TATE ALGEBRAS

RICARDO MENARES

ABSTRACT. These notes are intended to be a complement to section 1 of the notes by B. Conrad [Con] on non-archimedean geometry. Most proofs are taken from [BGR84].

CONTENTS

1. INTRODUCTION

Let k be a field and let $|\cdot| : k \to \mathbb{R}_{\geq 0}$ be a non-trivial, ultrametric norm such that the space $(k, |\cdot|)$ is complete. The unit disc $R := \{|x| \leq 1\}$ is a local ring, with maximal ideal $\mathfrak{m} = \{|x| < 1\}$. We denote by $\tilde{k} := R/\mathfrak{m}$ the residue field.

It is a consequence of the ultrametric property that "all triangles are isosceles", meaning that for all $x, y \in k$ such that $|x| \neq |y|$, we have that $|x + y| = \max\{|x|, |y|\}$. This in turn implies that R is a clopen[1] set. Hence, there is a basis of the topology of k made of clopen sets, so k is a totally disconnected topological space. This feature makes it non trivial to construct a useful function theory on k. For instance, a naive definition of "analytic function" $f : k \to k$ would be to ask f to admit an expansion as a convergent power series around every point. The function $f(x) = 1$ if $x \in R$ and $f(x) = 0$ if $x \notin R$ satisfies the naive definition but it's clearly pointless to call such a function analytic.

There are at least two, interrelated, solutions to this dilemma. Both solutions allow, more generally, to build a theory of analytic geometry over non-arquimedean complete fields. We may vaguely summarize them as follows:

[1] Closed and open

- Put a coarser topology on k, such that the naive definition does not include functions as the example above (Tate's approach)
- Embed R in a bigger space which is locally connected (Berkovich's approach)

The Tate algebra in n variables is

$$T_n(k) := \{f(X) = \sum_{J\in(\mathbb{Z}_{\geq 0})^n} a_J X^J : a_J \in k, |a_J| \to 0, |J| \to \infty\},$$

where for the multiindex $J = (j_1, \ldots, j_n)$ and $X = (x_1, \ldots, x_n)$ we define $X^J := \prod_{i=1}^n x_i^{j_i}$ and $|J| := j_1 + \ldots + j_n$. We will write T_n instead of $T_n(k)$ if the underlying field is specified by the context. Elements of T_n can be though of as convergent power series on the polydisk R^n. We define the Gauss norm $\|\cdot\| : T_n \to \mathbb{R}_{\geq 0}$ by $\|f(X)\| := \max |a_J|$. It is not difficult to check that the Gauss norm is an ultrametric norm that endows T_n with a structure of complete k-vector space.

Roughly speaking, the analytic functions we wish to consider are the elements of (a quotient of) some Tate algebra, in analogy with the fact that regular functions of algebraic varieties are elements of (a quotient of) some polynomial ring. In both approaches above the base space is constructed from knowledge of the regular functions on it. Thus, it is not a surprise that in both cases the spaces in question are constructed using ideas of Grothendieck.

This note is a complement to section 1 of the notes by B. Conrad [Con]. We provide proofs for Theorem 1.1.5. and Theorem 1.2.6. of loc. cit. We have taken most of these proofs from the book [BGR84].

This text is divided in two parts. In the first part we establish the basic ring theoretic properties of T_n (noetherianess, Jacobsoness and regularity, properties also enjoyed by polynomial rings $k[x_1, \ldots, x_n]$). Moreover, we will show that all ideals are closed, which allows the Gauss norm to induce a Banach structure on quotients of T_n. The overall strategy to establish these properties is to use induction on n. The device that will make this possible is the theory of *Weierstrass polynomials*, based on the *Weierstrass division theorem* and the *Weierstrass preparation theorem*. Of course, the theory of Weierstrass polynomials was first established over the complex numbers. The ring of holomorphic functions on a complex polydisk enjoys the same ring-theoretic properties as T_n, and this fact can be proved following the same argument that we will give below. Thus, the only difference between the ultrametric and arquimedean setting is in the construction of the Weierstrass theory. In the complex setting this is classically done using the Cauchy formula and Rouche's theorem. For us, the main tool will be the reduction morphism $k \to \tilde{k}$.

The second part concerns the study of quotients of the Tate algebras, the so-called affinoid algebras. These form a category analogous to the category of finite algebras over a field. In particular, they satisfy a version of the Noether normalization lemma. On the other hand, affinoid algebras are Banach spaces. Their analytic theory turns out to be quite simple. For instance, all morphisms are continuous (implying that all complete norms are equivalent). There is also a canonical seminorm $|\cdot|_{\sup}$ associated with any given affinoid algebra, that generalizes the Gauss norm, satisfies a maximum principle and can be used to characterize power-bounded elements.

In order to establish these properties of affinoid algebras, we need to combine tools from dimension theory of commutative rings and notions such as the spectral norm and weakly stable fields.

2. Basic theory of T_n

2.1. Properties of the Gauss norm. It is easy to check that $\|fg\| \leq \|f\|\|g\|$. In particular, the set $T_n^0 := \{\|f\| \leq 1\}$ is a subring of T_n and the canonical morphism $k \to \tilde{k}$ induces a surjective ring homomorphism

$$T_n^0 \to \tilde{k}[x_1, \ldots, x_n] \tag{2.1}$$

which to a given $f = \sum a_J X^J$ assigns the polynomial $\tilde{f} = \sum \tilde{a}_J X^J$.

Proposition 2.1. (1) *For all $f, g \in T_n$, we have that $\|fg\| = \|f\|\|g\|$.*

(2) *Let $B = \{(x_1, \ldots, x_n)$ such that $x_i \in \bar{k}$ and $|x_i| \leq 1\}$. Then we have that*

$$\|f\| = \sup_{X\in B} |f(X)|. \tag{2.2}$$

Proof: (1) By scaling with an appropiate element in k, we may assume $\|f\| = \|g\| = 1$. This means that $\tilde{f}$ and $\tilde{g}$ are nonzero polynomials, whence $\tilde{f}\tilde{g}$ is a nonzero polynomial, implying that $\|fg\| = 1$.

(2) Again we may suppose $\|f\| = 1$. Then we most exhibit some $A \in B$ with $|f(A)| = 1$. We have that

$$|f(X)| < 1 \iff \tilde{f}(\tilde{X}) = 0.$$

Since $\tilde{f}$ is a nonzero polynomial and $\tilde{\bar{k}} = \bar{\tilde{k}}$ is algebraically closed, there is some element $a \in (\tilde{\bar{k}})^n$ such that $\tilde{f}(a) \neq 0$ (e.g. by Hilbert's Nullstellensatz). Hence, we can take A to be any lifting of a ∎

Remarks 2.1.
(1) When the base field is not algebraically closed, the function induced by a nonzero polynomial can be the zero function. Because of this, property (2) of the Gauss norm is not true if k is not algebraically closed and we put k instead of $\bar{k}$ in the definition of B. For example, take $k = \mathbb{Q}_p$ and $f = x^p - x$.
(2) The proof shows that the supremum in (2.2) is actually a maximum and is reached at the "boundary" $\{|x_i| = 1, i = 1, \ldots, n\}$. Moreover, since the subset of $\tilde{\bar{k}}$ where $\tilde{f}$ does not vanish is Zariski open, the set where $|f|$ attains the maximum is quite big. For example, if $n = 1$ and $\bar{k} = k$, we have that

$$R = \bigsqcup_A D^0(A, 1),$$

where A runs trough a set of representatives of $\tilde{k}$ and $D^0(A, 1) = \{x \in k : |x - A| < 1\}$. Then the proof shows in this case that $|f|$ is a constant, equal to the maximum, at all but a finite number of the disks $D^0(A, 1)$. This can be used to give another proof of (1).
(3) Let $I \subset T_n$ be an ideal. We will show in section 2.6 that I is closed. Hence, there is a complete residue norm on $K := T_n/I$, given by

$$|\bar{f}|_K := \inf_{h \in I} \|f + h\|.$$

The function $|\cdot|_K$ is a submultiplicative norm such that the restriction to k is the original norm $|\cdot|$. However, $|\cdot|_K$ need not be multiplicative, even if I is maximal. Here is an example: let $a \in k$ such that $|a| < 1$ and consider the ideal $I = (x^2 - a) \subset T_1$. Since for all $q \in T_1$, we have that $\|x + (x^2 - a)q\| = \max\{\|x\|, \|x^2 - a\|\|q\|\}$ (cf. Theorem 2.1, (2)), we conclude $|\bar{x}|_K = 1$. However, we have that $|\bar{x}^2|_K < 1$. Indeed, we have that

$$\|x^2 - (x^2 - a)\| = |a| < 1.$$

2.2. Units in T_n.

Proposition 2.2. *Let $f \in T_n^0$. The following assertions are equivalent:*

(1) $f \in T_n^*$
(2) $\tilde{f}$ *is a nonzero constant*
(3) $|f(0)| = 1$ *and* $\|f - f(0)\| < 1$

Proof: (2) $\iff$ (3) is clear.

(1) $\Rightarrow$ (2) : $fg = 1$ implies $\tilde{f}\tilde{g} = 1$, so $\tilde{f} \in \tilde{k}[x_1, \ldots, x_n]^* = k^*$.

(2) $\Rightarrow$ (1) : there exists $g \in T_n^0$ s.t. $fg = 1 + u$, where $\|u\| < 1$. But then the series $\sum_{n \geq 0}(-u)^n$ is convergent and furnishes the inverse of $1 + u$, showing that f is invertible ∎

Remark 2.1. It follows from the last part of the proof that T_n^* is open.

Lemma 2.1. *Suppose $\|f\| = 1$. Then there exists $c \in k$ with $|c| = 1$ s.t. $f + c$ is not a unit.*

Proof: take $c = 1$ if $|f(0)| < 1$ and $c = f(0)$ otherwise ∎

Proposition 2.3.

$$\cap_{\mathfrak{m}} \mathfrak{m} = (0),$$

where $\mathfrak{m}$ runs through the set of maximal ideals of T_n.

Proof: if not, then there is some f in the intersection with $\|f\| = 1$. The lemma gives us $c \in k^*$ s.t. $f + c$ is not a unit. Hence, there is a maximal ideal $\mathfrak{m}$ with $f + c \in \mathfrak{m}$. But then $c \in \mathfrak{m}$, which is a contradiction ∎

2.3. **Weierstrass division theorem.**

Definition 2.1. An element $g \in T_n$, $\quad g = \sum_{t=0}^{\infty} g_t(x_1, \ldots, x_{n-1})x_n^t$ is called x_n-distinguished of degree s if the following conditions hold

(1) $g_s \in T_{n-1}^*$
(2) $\|g_s\| = \|g\|$ and $\|g_s\| > \|g_t\|$, for all $t > s$.

Remark 2.2. If $\|g\| = 1$, then g is x_n-distinguished of degree s if and only if $\tilde{g}$ is an unitary polynomial of degree s in $(\tilde{k}[x_1, \ldots, x_{n-1}])[x_n]$.

Theorem 2.1. *(**WDT**) Let $g \in T_n$ be x_n-distinguished of degree s. Then*

(1) *for all $f \in T_n$, there exists an unique $q \in T_n$ and an unique $r \in T_{n-1}[x_n]$ with $\deg r < s$ such that*

$$f = qg + r.$$

(2) *We have that $\|f\| = \max\{\|qg\|, \|r\|\}$.*
(3) *If $f, g \in T_{n-1}[x_n]$, then $q \in T_{n-1}[x_n]$.*

Proof: Assume the existence part of (1). We first prove (2). We may suppose that $\|g\| = 1$. Having done this, we may further suppose that

$$\max\{\|qg\|, \|r\|\} = 1. \tag{2.3}$$

We proceed by contradiction. Assume $\|f\| < 1$. Then we have that

$$0 = \tilde{q}\tilde{g} + \tilde{r}.$$

Since $\deg \tilde{g} = s > \deg r \geq \deg \tilde{r}$, we conclude $\tilde{g} = \tilde{r} = 0$, which is in contradiction with (2.3).

Now we prove uniqueness in (1). From $0 = qg + r$ and part (2), we conclude $q = r = 0$.

To prove part (3), apply euclidean division in $T_{n-1}[X_n]$ and use the uniqueness in (1).

Now we will prove the existence of the representation in (1). We begin with an intermediate result.

Lemma 2.2. *Let $B \subset T_n$ be an additive subgroup. Let $0 < \varepsilon < 1$ be such that for all $f \in T_n$, there exists $b \in B$ such that $\|f + b\| \leq \varepsilon\|f\|$. Then B is dense in T_n.*

Proof: if not, then there exists $f \in T_n$ such that

$$\delta := dist(f, B) > 0.$$

Let $b_1 \in B$ such that $\|f - b_1\| < \delta/\varepsilon$. Then we can choose $b_2 \in B$ such that

$$\|(f - b_1) + b_2\| \leq \varepsilon\|f - b_1\| < \delta.$$

Since the leftmost term is $\geq \delta$, this is a contradiction ■

Now we finish the proof of the existence. We are still supposing $\|g\| = 1$. Let

$$B := \{qg + r : q \in T_n, r \in T_{n-1}[x_n], \deg r < s\}.$$

Using part (2), it is easy to show that B is a closed additive subgroup of T_n. Let $0 < \varepsilon < 1$ be given by

$$\varepsilon := \max\left\{\max_{t>s} \|g_t\|, 1/2\right\}.$$

Let $k_\varepsilon := \{x \in k : |x| \leq \varepsilon\}$. This is an ideal of R. Consider the ring $\tilde{k}_\varepsilon = R/k_\varepsilon$. Let

$$\tau : T_n^0 \longrightarrow \tilde{k}_\varepsilon[x_1, \ldots, x_n]$$

be the natural epimorphism. The polynomial $\tau(g)$ is unitary, so for a given $f \in T_n^0$, we can perform euclidean division in $\tilde{k}_\varepsilon[x_1, \ldots, x_n]$ to obtain $q \in T_n^0$ and $r \in T_{n-1}[x_n]$ with $\deg r < s$ such that

$$\tau(f) = \tau(q)\tau(g) + \tau(r).$$

This means that $\|f - (qg + r)\| \leq \varepsilon$. We have proved that for an arbitrary $f \in T_n$, there is a $b \in B$ such that $\|f - b\| \leq \varepsilon\|f\|$. Lemma 2.2 tells us that B is dense, finishing the proof ■

2.4. **Weierstrass polynomials.**

Definition 2.2. A Weierstrass polynomial is an element $w \in T_{n-1}[x_n]$ which is monic and such that $\|w\| = 1$. We denote by W the set of all Weierstrass polynomials.

In order to stablish ring theoretic properties of T_n (e.g. Noetheriness, Jacobsoness) it will be useful to use that an ideal either contains a Weierstrass polynomial (Theorem 2.2) or that this holds up to an automorphism (Theorem 2.3). Then we will use Theorem 2.4 to pass from a situation in T_n to a situation in T_{n-1}.

Theorem 2.2. *(**WPT**) Let $g \in T_n$ be x_n-distinguished of degree s Then*

(1) *There exists a unique $w \in W$ and a unique $e \in T_n^*$ such that $g = we$.*
(2) *If $g \in T_{n-1}[x_n]$, then $e \in T_{n-1}[x_n]$.*

Proof: by the WDT, there exists $q \in T_n$ and $r \in T_{n-1}[x_n]$ with $\deg r < s$ such that

$$x_n^s = qg + r.$$

Moreover, we have that

$$1 = \max\{\|qg\|, \|r\|\}. \tag{2.4}$$

Define $w := x_n^s - r$. Then w is a monic polynomial in $T_{n-1}[x_n]$, impliying $\|w\| \geq 1$, and since (2.4) implies $\|w\| \leq 1$, we actually have $\|w\| = 1$. Hence, $w \in W$.

Now we may suppose $\|g\| = 1$. Then we have that $\|q\| \leq 1$ and $\tilde{w} = \tilde{q}\tilde{g}$. Since $\tilde{w}$ and $\tilde{g}$ are unitary polynomials of the same degree, we conclude that $\tilde{q}$ is a nonzero constant, i.e. q is a unit.

The unicity assertion is a consequence of the unicity of the representation

$$x_n^s = q^{-1}g + w - x_n^s,$$

coming from the unicity part of the WDT.

The second assertion of the WPT follows from the analogous assertion in the WDT ■

Theorem 2.3. *Let $f \in T_n - \{0\}$. Then there exists an automorphism $\sigma : T_n \to T_n$ such that $\sigma(f)$ is x_n-distinguished.*

Proof: We may suppose $\|f\| = 1$. Let t be the total degree of $\tilde{f}$ and write $f = \sum a_J X^J$. Let $m = (m_1, \dots, m_n) := \max\{J : |a_J| = 1\}$, where we take the lexicographical order on multiindexes.

We define positive integers $c_1, \dots, c_n$ by

$$c_{n-j} = (1+t)^j, \quad j = 1, \dots, n.$$

We put $s := \sum_{i=1}^n m_i c_i$.

We will show that the automorphism defined by

$$\sigma(x_n) = x_n, \quad \sigma(x_i) = x_i + x_n^{c_i}, \quad i = 1, \dots, n-1$$

turns f into an x_n-distinguished element of degree s. We prove first the following
Claim: Let $J = (j_1, \dots, j_n) \neq m$ be such that $|a_J| = 1$. Then $\sum_{i=1}^n j_i c_i < s$.

Indeed, we have that there exists $1 \leq p \leq n$ such that $m_i = j_i$ for $i = 1, \dots, p-1$ and $m_p > j_p$. Then we have that

$$\begin{aligned}\sum_{i=1}^n j_i c_i &\leq \sum_{i=1}^{p-1} m_i c_i + (m_p - 1)c_p + \underbrace{t \sum_{i=p+1}^n c_i}_{c_p - 1} \\ &= \sum_{i=1}^{p} m_i c_i - 1 \\ &< s,\end{aligned}$$

finishing the proof of the claim. Now we have that

$$\widetilde{\sigma(f)} = \sum_J \tilde{a_J} \sum_{\substack{0 \leq k_1 \leq j_1 \\ \dots \\ 0 \leq k_{n-1} \leq j_{n-1}}} \binom{k_1}{j_1} \cdots \binom{k_{n-1}}{j_{n-1}} x_1^{j_1 - k_1} \cdots x_{n-1}^{j_{n-1}-k_{n-1}} x_n^{c_1 k_1 + \ldots + c_{n-1}k_{n-1} + j_n}.$$

If $J \neq m$ and $\tilde{a_J} \neq 0$, the claim ensures that the degree in x_n of the corresponding monomials is strictly less that s. Moreover, the degree in x_n will be equal to s only if $J = m$ and $k_i = m_i = j_i$, showing that $\widetilde{\sigma(f)}$ is an unitary polynomial of degree s, justifying that $\sigma(f)$ is x_n-distinguished of degree s■

If $I \subset T_n$ is a principal ideal, it is a simple check to verify that it is closed. Hence, the quotient space T_n/I ca be endowed with the residue norm. The same remark holds for principal ideals in $T_{n-1}[x_n]$. On the other hand, we endowe a space of the form T_d^m with the norm $\|(t_0, \ldots, t_{m-1})\| := \max\{\|t_i\|\}$.

Theorem 2.4. *Let $w \in W$ have degree s. We define $j : T_{n-1}^s \to T_{n-1}[x_n]$ by*

$$j(t_0, \ldots, t_{s-1}) = \sum_{l=0}^{s-1} t_l x_n^l.$$

Then we have isometric isomorphisms

$$T_{n-1}^s \xrightarrow{\bar{j}} T_{n-1}[x_n]/wT_{n-1}[x_n] \xrightarrow{\bar{i}} T_n/wT_n,$$

where $\bar{j}$ (resp. $\bar{i}$) is the natural map induced by j (resp. the inclusion).

In particular, the natural morphism $T_{n-1} \to T_n/wT_n$ is finite (i.e. the T_{n-1}-module is finitely generated).

Proof: we clearly have that

$$\|j(t_0, \ldots, t_{s-1})\| \leq \|\sum_{l=0}^{s-1} t_l x_n^l\| = \max \|t_l\|.$$

Suppose that $\bar{j}$ is not an isometry, i.e. there exists $\vec{t} \in T_{n-1}^s$ and $q \in T_{n-1}[x_n]$ with

$$\|\sum_{l=0}^{s-1} t_l x_n^l + qw\| < \max \|t_l\|.$$

Then this contradicts (2) in the WDT. Since $\bar{j}$ is an isometry, it is injective. To check the surjectivity, take $f \in T_{n-1}[x_n]$ and perform euclidean division to obtain $f = qw + r$, where $q \in T_{n-1}[x_n]$ and r has the form $r = \sum_{l=0}^{s-1} t_l x_n^s$. Then clearly $\bar{j}(t_0, \ldots, t_{l-1}) = \bar{f}$.

The morphism $\bar{i}$ is an isometry because $wT_{n-1}[x_n]$ is dense in wT_n. The surjectivity is a consequence of the WDT ■

2.5. **T_n is noetherian.** We will prove this by induction on n. We have that $T_0 = k$ is a field, hence noetherian. Suppose that T_{n-1} is noetherian. Let $I \subset T_n$ be a nonzero ideal. We will show that it is finitely generated. The latter property is preserved by automorphisms, hence by combining Theorems 2.3 and 2.2, we may suppose that there is a Weierstrass polynomial $w \in I$.

For any noetherian ring A, we have that $A[x]$ is also noetherian. Hence, using the induction hypothesis, $T_{n-1}[x_n]$ is noetherian. Then by Theorem 2.4 the ideal $\bar{I} \subset T_n/wT_n$ is finitely generated, say by $\bar{\alpha_1}, \ldots, \bar{\alpha_r}$. Then a generating set for I is $\{\alpha_1, \ldots, \alpha_r, w\}$

2.6. **All ideals of T_n are closed.** Let $I \subset T_n$ be an ideal. Since T_n^* is an open set, the closure $\bar{I}$ is a proper ideal of T_n. We know that T_n is noetherian, so we can write $\bar{I} = (\alpha_1, \ldots, \alpha_t)$. Consider the linear function

$$\pi : T_n^t \to \bar{I}$$

given by $\pi(u_1, \ldots, u_t) = \sum_{i=1}^t u_i \alpha_i$. This is a continuous function because of the way the norms are defined. Hence, the Banach open mapping theorem ensures that π is an open map. In particular, if we define $T_n^{00} := \{f \in T_n : \|f\| < 1\}$, the set $\pi\big((T_n^{00})^t\big) = \sum_{i=1}^t T_n^{00} \alpha_i$ is a neighborhood of 0. This shows that

$$\bar{I} = I + \sum_{i=1}^t T_n^{00} \alpha_i. \tag{2.5}$$

Indeed, it is clear that the right hand side is contained in $\bar{I}$. On the other hand, since I is dense in $\bar{I}$, we have that for any $y \in \bar{I}$ there is an $x \in I \cap \left(y + \sum_{i=1}^t T_n^{00} \alpha_i\right)$, justifying (2.5).

Equality 2.5 implies that we can write

$$\alpha_i = f_i + \sum_{j=1}^{t} a_{i,j}\alpha_j, \quad f_i \in I, \quad \|a_{i,j}\| < 1.$$

In matrix notation, we have $(Id - A)\vec{\alpha} = \vec{f}$. We need to show that $Id - A$ is an invertible matrix. But this is a consequence of the fact that the determinant $\det(Id - A)$ has the form $1 + c$, with $\|c\| < 1$, hence is a unit. This shows that $\bar{I} = I$.

Remark 2.3. The last part of the proof is an incarnation of "Nakayama's lemma".

2.7. T_n **is Jacobson.** Again we proceed by induction. Since any field is Jacobson, we have that T_0 has this property. Assume that T_{n-1} is Jacobson. Let $\mathfrak{a} \subset T_n$ be an ideal. We have to show that the intersection of all maximal ideals containing $\mathfrak{a}$ equals the intersection of all prime ideals containing $\mathfrak{a}$. We may suppose that $\mathfrak{a}$ is a prime ideal. We introduce the following notation: for every ring R, we put

$$j(R) := \cap \mathfrak{m},$$

where $\mathfrak{m}$ runs through the maximal ideals of R. Then what we need to show is

$$j\big(T_n/\mathfrak{a}\big) = 0. \tag{2.6}$$

The case $\mathfrak{a} = 0$ has been settled in Proposition 2.3, so we may suppose $\mathfrak{a} \neq 0$. Property (2.6) is preserved by automorphisms, so by combining Theorems 2.3 and 2.2, we may assume that there is a $w \in W \cap \mathfrak{a}$. Let $a := \mathfrak{a} \cap T_{n-1}$. Since Theorem 2.4 ensures that $T_{n-1} \to T_n/wT_n$ is a finite morphism, we have that $T_{n-1}/a \to T_n/\mathfrak{a}$ is also a finite morphism (take the same system of generators). Suppose (2.6) is not true, i.e. there is a nonzero element $x \in j\big(T_n/\mathfrak{a}\big)$. Then the finiteness implies that there is an integral equation, that we take of minimal degree,

$$x^s + b_{s-1}x^{s-1} + \ldots + b_1 x + b_0 = 0, \quad b_i \in T_{n-1}/a.$$

Then $b_0 \in j\big(T_n/\mathfrak{a}\big) \cap T_{n-1}/a$. But this set is contained in $j\big(T_{n-1}/a\big)$ (which is zero by the induction hypothesis). Indeed, any maximal ideal of T_{n-1}/a lifts to a maximal ideal of $T_n/\mathfrak{a}$. Hence, $b_0 = 0$, contradicting the minimality of s.

2.8. **Noether normalization Lemma for k-affinoid algebras.**

Definition 2.3. A k-algebra A is called k-affinoid if there exists an ideal $I \subset T_n$ and an isomorphism of k-algebras $A \simeq T_n/I$.

Example 2.1. Let

$$A = \{\sum_{i=-\infty}^{\infty} a_i x^i, \quad a_i \in k, \quad |a_i| \to 0 \text{ if } |i| \to \infty\}.$$

The k-morphism determined by $x_1 \mapsto x$ and $x_2 \mapsto x^{-1}$ induces an isomorphism

$$A \simeq T_2/(x_1 x_1 - 1),$$

with the inverse map given by $\sum a_i x^i \mapsto \sum_{i\geq 0} a_i x_1^i + \sum_{i<0} a_i x_2^{-i}$.

Theorem 2.5. *(***NNL***)*

1. *For every k-affinoid algebra $A \neq 0$, there is an injective finite morphism $T_d \to A$ for some $d \geq 0$.*
2. *For every finite morphism $\alpha : T_n \to A$, there is a morphism $\tau : T_d \to T_n$ with $d \leq n$ such that $\alpha \circ \tau : T_d \to A$ is injective.*

Proof: The first assertion is a consequence of the second assertion (take the natural map $\alpha : T_n \to T_n/I \simeq A$). We will prove the second assertion by induction on n. If $n = 0$, since T_0 is a field, α is injective. Now suppose the assertion is proven for $n - 1$.

We may assume $\ker \alpha \neq 0$ (since otherwise there is nothing to prove). Then by Theorems 2.2 and 2.3, there is an automorphism $\sigma : T_n \to T_n$ and $w \in W \cap \sigma(\ker \alpha)$. Thus replacing α by $\alpha \circ \sigma^{-1}$, we may suppose that there is a Weierstrass polynomial $w \in \ker \alpha$. Then we have that the induced morphism

$$\bar{\alpha} : T_n/wT_n \to A$$

is finite. But Theorem 2.4 ensures that the natural map $\beta : T_{n-1} \to T_n/wT_n$ is finite, hence the map $\beta \circ \bar{\alpha} : T_{n-1} \to A$ is also finite. We conclude by applying the induction hypothesis to this last map ■

2.9. **Maximal ideals of T_n.** We begin by recalling the following elementary result from commutative algebra.

Lemma 2.3. *Let A, B be integral domains such that there is a finite injective morphism $A \hookrightarrow B$. Then*

(1) *if A is a field, then B is a field*
(2) *if B is a field, then A is a field.*

Proof: assume A is a field. Let $x \in B - \{0\}$. Then x is integral over A, i.e. there is an equation

$$x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0 = 0, \quad a_i \in A, \quad a_0 \neq 0.$$

Then we have that $-a_0^{-1}(x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_1) \in B$ is an inverse for x. Then B is a field.

To prove part (2), take $x \in A - \{0\}$, use that $x^{-1} \in B$ is integral over A and argue as before ■

Proposition 2.4. *Let $\mathfrak{m} \subset T_n$ be a maximal ideal. Then $T_n/\mathfrak{m}$ is a finite extension of k.*

Proof: by the NNL, we have that there is a finite injective morphism $T_d \hookrightarrow T_n/\mathfrak{m}$. By part (2) of Lemma 2.3, we have that T_d is a field. But then $d = 0$ and $T_0 = k$■

Corollary 2.1. *Let $f : A \to B$ be a morphism of k-affinoid algebras and let $\mathfrak{m} \subset B$ be a maximal ideal. Then $f^{-1}(\mathfrak{m})$ is a maximal ideal in A.*

Proof: Proposition 2.4 implies that $B/\mathfrak{m}$ is a k-vector space of finite dimension. Since $\mathfrak{m}$ is prime, the ideal $f^{-1}(\mathfrak{m})$ is also prime. Then $A/f^{-1}(\mathfrak{m}) \to B/\mathfrak{m}$ is an injective morphism between integral k-algebras. Hence, $A/f^{-1}(\mathfrak{m})$ is also a k-vector space of finite dimension. Lemma 2.3, (2.3) then ensures that $A/f^{-1}(\mathfrak{m})$ is a field, i.e. $f^{-1}(\mathfrak{m})$ is maximal ■

Remark 2.4. The preceding corollary is true if we replace "k-affinoid algebras" by "k-algebras of finite type" (e.g. coordinate rings of algebraic varieties). It is not true if we just ask for "k-algebras".

For a k-affinoid algebra A, we denote Specmax(A) the set of its maximal ideals.

Let

$$B^n := \{a = (a_1, \ldots, a_n) \in \bar{k}^n : |a_i| \leq 1\}$$

be the unit polydisk. For $a \in B^n$, we define

$$\tau(a) := \{f \in T_n : f(a) = 0\}.$$

Proposition 2.5. *The preceding rule defines a surjective function $\tau : B^n \to$ Specmax(T_n).*

Before proving this proposition, we establish a technical lemma.

Lemma 2.4. *Let $I \subset T_n$ be an ideal. We endow $L := T_n/I$ with the residue norm that we denote by $|\cdot|_L$. Then*

(1) *For all $y \in k$, we have that $|y| = |\bar{y}|_L$.*
(2) *Assume that I is maximal. Let K be a finite extension of k, that we endow with the induced multiplicative norm from k. Let $\varphi : L \to K$ be a morphism of k-algebras. Then φ is continuous and we have that*

$$|\varphi(\bar{f})| \leq |\bar{f}|_L, \quad \text{for all } f \in T_n.$$

Proof: supose $y \neq 0$ and suppose that there is $f \in I$ such that $\|y + f\| < |y|$. In particular, $|y + f(0)| < |y|$, implying $|f(0)| = |y|$. Moreover, we must have $\|f - f(0)\| < |y| = |f(0)|$. But then Proposition 2.2 implies that $f/f(0)$ is a unit, i.e. f is a unit. But then $I = T_n$, a contradiction. This proves the first part of the assertion.

Now we prove the second part. Note that the continuity of φ is automatic because by Proposition 2.4, L is also a k-vector space of finite dimension and φ is k-linear. Then we have that there exists $C > 0$ such that $|\varphi(\bar{f})| \leq C|\bar{f}|_L$. Applying this inequality to f^n, with n a positive integer, we have that $|\varphi(\bar{f}^n)| \leq C|\bar{f}^n|_L$. Using that $|\cdot|$ is multiplicative and that $|\cdot|_L$ is submultiplicative, we obtain

$$|\varphi(\bar{f})|^n \leq C|\bar{f}^n|_L \leq C|\bar{f}|_L^n,$$

implying that $|\varphi(\bar{f})| \leq C^{1/n}|\bar{f}|_L$. We conclude by letting $n \to \infty$■

Proof of Proposition 2.5 : for $a \in B^n$, consider the "evaluation morphism" $e_a : T_n \to k(a_1, \ldots, a_n)$ given by $e_a(f) := f(a)$. This map is surjective, inducing an isomorphism $T_n/\tau(a) \simeq k(a_1, \ldots, a_n)$. Hence, $\tau(a)$ is maximal.

Let $\mathfrak{m}$ $\in$Specmax(T_n). We consider $T_n/\mathfrak{m}$ as a normed k-vector space with the residue norm. Proposition 2.4 implies that there is an embedding $\iota : T_n/\mathfrak{m} \hookrightarrow \bar{k}$. By Lemma 2.4, ι is continuous and if we put $a_i := \iota(\bar{x_i})$, we have that $|a_i| \leq 1$. Hence $a = (a_1, \dots, a_n) \in B^n$.

On the other hand, the canonical map $l : T_n \to T_n/\mathfrak{m}$ is also continuous. The maps $\iota \circ e_a$ and $\iota \circ l$ are continuous and coincide on $(x_1, \dots, x_n)$, hence they are equal. We conclude $\tau(a) = \mathfrak{m}$ ■

Proposition 2.6. *Let* $\mathfrak{m} \in$ *Specmax* (T_n) *and let* $\mathfrak{m}' := \mathfrak{m} \cap k[x_1, \dots, x_n]$. *Then* m' *is a maximal ideal in* $k[x_1, \dots, x_n]$, *we have that* $\mathfrak{m} = \mathfrak{m}'T_n$ *and* $k[x_1, \dots, x_n]/\mathfrak{m}' \simeq T_n/\mathfrak{m}$.

Proof: using Proposition 2.5, write $\mathfrak{m} = \tau(a)$ with $a \in B^n$. Then $h_a : k[x_1, \dots, x_n] \to k(a) \simeq T_n/\mathfrak{m}$ given by $h_a(f) := f(a)$ induces an isomorphism $\bar{h}_a : k[x_1, \dots, x_n]/\mathfrak{m}' \simeq T_n/\mathfrak{m}$, showing that $\mathfrak{m}'$ is maximal.

We have a commutative diagram

$$\begin{array}{ccc} k[x_1, \dots, x_n]/\mathfrak{m}' & \xrightarrow{\iota} & T_n/\mathfrak{m}'T_n \\ \downarrow \bar{h}_a & \swarrow \pi & \\ T_n/\mathfrak{m} & & \end{array}$$

Since $\bar{h}_a$ is bijective, we have that π is surjective and ι is injective. Then $L := \iota\Big(k[x_1, \dots, x_n]/\mathfrak{m}'\Big)$ is a field of finite dimension over k. Hence, it is complete, implying that it is closed in $T_n/m'T_n$. On the other hand, L is dense because $k[x_1, \dots, x_n]$ is dense in T_n. Hence, $L = T_n/m'T_n$, i.e. ι is surjective. This implies that π is injective, showing that $\mathfrak{m} = \mathfrak{m}'T_n$ ■

Corollary 2.2. *Let* $\mathfrak{m} \in$ *Specmax*(T_n). *Then there exist* n *polynomials* $p_i \in k[x_1, \dots, x_i]$, *monic in* x_i, *such that*

(1) $\mathfrak{m} = (p_1, \dots, p_n)_{T_n}$ *and* $\mathfrak{m}' = (p_1, \dots, p_n)_{k[x_1, \dots, x_n]}$.
(2) *if we represent* $m = \tau(a)$, *then* $k[x_1, \dots, x_i]/(p_1, \dots, p_i) \simeq k(a_1, \dots, a_i)$.

Proof: use Proposition 2.6 to reduce the problem to the known case of maximal ideals of polynomial algebras ■

2.10. T_n **is a regular ring.** We begin by recalling some facts about dimension theory of rings. Let R be a ring and $\mathfrak{p}$ a prime ideal. We define the height of $\mathfrak{p}$ by

$$ht(\mathfrak{p}) := \sup\{n : \text{ there exists a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

We define the dimension of R by $\dim(R) := \sup\{ht(\mathfrak{p}) : \mathfrak{p} \subset R \text{ is a prime ideal}\}$.

Let A be a local ring with maximal ideal $\mathfrak{m}$. We have that $\dim A = ht(\mathfrak{m})$. The set $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field $A/\mathfrak{m}$.

Proposition 2.7. *([AM69], Corollary 11.15) If* A *is noetherian, then* $\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

Lemma 2.5. *We have that* $\dim R = \sup\{\dim R_{\mathfrak{m}} : \mathfrak{m}$ *is a maximal ideal of* $R\}$.

Proof: if the biggest ideal in a chain of prime ideals is not maximal, then the chain can be extended by adding a maximal ideal ■

Definition 2.4.
- A local ring A is called regular if $\dim A = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.
- A ring R is called regular if for all prime ideals $\mathfrak{p} \subset R$, the local ring $R_{\mathfrak{p}}$ is regular.

Remarks 2.2.
(1) Let X be a smooth manifold and let $P \in X$. Let A be the ring of germs at P. Then A is a local ring with maximal ideal $\mathfrak{m}$ consisting of germs of functions vanishing at P. The residue field is $\mathbb{R}$. The ideal $\mathfrak{m}^2$ consists of germs of functions vanishing to order 2 at P. By looking at Taylor expansions, we see that the $\mathbb{R}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ can be identified with the space of derivations at P, hence its dimension equals the dimension of X. On the other hand, if P were a singular point, then the space of derivations would have bigger dimension that $\dim X$. Hence, A being regular encodes the fact that X is smooth at P.
(2) If $\mathfrak{p}$ is a prime ideal of a regular local ring A, then $A_{\mathfrak{p}}$ is regular ([Mat80], p. 139). Hence, to verify that a given ring R is regular, it is enough to check the definition only for maximal ideals. Indeed, if $\mathfrak{m}$ is a maximal ideal and $\mathfrak{p} \subsetneq \mathfrak{m}$ is a prime ideal, we have that $R_{\mathfrak{p}} \simeq \big(R_{\mathfrak{m}}\big)_{\mathfrak{p}}$. In the preceding interpretation, this amounts to say that if the subvarieties of dimension 0 of the mainfold X are smooth, then all subvarieties of X are smooth.

Proposition 2.8. *For every maximal ideal* $\mathfrak{m} \subset T_n$*, we have that* $(T_n)_\mathfrak{m}$ *is a regular ring of dimension* n*.*

Proof: consider the polynomials $p_1, \dots, p_n$ given by Corollary 2.2. Since $\mathfrak{m}$ can be generated by n elements, we have [2] that $\dim_{T_n/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq n$. On the other hand, the n ideals $\mathfrak{m}_i := (p_1, \dots, p_i)$ for $i = 1, \dots, n$ are prime. Indeed, evaluation at $(a_1, \dots, a_i)$ shows that $T_n/\mathfrak{m}_i \simeq T_{n-i}\big(k(a_1, \dots, a_i)\big)$, which is an integral domain. Hence, $\mathrm{ht}(\mathfrak{m}) \geq n$. We conclude that

$$n \leq \dim(T_n)_\mathfrak{m} \leq \dim_{T_n/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq n,$$

proving the assertion ■

Corollary 2.3. *We have that* $\dim T_n = n$*.*

Proof: the chain of prime ideals $0 \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \dots \subsetneq (x_1, \dots, x_n)$ shows that $\dim T_n \geq n$. The combination of Lemma 2.5 and Proposition 2.8 implies $\dim T_n \leq n$■

3. Basic theory of affinoid algebras

It is a consequence of section 2.5 that affinoid algebras are Noetherian. Let $\mathscr{A}$ be a k-affinoid algebra. We denote by $M(\mathscr{A})$ the set of maximal ideals of $\mathscr{A}$. Because of Proposition 2.4, we have that for any $\mathfrak{m} \in M(\mathscr{A})$, the field $A/\mathfrak{m}$ is a finite extension of k. Hence, we can choose an embedding $\iota : A/\mathfrak{m} \hookrightarrow \bar{k}$. For $f \in \mathscr{A}$, we define

$$|f(\mathfrak{m})| := |\iota(f + \mathfrak{m})|.$$

The unicity of the norm on $\bar{k}$ implies that $|f(\mathfrak{m})|$ does not depend on the choice of the embedding.

Definition 3.1. (1) A k-Banach space is a k-vector space V toghether with a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ such that
- $\|v\| = 0 \Leftrightarrow v = 0$
- $\|v + v'\| \leq \max\{\|v\|, \|v'\|\}$ for all $v, v' \in V$
- $\|cv\| = |c|\|v\|$ for all $c \in k$ and $v \in V$
- $(V, \|\cdot\|)$ is complete

(2) A k-Banach algebra is a k-algebra $\mathscr{A}$ toghether with a function $\|\cdot\|$ such that $(\mathscr{A}, \|\cdot\|)$ is a k-Banach space and such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathscr{A}$.

Remarks 3.1.
- As a consequence of section 2.6, we have that a k-affinoid algebra $\mathscr{A} \simeq T_n/I$, toghether with the residue norm induced by the Gauss norm on T_n, is a k-Banach algebra.
- A k-linear map between k-Banach spaces $L : V \to V'$ is continuous if and only if there exists $C > 0$ such that $\|L(v)\|' \leq C\|v\|$ for all $v \in V$. The proof of this statement is the same as in the case of real Banach spaces. This proof depends on the possibility of scalling a vector $v \in V$ to put it inside an arbitrarily small ball. Hence, the proof does not work if the norm on k is trivial.

3.1. All morphisms are continuous.

Theorem 3.1. *Let* $\mathscr{A}$ *be a* k*-affinoid algebra, endowed with the* k*-Banach algebra structure induced by the Gauss norm. Let* $\mathscr{B}$ *be a* k*-affinoid algebra endowed with a* k*-Banach algebra structure. Let* $\varphi : \mathscr{A} \to \mathscr{B}$ *be a* k*-algebra morphism. Then* φ *is continuous.*

Remark 3.1. To derive further consequences, it is important not impose on $\mathscr{B}$ the k-Banach structure induced by a Tate algebra and work instead with an arbitrary structure.

A proof of this theorem will be given by the end of this section. First, we recall Krull's ideal theorem, which is an algebraic version of the fact that an analytic function such that all of its derivatives vanish at a point, must be zero in a neighborhood of the point (cf. Remark 2.2, (1)).

Theorem 3.2. *(Krull) Let* A *be a noetherian local ring with maximal ideal* $\mathfrak{m}$*. Then we have that*

$$\cap_{l \geq 1} \mathfrak{m}^l = (0).$$

Lemma 3.1. *We have that*

$$\cap_{\mathfrak{m} \in M(\mathscr{B})} \cap_{l \geq 1} \mathfrak{m}^l = (0).$$

[2]For a maximal ideal $\mathfrak{m}$ of a ring R, we have that $R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} \simeq R/\mathfrak{m}$.

Proof: Let f be an element in the intersection. Fix $\mathfrak{m} \in M(\mathscr{B})$. By Krull's theorem, we have that

$$f = 0 \text{ in } \mathscr{B}_{\mathfrak{m}},$$

i.e. there exists $t \notin \mathfrak{m}$ such that $tf = 0$.

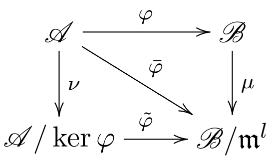
The set $Ann(f) := \{b \in \mathscr{B} : bf = 0\}$ is an ideal in $\mathscr{B}$, and we have proved that it is not contained in any maximal ideal. Then $1 \in Ann(f)$, i.e. $f = 0$ ∎

Remark 3.2. Since T_n is a Jacobson ring, the preceding result is new only if $\mathscr{B} \simeq T_n/I$ with I a non prime ideal.

Lemma 3.2. *Let $\mathfrak{m} \in M(\mathscr{B})$ ant let l be a positive integer. Then $\mathscr{B}/\mathfrak{m}^l$ is a k-vector space of finite dimension.*

Proof: by Noether's normalization lemma, we have that there is a finite injective morphism $\iota' : T_d \hookrightarrow \mathscr{B}/\mathfrak{m}^l$. We compose this morphism with the canonical map $\mathscr{B}/\mathfrak{m}^l \to \mathscr{B}/\mathfrak{m}$ to obtain a morphism $\iota : T_d \to \mathscr{B}/\mathfrak{m}$. We have that ι is also finite (take the same generators as for ι') and injective (use that T_d has no nilpotent elements). We conclude by Lemma 2.3 that T_d is a field, i.e. $d = 0$ and $T_0 = k$ ∎

Proof of Theorem 3.1: We will use the closed graph theorem. Let $(a_n) \subset \mathscr{A}$ be a sequence such that $\lim a_n = 0$ and $\lim \varphi(a_n) = b \in \mathscr{B}$. Consider the commutative diagram



The construction of the residue norm shows that the canonical map ν is 1-Lipschitz, hence continuous. The map $\tilde{\varphi}$ is injective, so by Lemma 3.2 we conclude that $\mathscr{A}/\ker\varphi$ is a k-vector space of finite dimension. Hence, $\tilde{\varphi}$ is continuous. This implies that $\bar{\varphi}$ is continuous.

We have that $\lim \bar{\varphi}(a_n) = 0 = \mu(b)$, that is $b \in \mathfrak{m}^l$. But then $b = 0$ because of Lemma 3.1 ∎

Corollary 3.1. *Let $\mathscr{A}$ be a k-affinoid algebra. Then any two k-Banach algebra structures $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, that is there exist $c_1, c_2 > 0$ such that*

$$\|v\| \leq c_1\|v\|' \leq c_2\|v\| \textit{ for all } v \in \mathscr{A}.$$

Proof: it is enough to show that any k-Banach algebra structure is equivalent to the structure induced by the Gauss norm on T_n. This can be proved by applying Theorem 3.1 to the identity map ∎

3.2. **Finite algebras over an affinoid algebra are affinoid.** The goal of this section is to prove the following

Theorem 3.3. *Let $\mathscr{A}$ be a k-affinoid algebra and let B be a $\mathscr{A}$-algebra such that the structure morphism $\varphi : \mathscr{A} \to B$ turns B into a finitely generated $\mathscr{A}$-module. Then B is k-affinoid.*

Remark 3.3. As an example, the above theorem shows that powers of the form T_n^r, with the algebra structure given by component wise multiplication, are affinoid. Moreover, this particular case of the Theorem implies the general case.

The analogous of Theorem 3.3 for an algebra A of finite type over a field K and a A-algebra A' such that the structural morphism $\psi : A \to A'$ turns A' into a finitely generated A-module is usually proved as follows: take generators $\{a_1, \ldots, a_r\}$ of A' as an A-module. Then

(1) consider a surjective morphism of K-algebras $\nu : K[y_1, \ldots, y_s] \to A$
(2) use that there is a unique morphism of K-algebras

$$\tilde{\psi} : (K[y_1, \ldots, y_s])[x_1, \ldots, x_r] \to A'$$

such that $\tilde{\psi}|_{K[y_1,\ldots,y_s]} = \psi \circ \nu$ and $\tilde{\psi}(x_i) = a_i$.

(3) $\tilde{\psi}$ is surjective by assumption.

To prove Theorem 3.3 we will follow analogous steps, replacing polynomial algebras by Tate algebras. However, the analogous to step (2) is not straightforward. Since we are dealing with series rather than polynomials, in order to extend φ to a morphism defined by its value on the indeterminates, we need a topology in B such that the map φ is continuous. This is provided by the following

Theorem 3.4. *Assume the hypothesis of Theorem 3.3 and suppose $\mathscr{A} = T_n$. Then there exists a norm $|\cdot|_B$ on B such that $(B, |\cdot|_B)$ is a k-Banach algebra, φ is continuous and $|\varphi(f)b|_B \leq \|f\||b|_B$ for all $f \in T_n$ and $b \in B$ (i.e. φ is 1-Lipschitz).*

Proof of Theorem 3.3: taking an epimorphism $\nu : T_n \to \mathscr{A}$ and replacing φ by $\varphi \circ \nu$ we reduce the problem to the case $\mathscr{A} = T_n$. Consider the norm $|\cdot|_B$ given by Theorem 3.4. Let $\{b_1, \dots, b_r\}$ be a set of generators of B as a T_n-module. We may assume $|b_i|_B \leq 1$ for all $i = 1, \dots, r$. We extend φ to a k-algebra morphism $\tilde{\varphi} : T' := T_n[y_1, \dots, y_r] \to B$ by $\tilde{\varphi}(y_i) = b_i$. Then we have that $\tilde{\varphi}$ is continuous with respect to the topology on T' inherited from T_{n+r}. Indeed, take $f = \sum_{|J|\leq m} f_J Y^J \in T'$. Then we have that

$$\begin{aligned} |\tilde{\varphi}(f)|_B &= |\sum_{|J|\leq m} \varphi(f_J) b^J|_B \\ &\leq \max_{|J|\leq m} |\varphi(f_J) b^J|_B \\ &\leq \max_{|J|\leq m} |\varphi(f_J)|_B \underbrace{|b^J|_B}_{\leq 1} \\ &\leq \max_{|J|\leq m} \|f_J\| \\ &= \|\sum_{|J|\leq m} f_J Y^J\|. \end{aligned}$$

Since $\tilde{\varphi}$ is continuous and T' is dense in T_{n+r}, then there is a unique extension $\tilde{\varphi} : T_{n+r} \to B$. This map is surjective, thus showing that B is k-affinoid ■

Now we will prove some intermediate results leading to a proof of Theorem 3.4.

Lemma 3.3. *For every positive integer r, we endow $T := T_n^r$ with the product topology and with the algebra structure given by component wise multiplication. Then T is Noetherian, the group of invertible elements T^* is open and all ideals of T are closed.*

Proof: the T_n-module T is finitely generated, hence noetherian. We have that $T^* = (T_n^*)^r$ is a product of open sets, hence open. Then, the argument given in section 2.6 applies to show that all ideals of T are closed ■

Lemma 3.4. *Assume the hypothesis of Theorem 3.3 and suppose $\mathscr{A} = T_n$. Then there exists a norm $|\cdot|'$ on B such that $(B, |\cdot|')$ is a Banach space, φ is continuous and there exists $K > 0$ such that*

$$|xy|' \leq K|x|'|y|'$$

for all $x, y \in B$. Moreover, we have that $|\varphi(f)x|' \leq \|f\||x|'$ for all $f \in T_n$ and $x \in B$.

Proof: we have a surjective k-algebra morphism $T_n^r \to B$ over φ for some $r \geq 0$. Then we endow B with the quotient topology. Consider on T_n^r the norm $\|(t_1, \dots, t_r)\| := \max \|t_i\|$. Then we define $|\cdot|'$ as the corresponding residue norm on B. More explicitely, let $x \in B$ and take a representation

$$x = \sum_{i=1}^{r} \varphi(\alpha_i) b_i, \quad \alpha_i \in T_n. \tag{3.7}$$

Then

$$|x|' = \inf_{\varphi(t)=0} \max_{i=1}^{r} \|\alpha_i + t_i\|. \tag{3.8}$$

We have that $|\varphi(f)x|' \leq \|f\||x|'$ for all $f \in T_n$ (in particular, φ is continuous). Indeed,

$$\begin{aligned} |\varphi(f)x|' &= \inf_{\varphi(t)=0} \max_{i=1}^{r} \|f\alpha_i + t_i\| \\ &\leq \inf_{\varphi(t)=0} \max_{i=1}^{r} \|f\alpha_i + f t_i\| \\ &= \|f\||x|'. \end{aligned}$$

A representation of the form (3.7) will be called admissible if $\max \|\alpha_i\| \leq 2|x|'$. From (3.8), it is easy to see that an admissible representation for a given $x \in B$ always exists.

Let $x_1, x_2 \in B$. We choose admissible representations $x_k = \sum_{i=1}^r \varphi(\alpha_{i,k})b_i$ for $k = 1, 2$. Let $C := \max_{i,j=1}^r |b_i b_j|'$. Then we have that

$$\begin{aligned}
|x_1 x_2|' &= |\sum_{i,j=1}^{r} \varphi(\alpha_{i,1}\alpha_{j,2})b_i b_j|' \\
&\leq \max_{i,j=1}^{r} |\varphi(\alpha_{i,1}\alpha_{j,2})b_i b_j|' \\
&\leq \max_{i,j=1}^{r} \|\alpha_{i,1}\alpha_{j,2}\| |b_i b_j|' \\
&\leq C \max \|\alpha_{i,1}\| \|\alpha_{j,2}\| \\
&\leq C \max \|\alpha_{i,1}\| \max \|\alpha_{j,2}\| \\
&\leq 4C|x_1|'|x_2|' \quad \blacksquare
\end{aligned}$$

Proof of Theorem 3.4: using the preceeding lemma, we define

$$|x|_B = \sup_{y \neq 0} \frac{|xy|'}{|y|'}, \quad x \in B.$$

It is easy to check that $|\cdot|_B$ is a k-Banach algebra norm on B that is equivalent to $|\cdot|'$ and satisfies the required properties $\blacksquare$

3.3. **The sup norm.** Let $\mathscr{A}$ be a k-affinoid algebra. We define

$$|f|_{\sup} := \sup_{x \in M(\mathscr{A})} |f(x)|, \quad f \in \mathscr{A}.$$

This function does not always define a norm on $\mathscr{A}$, for a nilpotent element is sent to zero. We summarize the properties we want to establish.

Theorem 3.5. (1) *(Maximum principle) We have that*

$$|f|_{sup} = \max_{x \in M(\mathscr{A})} |f(x)| < \infty.$$

(2) $|\cdot|_{sup}$ *is submultiplicative. Moreover,* $|f^n|_{sup} = |f|_{sup}^n$ *for all positive integers* n
(3) $|\cdot|_{sup}$ *is a norm if and only if* $\mathscr{A}$ *is reduced. In this case,* $(\mathscr{A}, |\cdot|_{sup})$ *is a* k*-Banach algebra*
(4) *We have that* $\{f : (f^n)_{n\geq 0}$ *is bounded* $\} = \{f : |f|_{sup} \leq 1\}$.

Remark 3.4. If $\mathscr{A}$ is reduced, Theorem 3.5 shows that $|\cdot|_{\sup}$ induces the canonical topology on $\mathscr{A}$. Thus, we may regard $|\cdot|_{\sup}$ as a canonical norm, independent of the choice of an epimorphism $T_n \to \mathscr{A}$.

The basic idea to prove the maximum principle for $\mathscr{A}$ is to reduce the problem to the corresponding statement for Tate algebras with the Gauss norm via a finite injective morphism

$$T_d \to \mathscr{A}, \tag{3.9}$$

given by NLL. To carry this idea on, we need to relate the sup norm on $\mathscr{A}$ with the Gauss norm on T_d.

Let $(K, |\cdot|)$ be a valued field[3] and let $K \to A$ be a K-algebra which is integral over K. In this situation, there exists a *spectral norm* on A (cf. section 3.3.2), which is a real valued function on A, canonically attached to $|\cdot|$ and that turns out to be a norm if A is reduced. Hence, a natural idea is to relate the spectral norm to the sup norm in the situation (3.9) above. Since T_d is not a field, and since we do not want to assume $\mathscr{A}$ reduced, we will relate the sup norm on $\mathscr{A}$ with the sup norm on the k-algebra $(\mathscr{A}/y)_{red}$, where $y \in M(\mathscr{A})$ and $(\cdot)_{red}$ is the biggest reduced quotient of $(\cdot)$. The key result to make useful the relation between these norms is Theorem 3.7 : the norms are equal when $\mathscr{A}$ is both k-affinoid and reduced, integral over k. The proof of the maximum principle is achieved in 3.3.3.

To prove the completeness of $|\cdot|_{\sup}$ stated in part (3), we wish to use in (3.9) the fact that T_d is complete and the morphism is finite. However, we need to show that the product topology induced on $\mathscr{A}$ is the same as the topology induced by $|\cdot|_{\sup}$. Note that this is nontrivial because the field $Q(T_d)$ is not complete, implying that inequivalent norms on $\mathscr{A}_{Q(T_d)}$ may exist! However, since $Q(T_d)$ is the quotient field of a complete ring, a form of completeness still survives. This notion is called *weak stability* (cf. section 3.3.4) and turns out to be enough to

[3] i.e. $|\cdot|$ is a multiplicative ultrametric norm on K.

show the completeness of $\mathscr{A}$, as shown in section 3.3.6. Again, a fundamental role is played by the spectral norm.

Recall that for a set $S \subset \mathscr{A}$ being bounded means that for any complete norm $|\cdot|_{\mathscr{A}}$ on $\mathscr{A}$ there exists $C_{\mathscr{A}} > 0$ such that $|f|_{\mathscr{A}} \le C_{\mathscr{A}}$ for all $f \in S$. Assuming that $\mathscr{A}$ is reduced and part (3), the proof of part (4) bowls down to take $|\cdot|_{\mathscr{A}} = |\cdot|_{\mathrm{sup}}$ and using the fact that the sup norm is power multiplicative. Hence, a non trivial argument is necessary only when $\mathscr{A}$ is not assumed to be reduced. Such an argument is given in section 3.3.7 and it is a consequence of a closer study of the spectral norm.

3.3.1. *Sup norm and minimal prime ideals.* Since $\mathscr{A}$ is noetherian, there are only a finite number of minimal prime ideals $\{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$[4]. We denote by $\pi_i : \mathscr{A} \to \mathscr{A}/\mathfrak{p}_i$ the canonical morphism.

We denote by $\mathscr{A}_{red} := \mathscr{A}/j(\mathscr{A})$, where $j(\mathscr{A}) = \{f \in \mathscr{A} : f \text{ is nilpotent }\}$ and we denote by red: $\mathscr{A} \to \mathscr{A}_{red}$ the canonical morphism.

Lemma 3.5. (1) *For all $f \in \mathscr{A}$, we have that*

$$|f|_{sup} = \max_{i=1}^{r} |\pi_i f|_{sup}$$

(2) *we have that $|f|_{sup} = |red(f)|_{sup}$.*

Proof: take a sequence $(x_n) \subset M(\mathscr{A})$ such that $|f(x_n)| \to |f|_{\mathrm{sup}}$. There exists i such that $\mathfrak{p}_i \subseteq x_n$ for an infinity of n. Taking a subsequence, we may assume that this happens for all n. Then we have that

$$\begin{aligned} |\pi_i(f)(x_n/\mathfrak{p}_i)| &= \Big|\pi_i(f)\Big|_{(\mathscr{A}/\mathfrak{p}_i)/(x_n/\mathfrak{p}_i)} \\ &= |f|_{\mathscr{A}/x_n} \\ &= |f(x_n)| \\ &\to |f|_{\mathrm{sup}}. \end{aligned}$$

This proves the first assertion. The second assertion is a consequence of the canonical identification $M(\mathscr{A}) = M(\mathscr{A}_{red})$ given by the fact that $j(\mathscr{A}) = \cap_{x \in M(\mathscr{A})} x$ ■

Remark 3.5. The first assertion allows us to reduce the proof of Theorem 3.5, (1), to the case where $\mathscr{A}$ is a domain.

3.3.2. *Spectral value and spectral norm.* Let $(A, |\cdot|)$ be a ring, together with a nontrivial, submultiplicative, ultrametric norm. Let

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_n \in A[x].$$

We define the spectral value of p by

$$\sigma(p) := \max_{i=1}^{n} |a_i|^{1/i}.$$

Proposition 3.1. (1) *Suppose that $|\cdot|$ is multiplicative. Then we have that $\sigma(pq) = \max\{\sigma(p), \sigma(q)\}$ for all $p, q \in A[x]$. In particular, $\sigma(p^n) = \sigma(p)$ for all positive integers n*

(2) *Let K be a valued field. Suppose that A is a normed K-algebra and that $|\cdot|$ is power multiplicative (i.e. $|a^n| = |a|^n$ for all $a \in A$ and $n \ge 0$). Suppose that we can write $p(x) = \prod_{i=1}^{n}(x - \alpha_i)$, with $\alpha_i \in A$. Then*

$$\sigma(p) = \max |\alpha_i|.$$

Proof: write $q(x) = x^m + b_1 x^{m-1} + \ldots + b_m$ and $pq(x) = x^{m+n} + c_1 x^{m+n-1} + \ldots + c_{m+n}$, where $c_k = \sum_{i+j=k} a_i b_j$ (with conventions $a_0 = b_0 = 1$, etc.). Suppose $\sigma(p) \le \sigma(q)$. Then

$$|c_k| \le \max_{i+j=k} |a_i||b_j| \le \max_{i+j=k} \sigma(p)^i \sigma(q)^j \le \sigma(q)^k.$$

This shows that $\sigma(pq) \le \sigma(q)$.

Suppose that $\sigma(p) = \sigma(q)$. Let $i_0 = \min\{l \ge 1 : |a_l| = \sigma(p)^l\}, j_0 = \min\{l \ge 1 : |b_l| = \sigma(q)^l\}$. Let $k_0 = i_0 + j_0$. Then we have that

$$c_{k_0} = a_{i_0} b_{j_0} + \text{ terms of strictly lower absolute value },$$

[4] Suppose that there is a countable collection of minimal prime ideals $(\mathfrak{p}_i)_{i=1}^{\infty}$. Then $I_n := \cap_{i \ge n} \mathfrak{p}_i$ defines a strictly increasing sequence of proper ideals, contradicting the noetherian property.

implying $|c_{k_0}| = \sigma(q)^{k_0}$ (here's where we use the multiplicativity). Hence, $\sigma(pq) \geq \sigma(q)$.

Suppose that $\sigma(p) < \sigma(q)$. Choose k_0 such that $|b_{k_0}| = \sigma(q)^{k_0}$. Then we have that

$$c_{k_0} = b_{k_0} + \text{ terms of strictly lower absolute value },$$

implying $|c_{k_0}| = \sigma(q)^{k_0}$. Hence, $\sigma(pq) \geq \sigma(q)$. This proves the first assertion.

To prove the second assertion, note that the coefficient a_k is a sum of terms of the form $\pm\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_k}$. Hence, $|a_k| \leq (\max|\alpha_i|)^k$, implying $\sigma(p) \leq \max|\alpha_i|$.

Suppose $\sigma(p) < \max|\alpha_i|$, i.e. there exists $\alpha = \alpha_i$ s.t. $|\alpha|^k > |a_k|$ for all k. Then we have that

$$|\alpha|^n = |\alpha^n| \leq \max\{|a_k||\alpha|^{n-k}\} < |\alpha|^n,$$

a contradiction ■

Let $(K, |\cdot|)$ be a valued field and let $(A, |\cdot|)$ be a K-algebra which is reduced and integral over K. For an element $a \in A$, consider an integral equation of minimal degree

$$q(a) = 0, \quad q(x) = x^n + t_1x^{n-1} + \ldots + t_{n-1}x + t_n.$$

The polynomial q is uniquely determined by a. We define the spectral norm

$$|a|_{\text{sp}} := \sigma(q).$$

Lemma 3.6. *Let A, K be as above. For every prime ideal $\mathfrak{p} \subset A$, we denote by $\pi_\mathfrak{p} : A \to A/\mathfrak{p}$ the canonical morphism. Then we have that*

$$|a|_{sp} = \max_{\mathfrak{p}\in Spec(A)} |\pi_\mathfrak{p}(a)|_{sp}.$$

In particular, $|\cdot|_{sp}$ is a submultiplicative and power multiplicative norm on A.

Proof: let $I_a = \{f \in K[x] : f(a) = 0\}$. We have that I_a is a proper ideal of $K[x]$. Since $K[x]$ is a PID, there exists a unique monic polynomial $q \in K[x]$ such that $I_a = (q)$ [5]. Similarly, we have that $I_{\pi_\mathfrak{p}(a)} = (q_\mathfrak{p})$ for a unique monic $q_\mathfrak{p} \in K[x]$. Furthermore, since $A/\mathfrak{p}$ is a domain, $q_\mathfrak{p}$ is irreducible.

Since $q(\pi_\mathfrak{p}(a)) = \pi_\mathfrak{p}(q(a)) = 0$, we have that $q_\mathfrak{p}$ divides q. In particular, there are only a finite number of pairwise different $q_\mathfrak{p}'s$, say $q_{\mathfrak{p}_1}, \ldots, q_{\mathfrak{p}_r}$. Let $q' := \prod_{i=1}^r q_{\mathfrak{p}_i}$. We have that

$$f \in K[x], f(a) = 0 \Leftrightarrow \forall\mathfrak{p}, \pi_\mathfrak{p}(f(a)) = 0 \Leftrightarrow \forall\mathfrak{p}, f(\pi_\mathfrak{p}(a)) = 0 \Leftrightarrow \forall\mathfrak{p}, q_\mathfrak{p}|f \Leftrightarrow q'|f.$$

This implies that $I_a = (q')$. Hence, $q' = q$. Now we have that

$$\begin{aligned} |a|_{\text{sp}} &= \sigma(q) \\ &= \max_i \sigma(q_{\mathfrak{p}_i}) \\ &= \max_\mathfrak{p} \sigma(q_\mathfrak{p}) \\ &= \max_\mathfrak{p} |\pi_\mathfrak{p}(a)|_{\text{sp}} \blacksquare \end{aligned}$$

Theorem 3.6. *Let K be a finite extension of k. Let $|\cdot|$ be the norm on K induced by the norm on k.*

- *We have that $|a|_{sp} = |a|$ for all $a \in K$. In particular, $|\cdot|_{sp}$ is a norm and we have that $|a|_{sp} = |t_n|^{1/n}$.*
- *Let $|\cdot|_1$ be a submultiplicative norm on K, extending the norm on k. Then*

$$|a| = \inf_{i\geq 1} |a^i|_1^{1/i} = \lim_{i\to\infty} |a^i|_1^{1/i}, \quad \forall a \in K. \tag{3.10}$$

Proof: let $p \in k[x]$ be the minimal polynomial of a over k. Let K' be its splitting field, endowed with the induced norm from k. Let $a_1 = a, a_2, \ldots, a_n$ be the roots of p. Since k is complete, we have that $|a_i| = |a_j|$ for all i, j. Moreover, $|\cdot|$ is power multiplicative, hence Proposition 3.1, (2) implies $|a|_{\text{sp}} = |a|$, justifying the first assertion.

[5] We remark that q may be reducible if A is not a domain.

To prove the second assertion, first note that the limit in (3.10) exists and equals the inf because for fixed $a \in K$, the sequence $i \mapsto |a^i|_1$ is submultiplicative (cf. [BGR84], section 1.3.2.). Let $|a|_2 := \lim_{i\to\infty} |a^i|_1^{1/i}$. We will argue that $|\cdot|_2 = |\cdot|$.

The function $|\cdot|_2$ is an ultrametric, submultiplicative norm (the triangle inequality is elementary, but tricky, and we refer to [BGR84], Proposition 1.3.2.1). From the inf formula we deduce $|\cdot|_2 \leq |\cdot|$. From the limit formula it is easy to see that $|\cdot|_2$ is also power multiplicative. Let K' be the normal closure of K. Since K' is a finite extension of K, all norms can be extended to K' while keeping the multiplicativy or power multiplicativity properties. Let $\sigma : K' \to K'$ be a k-automorphism. The we have that the restriction to K of $|\cdot|_2 \circ \sigma$ is another power multiplicative norm. Since k is complete, these norms must be equivalent. In particular, there exists $C_\sigma > 0$ such that

$$|\sigma(a)|_2 \leq C_\sigma |a|_2 \quad \forall a \in K.$$

This implies that for all positive integers n, we have that $|\sigma(a)|_2 \leq C_\sigma^{1/n}|a|_2$. Hence, we can take $C_\sigma = 1$.

Applying Proposition 3.1, (2) to $|\cdot|_2$, we obtain

$$|a| \leq |a|_2, \quad \forall a \in K,$$

finishing the proof ■

Theorem 3.7. *Let $\mathscr{A}$ be a k-affinoid algebra which is reduced and integral over k. Then we have that*

$$|a|_{sp} = |a|_{sup} \quad \textit{for all } a \in \mathscr{A}.$$

Proof: for every non zero prime ideal $\mathfrak{p} \subset \mathscr{A}$, the domain $\mathscr{A}/\mathfrak{p}$ is a finite extension of k. Hence, it is a field. Let $a \in \mathscr{A}$. Using the notations of Lemma 3.6, we have that

$$\begin{aligned} |a|_{\mathrm{sp}} &= \max_{\mathfrak{p}} |\pi_{\mathfrak{p}}(a)|_{\mathrm{sp}} \quad (\text{Lemma } 3.6) \\ &= \max_{\mathfrak{p}} |\pi_{\mathfrak{p}}(a)| \quad (\text{Theorem } 3.6) \\ &= \max_{\mathfrak{p}} |a(\mathfrak{p})| \\ &= \max_{\mathfrak{p}} |\pi_{\mathfrak{p}}(a)|_{\mathrm{sup}} \\ &= |a|_{\mathrm{sup}} (\text{Lemma } 3.5, (1)) \blacksquare \end{aligned}$$

Lemma 3.7. *Let $\mathscr{A}$ be a k-affinoid algebra which is a domain and let $T_d \hookrightarrow \mathscr{A}$ be a finite monomorphism given by NNL. Let $y \in M(T_d)$ and let $\alpha \in \mathscr{A}$. We denote by*

$$\alpha_y := \textit{ image of } \alpha \textit{ in } \mathscr{A}/y\mathscr{A}.$$

$$\bar{\alpha}_y := \textit{ image of } \alpha \textit{ in } (\mathscr{A}/y\mathscr{A})_{red}.$$

Then we have that

$$|\alpha|_{sup} = \sup_{y\in M(T_d)} |\bar{\alpha}_y|_{sup}.$$

Proof: since $\mathscr{A}$ is integral over T_d, the Cohen-Seidenberg theorem ensures that every maximal ideal in T_d lifts to a maximal ideal of $\mathscr{A}$. We have that

$$\begin{aligned} |\alpha|_{\mathrm{sup}} &= \sup_{y\in M(T_d)} \sup_{x\in M(\mathscr{A}):x\cap T_d=y} |\alpha(x)| \\ &= \sup_{y\in M(T_d)} \sup_{x\in M(\mathscr{A}/y\mathscr{A})} |\alpha(x)| \\ &= \sup_{y\in M(T_d)} |\alpha_y|_{\mathrm{sup}}. \end{aligned}$$

We conclude by the second assertion in Lemma 3.5 ■

3.3.3. *Maximum principle.* **Proof of Theorem 3.5, (1)**: we assume $\mathscr{A}$ is a domain (cf. Remark 3.5). We use the notations of the previous lemma.

Claim 1: for $y \in M(T_d)$, the set $E_y = \{x \in M(\mathscr{A}) : x \cap T_d = y\}$ is finite.

Indeed, let $x \in E_y$ and let $y\mathscr{A} \subseteq \mathfrak{p} \subseteq x$ be a prime ideal of $\mathscr{A}$. Then we have that $\mathfrak{p} \cap T_d = y$. Since the extension $T_d \hookrightarrow \mathscr{A}$ is integral, the Cohen-Seidenberg theorem ensures that $\mathfrak{p}$ is maximal, that is $\mathfrak{p} = x$. Hence, the set E_y injects into the set F of minimal prime ideals of $\mathscr{A}/y\mathscr{A}$. This ring being noetherian, the set is F finite.

Claim 2: for $y \in M(T_d)$, the ring $(\mathscr{A}/y\mathscr{A})_{red}$ is integral over k.

Indeed, we have an injection

$$(\mathscr{A}/y\mathscr{A})_{red} \hookrightarrow \prod_{x \in E_y} \mathscr{A}/x$$

taking $a + y\mathscr{A}$ to $(a + x\mathscr{A})_{x \in E_y}$. Since every field in this finite product is a finite extension of k, this proves Claim 2.

Let

$$\alpha^n + f_1\alpha^{n-1} + \ldots + f_{n-1}\alpha + f_n = 0, \quad f_i \in T_d$$

be the monic integral equation of minimal degree for α over T_d (cf. Claim 2).

We have that

$$\begin{aligned} |\alpha|_{\sup} &= \sup_{y \in M(T_d)} |\bar{\alpha}_y|_{\sup} \quad \text{(Lemma 3.7)} \\ &= \sup_{y \in M(T_d)} |\bar{\alpha}_y|_{\mathrm{sp}} \quad \text{(Theorem 3.7)} \\ &= \sup_{y \in M(T_d)} \max_{i=1}^{n} |f_i(y)|^{1/i}. \end{aligned}$$

Since the maximum principle holds for T_d, there exists $y_0 \in M(T_d)$ such that

$$|\alpha|_{\sup} = \max_{i=1}^{n} |f_i(y_0)|^{1/i} = |\bar{\alpha}_{y_0}|_{sp} = |\bar{\alpha}_{y_0}|_{\sup}.$$

Claim 1 implies that there exists $x_0 \in M(\mathscr{A})$ such that $x_0 \cap T_d = y_0$ and $|\alpha(x_0)| = |\bar{\alpha}_{y_0}|_{\sup}$, finishing the proof ■

3.3.4. *Weak stability.* Let $(A, |\cdot|)$ be a valued ring (i.e. $|\cdot|$ is a nontrivial ultrametric multiplicative norm). Let $(M, |\cdot|)$ be a normed A-module (i.e. $|am| = |a||m|$ for all $a \in A$ and $m \in M$). The module M is said to be b-separable if for all $m \in M - \{0\}$, there exists a bounded A-linear map $\lambda : M \to A$ such that $\lambda(m) \neq 0$.

Let $(K, |\cdot|)$ be a valued field and let V be a vector space over K. Let

$$F(V) = \{U \subseteq V : U \text{ is a finite dimensional vector space over } K\}.$$

Theorem 3.8. *The following statements are equivalent*

1. *For all $U \in F(V)$, there exists a linear homeomorphism $U \to K^n$ for $n = \dim_K U$, where we endow K^n with the product topology*
2. *Every $U \in F(V)$ is closed*
3. *Every $U \in F(V)$ is b-separable*

Proof: Exercice

Definition 3.2.
- V is said to be weakly cartesian if the conditions of the preceeding theorem are fullfilled
- K is said to be weakly stable if for all finite field extensions L/K, we have that $(L, |\cdot|_{\mathrm{sp}})$ is weakly cartesian (in other words, we are asking that $(\bar{K}, |\cdot|_{\mathrm{sp}})$ is weakly cartesian).

Theorem 3.9. *Let K be a valued field which is perfect. Then K is weakly stable. In particular, all valued fields of characteristic 0 are stable.*

Proof: let L/K be a finite extension. Let $u \in L^*$. Since L/K is separable, the trace form $T_{L/K} : L \times L \to K$ is non degenerate. Then there exists $v \in L$ s.t. $T_{L/K}(uv) \neq 0$. Define $\lambda : L \to K$ by $\lambda(x) = T_{L/K}(xv)$. Then λ is a K-linear function with $\lambda(u) \neq 0$.

Let $p(t) \in K[t]$ be the characteristic polynomial of the K-linear map $l : L \to L$ given by $l(y) = yvx$. Let $p_1(t) \in K[t]$ be the minimal polynomial of vx. We have that $p(t) = p_1(t)^m$ for some positive integer m. Writing $p(t) = t^n + a_1t^{n-1} + \ldots + a_n$ we have that

$$|\lambda(x)| = |-a_1| \leq \sigma(p) = \sigma(p_1) = |xv|_{sp} \leq |x|_{\mathrm{sp}}|v|_{\mathrm{sp}},$$

showing that λ is continuous. This shows that $(L, |\cdot|_{\text{sp}})$ is weakly cartesian ■

As the previous theorem suggest, there are (non-perfect) fields in characteristic p which are not weakly stable. There is such an example in appendix A.

Theorem 3.10. *Suppose char $K = p > 0$. Let $K_1 := \{x \in \bar{K} : x^p \in K\}$. Then K is weakly stable if and only if $(K_1, |\cdot|_{sp})$ is weakly cartesian.*

To prove this Theorem we will use the following

Lemma 3.8. *([BGR84], section 2.3.3., Propositions 1 and 2)*

(1) *If V_1, V_2 are weakly cartesian, then $V_1 \times V_2$ is weakly cartesian*
(2) *Let K'/K be an algebraic extension such that K' is K-weakly cartesian. Let V a K'-vector space which is K'-weakly cartesian. Then V, seen as a K vector space, is weakly cartesian.*

Proof of Theorem 3.10: we begin by proving that $K_n := \{x \in \bar{K} : x^{p^n} \in K\}$ is K-weakly cartesian, for all $n \geq 1$.

We proceed by induction. The case $n = 1$ is the hypothesis. Suppose the assertion is true for n. Using Lemma 3.8, (2), we reduce the problem to show that every $U \in F_{K_n}(K_{n+1})$ is closed in K_{n+1}. Consider the Frobenius morphism

$$Fr : K_{n+1} \to K_1$$

defined by $Fr(x) = x^{p^n}$. This morphism is a field isomorphism. Hence, $Fr(U) \in F_K(K_1)$. Moreover, Fr is continuous. Indeed, since the spectral norm is power multiplicative, we have that $|Fr(x)|_{\text{sp}} = |x|_{\text{sp}}^{p^n}$. This proves that a null sequence is taken into a null sequence, i.e. Fr is continuous at 0. A similar argument shows that the inverse of Fr is also continuous, so Fr is an homeomorphism. Since $Fr(U)$ is closed in K_1, we have that U is closed.

Now let $K_\infty = \cup_{n=1}^{\infty} K_n$. The above assertion implies that K_∞ is K-weakly cartesian. On the other hand, since K_∞ is perfect, we have that $\bar{K}$ is K_∞-weakly cartesian. Using Lemma 3.8, (2) we conclude that $\bar{K}$ is K-weakly cartesian ■

Corollary 3.2. *Let A be a valued k-algebra which is a domain and such that char $p > 0$. Let $K = Q(A)$ be the quotient field and let $A_1 := \{x \in K_1 : x^p \in A\}$. Suppose that every finitely generated A-submodule of A_1 is b-separable. Then K is weakly stable.*

Proof: we begin by showing that

$$K_1 = \{\frac{x}{a} : x \in A_1, a \in A - 0\}. \tag{3.11}$$

The inclusion " $\supseteq$" is clear. To prove the opposite inclusion, take $z \in K_1$. Since z^p belongs to K, we can write $z^p = b/a$ with $b, a \in A$ and $a \neq 0$. Let $x := az$. We have that $x^p = a^{p-1}b^p \in A$, hence, $x \in A_1$.

Let $U \in F(K_1)$ and let $\{v_1, \ldots, v_n\}$ be a basis. Because of (3.11), we can assume that $v_i \in A_1$. The finitely generated module A-module $N := \oplus A v_i$ is contained in A_1, hence, it is b-separable.

Let $u \in U - \{0\}$. Then there exists $c \in k$ such that $cu \in N$. Let $\lambda' : N \to A$ be a bounded A-linear map such that $\lambda'(cu) \neq 0$. But then there is a unique bounded K-linear extension $\lambda : U \to K$ and we have that $\lambda(u) \neq 0$■

Proposition 3.2. *Let $(V, |\cdot|)$ be a normed k-vector space of countable dimension[6]. Then V is b-separable.*

The proof of the Proposition is a consequence of the following "Gram-Schmidt algorithm":

Lemma 3.9. *Let $\{v_i\}$ be a basis of V. For all $\alpha, \rho > 1$, there exists a basis $Y = \{y_i\}$ such that*

(1) *$1 \leq |y_i| \leq \rho$, for all i*
(2) *define $U_n := <v_1, \ldots, v_n>_k$. Then $U_n = <y_1, \ldots, y_n>_k$*
(3) *Y is α-cartesian, i.e. for all $n_0 \in \mathbb{N}$ and $a_1, \ldots, a_{n_0} \in k$, we have that*

$$\max_{n \leq n_0} |a_n y_n| \leq \alpha \Big| \sum_{n=1}^{n_0} a_n y_n \Big|.$$

[6]That is, there is a countable l.i. set such that the vector space it spans is dense in V. Such a set is called a basis of V.

Proof: Step 1. Let $\alpha_1 = 1 < \alpha_2 < \cdots$ be a strictly increasing sequence such that $\alpha_n \to \alpha$. Let $Y = \{y_i\}$ be a basis of V such that for all $n \geq 1$, we have property P_n defined by

$$(P_n) : \alpha_n \max\{|u|, |ay_{n+1}|\} \leq \alpha_{n+1}|u + ay_{n+1}|, \text{ for all } a \in k, u \in U_n. \tag{3.12}$$

Then Y satisfies condition (3) of Lemma 3.9.

Indeed, it is easily seen by induction that in this case we have that

$$\max_{m \leq n} |a_m y_n| \leq \alpha_n \Big| \sum_{m \leq n} a_m y_m \Big|$$

for all n. Since $\alpha_n \leq \alpha$, this justifies the claim.

Step 2. Let $U \subset V$ be a k-vector space and let $x \in V \backslash \bar{U}$. Then for all $\beta > 1$, there exists $y \in U' := U + kx$ such that

- $U' = U + ky$
- $\max\{|u|, |ay|\} \leq \beta|u + ay|$, for all $u \in U$ and $a \in k$.

Indeed, let $u_0 \in U$ such that $|x + u_0| \leq \beta d(x, U)$. Then $y := x + u_0$ satisfies the first required property. If $|y| \neq |ay|$, then clearly the second property is also satisfied. Suppose $|y| = |ay|$. Then we need to show that

$$|ay| \leq \beta|u + ay|.$$

But this is clear from the definition of u_0.

Step 3. Suppose that we have found $\{y_1, \ldots, y_n\}$ satisfying properties (1) and (2) in Lemma 3.9 and property P_{n-1} in (3.12).

Since k is complete, V is weakly cartesian, so U_n is closed. Hence, $v_{n+1} \notin \overline{U_n} = U_n$. Using step 2 with $\beta = \alpha_{n+1}/\alpha_n$, we have that there exists $y'_{n+1} \in U_{n+1}$ such that $U_{n+1} = U_n + ky'_{n+1}$ and satisfying property P_n. Then we choose $c \in k$ such that $y_{n+1} := cy'_{n+1}$ satisfies $1 \leq |y_{n+1}| \leq \rho$. It is easily checked that this choice satisfies property P_n and (2) in Lemma 3.9. Hence, using this procedure we can obtain a basis Y with the required properties ■

Proof of Proposition 3.2: choose any $\alpha, \rho > 1$ and the basis Y given by the preceding lemma. Let V' be the vector space spanned by Y. Then for all i, the k-linear map $F_i : V' \to k$ given by $F_i(\sum_j a_j y_j) = a_i$ is continuous. Indeed, putting $u = \sum_j a_j y_j \in V'$, we have that

$$\begin{aligned} |F_i(u)| &= |a_i| \\ &\leq |a_i y_i| \\ &\leq \max_j |a_j y_j| \\ &\leq \alpha|\sum_j a_j y_j| \\ &= \alpha|u|. \end{aligned}$$

Since k is complete, we conclude that F_i extends to a bounded k-linear map $F_i : V \to k$ with operator norm bounded by α. In particular, the bound does not depend on i.

Let $u \in V$. Suppose that $F_i(u) = 0$ for all i. Take a sequence $(u_n) \subset V'$ that converges to u. Let $a_{i,n} := F_i(u_n)$. Since F_i is continuous, we have that

$$\lim_{n \to \infty} a_{i,n} = F_i(u) = 0.$$

Let $\varepsilon > 0$. Since u_n is a Cauchy sequence, we have that $|u_n - u_m| \leq \varepsilon/\alpha$ for all $n, m \geq m_0$. This implies that for all i and $n, m \geq m_0$ we have that $|a_{i,n} - a_{i,m}| = |F_i(u_n - u_m)| \leq \varepsilon$. Hence,

$$|a_{i,n}| = \lim_{m \to \infty} |a_{i,n} - a_{i,m}| \leq \varepsilon, \qquad \text{for all } i \text{ and for all } n \geq m_0.$$

Writing $u_n = \sum_j a_{j,n} y_j$ we see that $|u_n| \leq \rho\varepsilon$ for all $n \geq m_0$. We conclude that $u_n \to 0$, i.e. $u = 0$. This shows that V is b-separable ■

3.3.5. *Weak stability of $Q(T_n)$.* Suppose char $k = p > 0$. We will prove the weak stability of $Q(T_n)$ using Corollary 3.2. We need to show that every finitely generated T_n-module $M \subset T_{n,1}$ is b-separable.

Write $T_n = k<X>$, with $X = (x_1, \dots, x_n)$. Then the natural embedding $T_n \to T_{n,1}$ can be described as

$$k<X> \longrightarrow k_1<Y>$$

$$x_i \longrightarrow y_i^p.$$

Indeed, the p-power of any element in $k_1<Y>$ lies in $k<X>$ and every element in $k<X>$ has a p-th root in $k_1<Y>$.

On the other hand, the field k_1 is complete. To see this, take a Cauchy sequence $(a_n) \subset k_1$. Then $(a_n^p) \subset k$ is also Cauchy, because $|a_n^p - a_m^p| = |a_n - a_m|^p$. Then there is $b \in k$ such that a_n^p converges to b. Let a be a p-th root of b. Then we have that $|a_n - a| = |a_n^p - b|^{1/p}$, implying that a_n converges to a.

We conclude that $T_{n,1}$ is also a Tate algebra.

Let $\{m_1, \dots, m_r\} \subset M$ be a maximal l.i. set. Write

$$m_i = \sum_J c_{i,J} Y^J.$$

We denote by k' the completion of the field $k\Big((c_{i,J})_{\substack{i=1,\dots,r,\\ J\subset\mathbb{Z}^n}}\Big)$. Since k' is a k-vector space of countable dimension, Proposition 3.2 ensures that k' is b-separable.

Claim 1: $k'<Y>$ is b-separable as a $k<Y>$-module. Take a nonzero series $\sum_J a_J Y^J \in k'<Y>$. Choose J such that $a_J \neq 0$. Let $\lambda : k' \to k$ be a continuous k-linear map such that $\lambda(a_J) \neq 0$. Then this maps extends to a continuous map $\lambda' : k'<Y> \to k<Y>$ by the rule

$$\lambda'\Big(\sum_J b_J Y^J\Big) = \sum_J \lambda(b_J) Y^J.$$

This $k<Y>$-linear map is continuous. Indeed, we have that $\|\lambda'(\sum_J b_J Y^J)\| \leq \|\lambda\| \|\sum_J b_J Y^J\|$.

Claim 2: $k<Y>$ is b-separable as a $k<X>$-module. We have a direct sum decomposition

$$k<Y> = \bigoplus_{\substack{J=(j_1,\dots,j_n)\\ 0\leq j_i<p}} k<X>Y^J.$$

If we put the max norm on the right hand side, this decomposition is an isometry. Since every $k<X>Y^J$ is b-separable as a $k<X>$-module, this justifies the claim.

Claim 3: $k'<Y>$ is b-separable as a $k<X>$-module. Since the composition of bounded linear maps is bounded, the claims follows by putting together Claim 1 and 2.

Claim 4: $M \subset k'<Y>$. Since for all $m \in M$ we have that $m^p \in k<X>$, we have that M is integral over $k<X>$. On the other hand,

$$M \subset k'<Y> \otimes_{k<X>} Q(k<X>) \subseteq Q(k'<Y>).$$

Since $k'<Y>$ is a Tate algebra, it is integrally closed, i.e.

$$Q(k'<Y>) \cap \overline{k'<Y>} = k'<Y>.$$

This justifies the claim.

Since $k'<Y>$ is b-separable as a $k<X>$-module, the same is true for M, finishing the proof ■

3.3.6. *Completeness of the sup norm.* The goal of this section is to prove part (3) of Theorem 3.5. Suppose first that $\mathscr{A}$ is a domain. Let $T_d \to \mathscr{A}$ by an injective finite morphism given by NNL. Then we have a finite extension $Q(T_d) \to Q(\mathscr{A})$. Since $Q(T_d)$ is weakly stable, the spectral norm on $Q(\mathscr{A})$ induces the product topology. On the other hand, we have that $|\cdot|_{\text{sp}} = |\cdot|_{\text{sup}}$ (Theorem 3.7).

Let $\{a_1, \dots, a_n\}$ be a basis of the extension. By the paragraph above, $\big(Q(\mathscr{A}), |\cdot|_{\text{sup}}\big)$ is weakly cartesian, i.e. there exists $\alpha > 0$ such that

$$\max_{i=1}^{n}\{\|a_i t_i\|\} \leq \alpha \Big|\sum_{i=1}^{n} a_i t_i\Big|_{\text{sup}}, \text{ for all } t_i \in Q(T_d). \tag{3.13}$$

On the other hand, there exists a nonzero $t \in T_d$ s.t.

$$\mathscr{A} \subset \bigoplus_{i=1}^{n} T_d \frac{a_i}{t} =: \mathscr{A}'.$$

Since T_d is complete, the module $\mathscr{A}'$, together with the max norm, is complete. Using (3.13), we conclude that $(\mathscr{A}', |\cdot|_{\sup})$ is complete. Since $\mathscr{A}$ is a submodule of $\mathscr{A}'$, it is closed (same proof as in section 2.6), hence complete.

Now we treat the general case where $\mathscr{A}$ is supposed to be a reduced k-algebra. Let $\mathfrak{p}_1, \dots \mathfrak{p}_r$ be the collection of minimal prime ideals (which is finite because $\mathscr{A}$ is noetherian). Then we have an injection

$$\mathscr{A} \hookrightarrow \bigoplus_{i=1}^{r} \mathscr{A}/\mathfrak{p}_i =: \mathscr{A}'.$$

Because of the special case just proved, we have that $(\mathscr{A}/\mathfrak{p}_i, |\cdot|_{\sup})$ is complete for all i. Then $\mathscr{A}'$, endowed with the max norm, is complete. Using Lemma 3.5, (1), we conclude that the max norm induces the $|\cdot|_{\sup}$ norm on $\mathscr{A}$. Since $\mathscr{A}$ is a submodule, it must be closed, hence complete ■

3.3.7. *Power bounded elements.* The goal of this section is to prove Theorem 3.5, (4).

Lemma 3.10. *Let $T_d \to \mathscr{A}$ be a finite injective morphism and let $\alpha \in \mathscr{A}$. Then there exists a monic polynomial $q \in T_d[x]$ such that $q(\alpha) = 0$ and $|\alpha|_{sup} = \sigma(q)$.*

Proof: Suppose first that $\mathscr{A}$ is a domain. Without lost of generality, we may suppose $\mathscr{A} = T_d[\alpha]$. Let $q \in T_d[x]$ be the unique monic polynomial such that $q(\alpha) = 0$ and of minimal degree (i.e. q is the minimal polynomial of α).

Claim. We have that $\mathscr{A} \cong T_d[x]/(q)$.

The field extension $Q(T_d) \to Q(\mathscr{A})$ being finite, we denote by $t \in Q(T_d)[x]$ the minimal polynomial of α (it is monic). We have that $Q(\mathscr{A}) \cong Q(T_d)[x]/(t)$. Moreover, t has coefficients in T_d. Indeed, since α is integral over T_d, all the conjugates of α are integral as well. Since the coefficients of t are symmetric functions of these conjugates, they are integral too. But T_d, being a unique factorization domain ([BGR84], Theorem 5.6.2.1), is integrally closed in $Q(T_d)$. Hence, $t \in T_d[x]$. We conclude that $t = q$, justifying the claim.

We have that

$$\begin{aligned} |\alpha|_{\sup} &= \sup_{y \in M(T_d)} |\bar{\alpha}_y|_{\sup} \\ &= \sup_{y \in M(T_d)} |\bar{\alpha}_y|_{\mathrm{sp}}. \end{aligned}$$

We denote by q_y the image of q in $(T_d/y)[x]$ and we write $q_y = q_1^{n_1} \cdots q_r^{n_r}$ where the q_i's are pairwise distinct irreducible polynomials in $(T_d/y)[x]$. Using the Claim, we have that

$$\begin{aligned} \mathscr{A}/y\mathscr{A} &\cong \bigoplus_{i=1}^{r} (T_d/y)[x]/(q_i^{n_i}) \\ (\mathscr{A}/y\mathscr{A})_{red} &\cong \bigoplus_{i=1}^{r} (T_d/y)[x]/(q_i). \end{aligned}$$

Then we have that

$$\begin{aligned} |\bar{\alpha}_y|_{\mathrm{sp}} &= \max_{i=1}^{r} |\bar{\alpha}_{y,i}|_{\mathrm{sp}} \\ &= \max_{i=1}^{r} \sigma(q_i) \\ &= \sigma(q_1 \cdots q_r) \\ &= \sigma(q_y). \end{aligned}$$

Finally

$$\begin{aligned} |\alpha|_{\mathrm{sp}} &= \sup_{y \in M(T_d)} \sigma(q_y) \\ &= \sigma(q). \end{aligned}$$

This finishes the proof in the case where $\mathscr{A}$ is an integral domain. Now we prove the general case. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal prime ideals of $\mathscr{A}$. Let $\pi_i : \mathscr{A} \to \mathscr{A}/\mathfrak{p}_i$ be the canonical morphism. Let $q_i \in T_d[x]$ be the minimal polynomial of $\pi_i(\alpha)$ (note that $T_d \to \mathscr{A}/\mathfrak{p}_i$ is finite). Let $\tilde{q} := q_1 \cdots q_r$.

Since $\pi_i(\tilde{q}(\alpha)) = 0$ for all i, we have that $\tilde{q}(\alpha)$ is a nilpotent element. Let e be an integer such that $\tilde{q}(\alpha)^e = 0$ and set $q := \tilde{q}^e$. Then we have that

$$|\alpha|_{\sup} = \max_i |\pi_i(\alpha)|_{\sup} = \max_i \sigma(q_i) = \sigma(\tilde{q}) = \sigma(q) \quad \blacksquare$$

Lemma 3.11. *Let $|\cdot|_{\mathscr{A}}$ be a k-Banach algebra norm on $\mathscr{A}$. Then we have that*

$$|f|_{sup} \leq |f|_{\mathscr{A}}, \quad \forall f \in \mathscr{A}.$$

Proof: let $x \in M(\mathscr{A})$. Let $|\cdot|_r$ be the residue norm induced by $|\cdot|_{\mathscr{A}}$ on $\mathscr{A}/x$. Since this norm is submultiplicative, we can apply Theorem 3.6 to it to obtain

$$|f(x)| = \inf_{i \geq 1} |f(x)^i|_r^{1/i} \leq |f(x)|_r \leq |f|_{\mathscr{A}}.$$

Since x is arbitrary, this implies $|f|_{\sup} \leq |f|_{\mathscr{A}}$ $\blacksquare$

Proof of Theorem 3.5, (4): Let $f \in \mathscr{A}$ such that $|f|_{\sup} \leq 1$. Let $\iota : T_d \to \mathscr{A}$ be a finite injective morphism given by NNL. Let $q(x) = x^n + t_1 x^{n-1} + \cdots + t_n \in T_d[x]$ be polynomial given by Lemma 3.10 applied to f. Then we have that $\|t_i\| \leq 1$ for all $i = 1, \ldots, n$. Hence, f^n lives in the bounded set $f^{n-1}\iota(B_1) + \cdots + \iota(B_1)$, where $B_1 = \{t \in T_d : \|t\| \leq 1\}$. It follows by induction that for all positive integers k the powers f^{n+k} are bounded as well.

Suppose now that f is power bounded. That is, if $|\cdot|_{\mathscr{A}}$ is a k-Banach algebra norm on $\mathscr{A}$, then there exists $C > 0$ such that $|f^n|_{\mathscr{A}} \leq C$ for all integers $n \geq 1$. Using Lemma 3.11, we have that

$$|f|_{\sup}^n = |f^n|_{\sup} \leq |f^n|_{\mathscr{A}} \leq C.$$

Hence, $|f|_{\sup} \leq C^{1/n}$ for all n, implying $|f|_{\sup} \leq 1$ $\blacksquare$

Appendix A. An example of a non weakly cartesian field

Lemma A.1. (1) *Let K be a valued field and let $f, g \in K[x]$ be monic polynomials of degree n. Let $\alpha \in \bar{K}$ be such that $f(\alpha) = 0$. Then $|g(\alpha)|_{sp} \leq \|f - g\| \|f\|^{n-1}$.*

(2) *Suppose K is complete. Then there exists $\beta \in \bar{K}$ s.t. $g(\beta) = 0$ and*

$$|\alpha - \beta|_{sp} \leq \|f - g\|^{1/n} \|f\|.$$

Proof: let q be the minimal polynomial of α. Then we have a factorization $f = qr$ with $r \in K[x]$. Since f is monic, $\|f\| \geq \sigma(f)$. On the other hand, $\sigma(f) \geq \sigma(q) = |\alpha|_{\text{sp}}$. We conclude that $|\alpha|_{\text{sp}} \leq \|f\|$.

Now write

$$f = x^n + \sum_{j=1}^{n} f_j x^{n-j}, \quad g = x^n + \sum_{j=1}^{n} g_j x^{n-j}.$$

Then $g(\alpha) = g(\alpha) - f(\alpha) = \sum_{j=1}^{n} (g_j - f_j)\alpha^{n-j}$ implies

$$|g(\alpha)|_{\text{sp}} \leq \|g - f\| \max_{j=1}^{n} |\alpha|_{\text{sp}}^{n-j} \leq \|g - f\| \max_{j=1}^{n} \|f\|^{n-j} \leq \|g - f\| \|f\|^{n-1},$$

since $\|f\| \geq 1$. This proves part (1).

Suppose that assertion (2) does not hold. Write $g = \prod_{i=1}^{n}(x - \beta_i)$. Then $|\alpha - \beta_i|_{\text{sp}} > \|f - g\|^{1/n} \|f\|$, for all $i = 1, \ldots, n$ (we use the completeness here to ensure that the spectral norm is multiplicative). Then we have that

$$|g(\alpha)|_{\text{sp}} = \prod_{i=1}^{n} |x - \beta_i|_{\text{sp}} > \|f - g\| \|f\|^n.$$

Since $\|f\| \geq 1$, this contradicts part (1). $\blacksquare$

Theorem A.1. *Let K be a complete valued field and let K_{sep} be the separable closure of K. Then K_{sep} is dense in $\bar{K}$.*

Proof: let $\alpha \in \bar{K}$ and let $n = [K(\alpha) : K]$. Let $f(x) = x^n + \sum_{j=1}^{n} f_j x^{n-j}$ be the minimal polynomial of α. Let $\varepsilon > 0$ and let $\delta := (\varepsilon/\|f\|)^n$. We will show that there exists a monic separable polynomial $g \in K[x]$ such that $\|f - g\| < \delta$. Using Lemma A.1, (1), this will complete the proof.

Write $g_z = x^n + z_1x^{n-1} + \ldots + z_{n-1}x + z_n$, where $z = (z_1, \ldots, z_n) \in K^n$ is to be defined. Let $r_1, \ldots, r_n$ be the roots of g_z (possibly repeated). Define

$$\Delta(z) := \prod_{i \neq j}(r_i - r_j)^2.$$

We have that $\Delta(z) \neq 0$ if and only if g_z is separable. Since $\Delta(z)$ is symmetric in $(r_1, \ldots, r_j)$, we have that $\Delta(z)$ is a polynomial in the elementary symmetric functions of $(r_1, \ldots, r_j)$, i.e. $\Delta(z)$ is a polynomial in z.

The maximum principle (Proposition 2.1, (2)) shows that the zero set of Δ is nowhere dense in K^n. In particular, there exists $\tilde{z} \in D\big((f_1, \ldots, f_n), \delta\big)$ such that $\Delta(\tilde{z}) \neq 0$. Putting $g := g_{\tilde{z}}$ we obtain a polynomial with the required properties ■

Corollary A.1. *There exists non-weakly cartesian valued fields.*

Proof: let K be a non-perfect complete valued field. For example, consider $\mathbb{F}_p[x]$ with the valuation $|q(x)| := 2^{\deg q}$ and let K be the completion of $Q(\mathbb{F}_p[x])$ w.r.t. this valuation. Let $y \in \bar{K}$ such that $y^p = x$. Then $|y| = |x|^{1/p} = 2^{1/p}$. Since the value group does not change after completion, this implies that $y \notin K$. Hence, K is non-perfect.

The field K_{sep}, which is properly contained in $\bar{K}$, is not weakly stable because the Theorem ensures that it is dense in $\bar{K}$. Hence, it does not satisfy condition (2) in Theorem 3.8 ■

References

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. 2.7

[BGR84] S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry. (document), 1, 3.3.2, 3.8, 3.3.7

[Con] Brian Conrad. Several approaches to non-archimedean geometry. *http://math.stanford.edu/ conrad/papers/aws.pdf*. (document), 1

[Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980. 2

Facultad de Matemáticas, PUC, Vicuña Mackenna 4860, Santiago, Chile

E-mail address: rmenares@mat.puc.cl