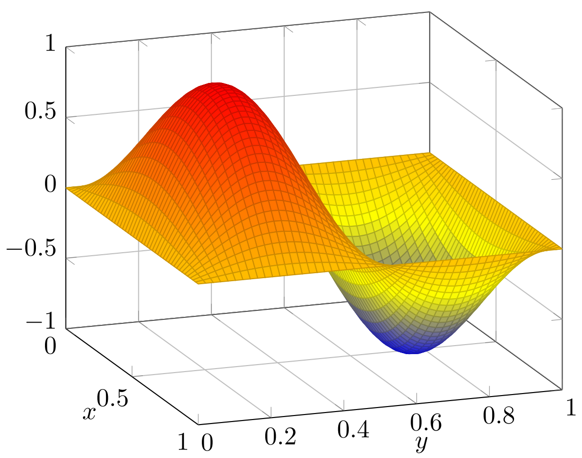
Test case

Physical parameters

$$\boldsymbol{\beta} = \begin{pmatrix} y - 1/2 \\ 1/2 - x \\ z + 1 \end{pmatrix}, \; \mu = 1$$

Friedrichs tensor $\sigma_{\boldsymbol{\beta},\mu} = \mu - \dfrac{1}{2}$



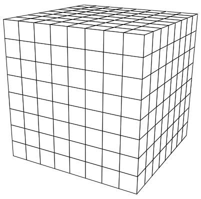


Smooth solution

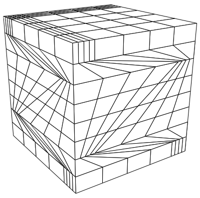
$$u(x, y, z) = \sin(\pi x)\sin(2\pi y)\sin(\pi z)$$

Computational setting

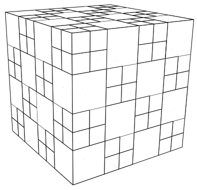
Hexahedral structured meshes



Regular mesh
$\#\mathrm{V} \sim 280\mathrm{k}$

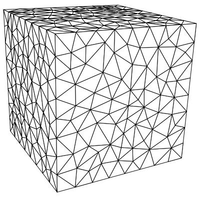


Kershaw mesh
$\#\mathrm{V} \sim 280\mathrm{k}$

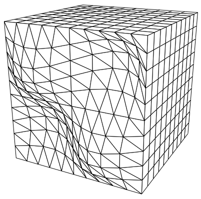


Checkerboard mesh
$\#\mathrm{V} \sim 260\mathrm{k}$

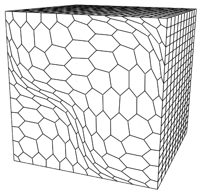
Polyhedral unstructured meshes



Tetrahedral mesh
$\#\mathrm{V} \sim 210\mathrm{k}$



Triangle prismatic mesh
$\#\mathrm{V} \sim 70\mathrm{k}$

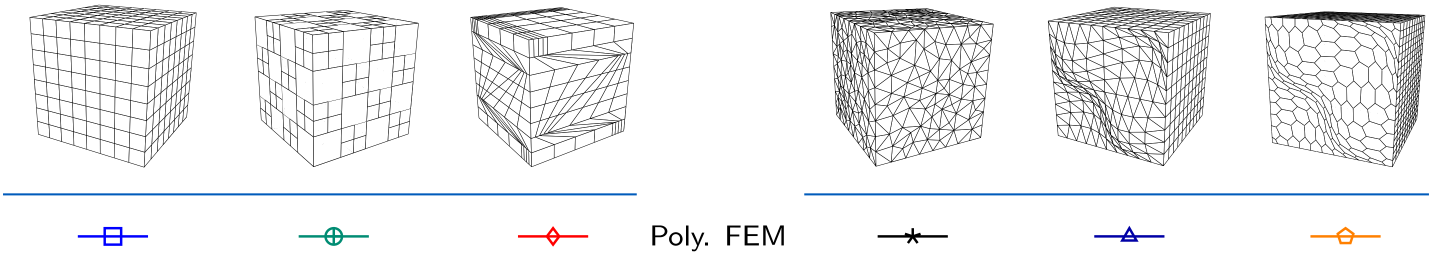


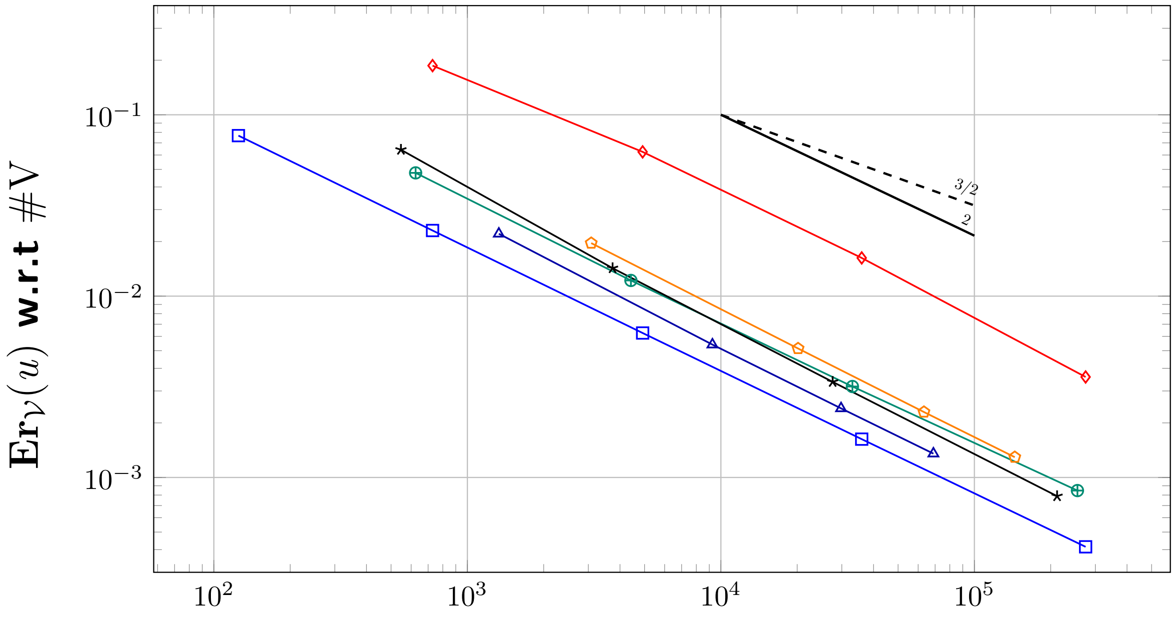
Polygonal prismatic mesh
$\#\mathrm{V} \sim 140\mathrm{k}$

Discrete relative L^2-error

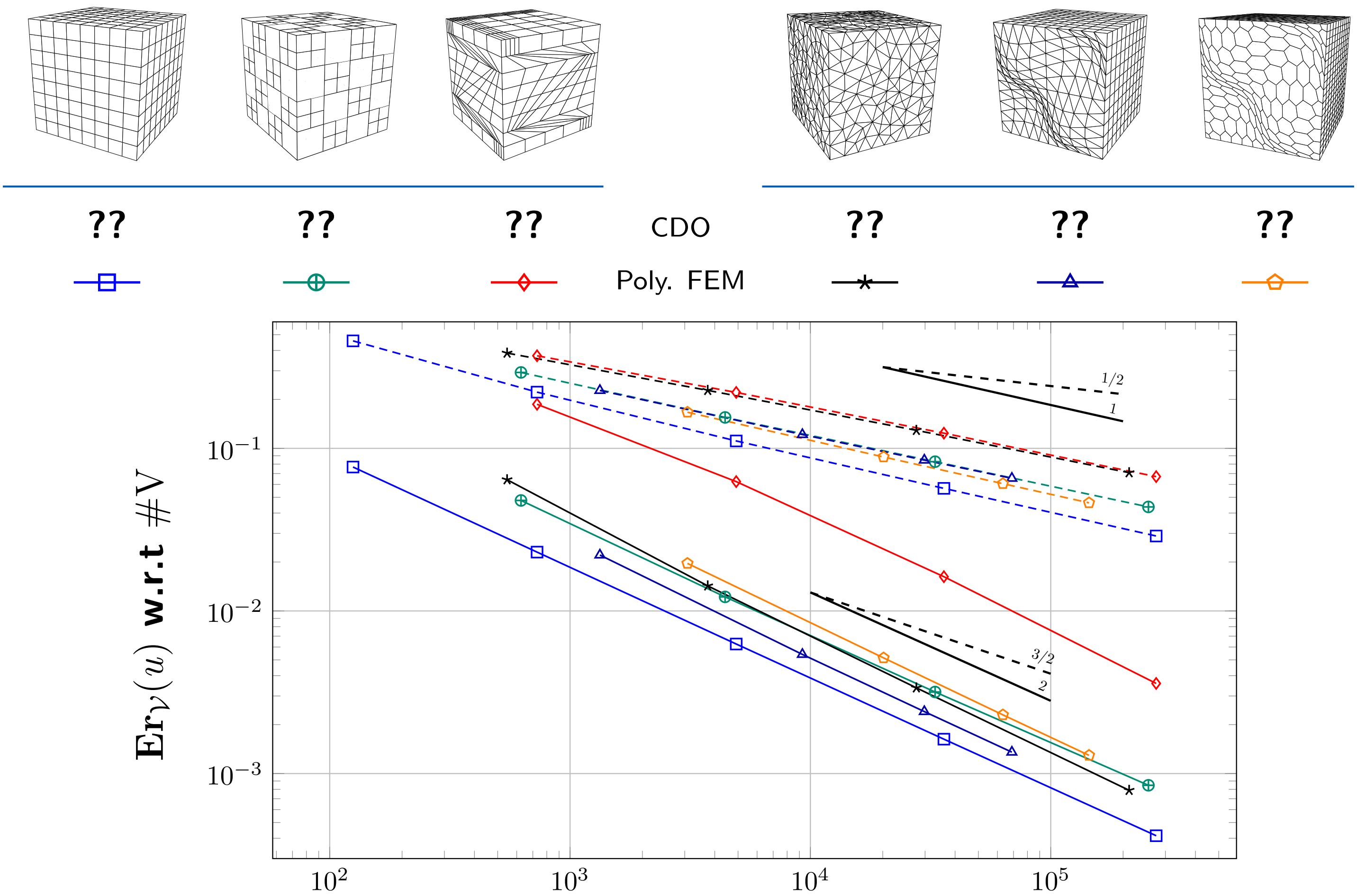
$$\mathbf{Er}_{\mathcal{V}}(u) := \left(\frac{\sum_{\mathrm{v}\in\mathrm{V}} |\tilde{\mathrm{c}}(\mathrm{v})| \, |\mathrm{u}_{\mathrm{v}} - u(\boldsymbol{x}_{\mathrm{v}})|^2}{\sum_{\mathrm{v}\in\mathrm{V}} |\tilde{\mathrm{c}}(\mathrm{v})| \, |u(\boldsymbol{x}_{\mathrm{v}})|^2} \right)^{1/2}$$

$\mathbb{P}_1$-polyhedral finite element scheme





Comparison

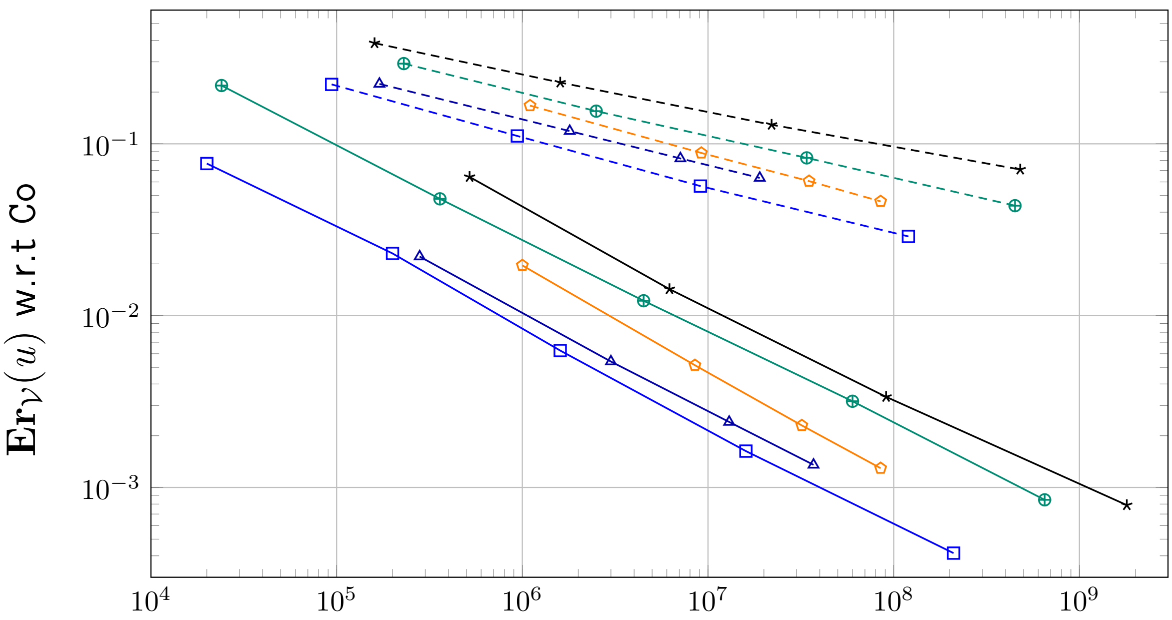


Computational efficiency

Computational cost $\mathtt{Co} = \mathtt{nnz} \times \mathtt{n}_{\mathrm{ITE}}$

$\mathtt{nnz}$ Total number of non-zeros in the final matrix

$\mathtt{n}_{\mathrm{ITE}}$ Number of iterations for a residual $\epsilon = 10^{-12}$ using a biCG/LU solver



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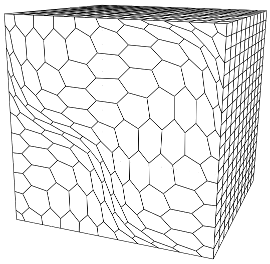

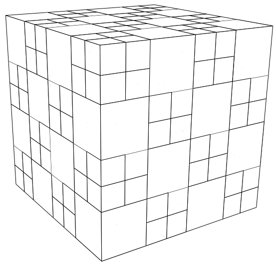

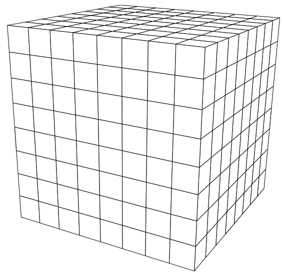

- P. Cantin, J. Bonelle, E. Burman & A. Ern,
"A vertex-based scheme on polyhedral meshes for advection-reaction equations with sub-mesh stabilization", Computers and Mathematics with Applications, 2016.