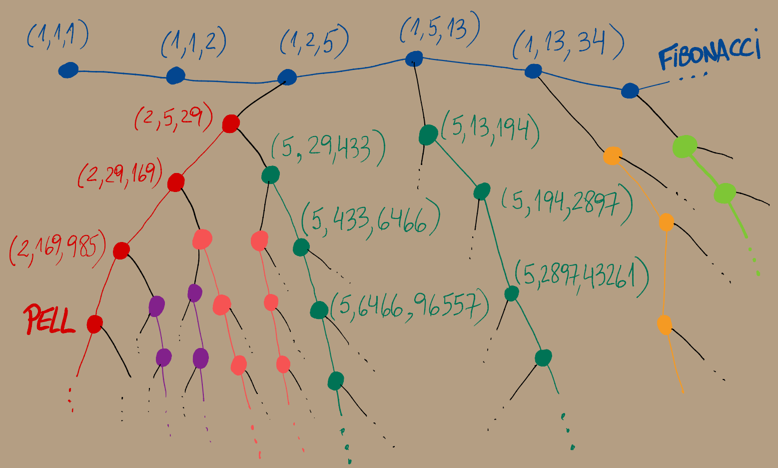
«Wahl singularities in degenerations of del Pezzo surfaces»

(joint with Juan Pablo Zúñiga, PhD student UC Chile)



Giancarlo Urzúa

UC Chile

17 July 2025 (4:10-5)

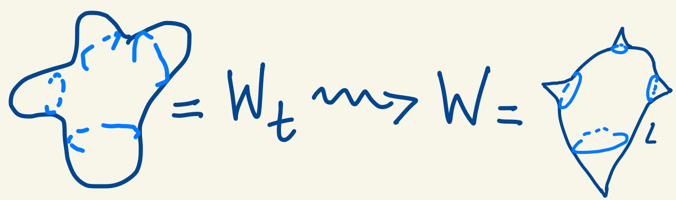
Big10 AG

- W = normal projective surface $/_{\mathbb{C}}$ with K_W $\mathbb{Q}$-Cartier and $K_W^2 > 0$.

Then K_W or $-K_W$ is big.

- Assume that W has a smoothing $W_t \rightsquigarrow W$ inducing #Milnor $= 0$ smoothings at each singularity and over $\mathbb{D} = \{t \in \mathbb{C} : |t| << \varepsilon\}$.

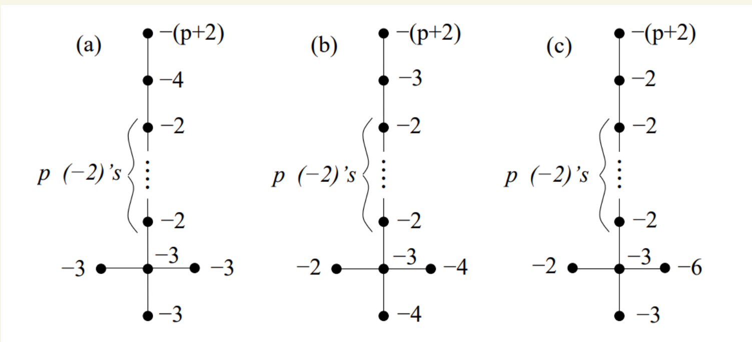
[then on the 3-fold $\mathcal{W} \to \mathbb{D}$ we have $K_{\mathcal{W}}$ is $\mathbb{Q}$-Cartier]

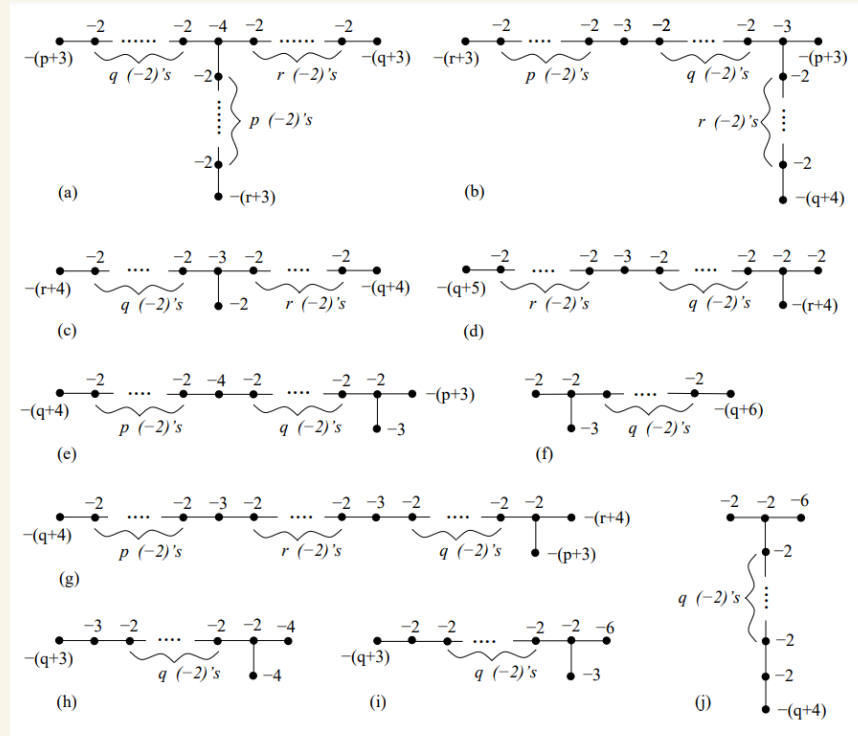


List of (minimal resolutions of) known QHD:

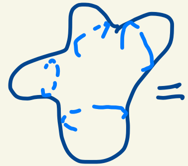
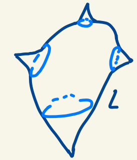
(due to Wohl, Stipsicz–Szabó–Wohl, Bhupal–Stipsicz)

Wohl chains





Wohl conjecture:
This is the complete list of families of QHD.

 $= W_t \rightsquigarrow W =$ 

(R) $-K_W$ big

$\Downarrow$

W_t is rational

(GT) K_W big

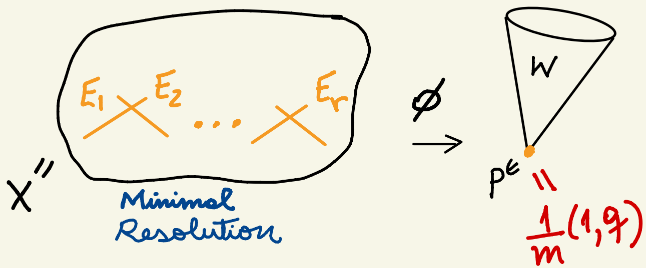
$\Downarrow$

W_t is of general type

$$\boxed{\text{Our goal is to classify the surfaces } W.}$$

- when the singularities of W are log-terminal
  $\Rightarrow$ they must be klt singularities.

- A cyclic quotient singularity is the germ at $(0,0)$ of the quotient of $\mathbb{C}^2$ by $(x,y) \overset{T}{\longmapsto} (\zeta x, \zeta^q y)$ where $\zeta^m = 1$, ζ primitive, and $0 < q < m$ coprime.



$E_i \simeq \mathbb{P}^1_{\mathbb{C}}$ $\quad E_i^2 = -e_i \leq -2$

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}} = [e_1, \ldots, e_r]$$

$\frac{m}{m-q} = [b_1, \cdots, b_s]$ dual of $\frac{m}{q}$

- A T-singularity is a quotient singularity that admits a $\mathbb{Q}$-Gorenstein smoothing over $\mathbb{D}$: [KSB] they are Du Val or $\frac{1}{dn^2}(1, dna-1)$ where $\gcd(n,a) = 1$. [$\Leftrightarrow$ log-terminal admitting $\mathbb{Q}$-Gorenstein smoothing]

- A Wahl singularity is $\frac{1}{n^2}(1, na-1)$ where $\gcd(n,a) = 1$.

Symmetry
&
Mutation

$$[e_1,\dots,e_r] \mapsto [e_1+1, e_2, \dots, e_r, 2]$$

The Wahl tree :

- [4] — [5,2]
  - [6,2,2]
    - [7,2,2,2] ⟨ …, …
    - [2,6,2,3] ⟨ …, …
  - [2,5,3]
    - [3,5,3,2] ⟨ …, …
    - [2,2,5,4] ⟨ …, …

The dual Wahl tree :

- [2,2,2] — [3,2,2,2]
  - [4,2,2,2,2]
    - [5,2,2,2,2,2] ⟨ …, …
    - [2,4,2,2,2,3] ⟨ …, …
  - [2,3,2,2,3]
    - [3,3,2,2,3,2] ⟨ …, …
    - [2,2,3,2,2,4] ⟨ …, …

(GT) K_W big [$\Rightarrow W_t$ is of general type in case there is smoothing. otherwise, think of W_t as rational blow-down.]

This is the hardest case to classify. Two examples:

(1) If $K_W^2 \leq 2p_g(W) - 3$, then we can classify all of them.

[This is the joint work with V. Monreal and J. Negrete]

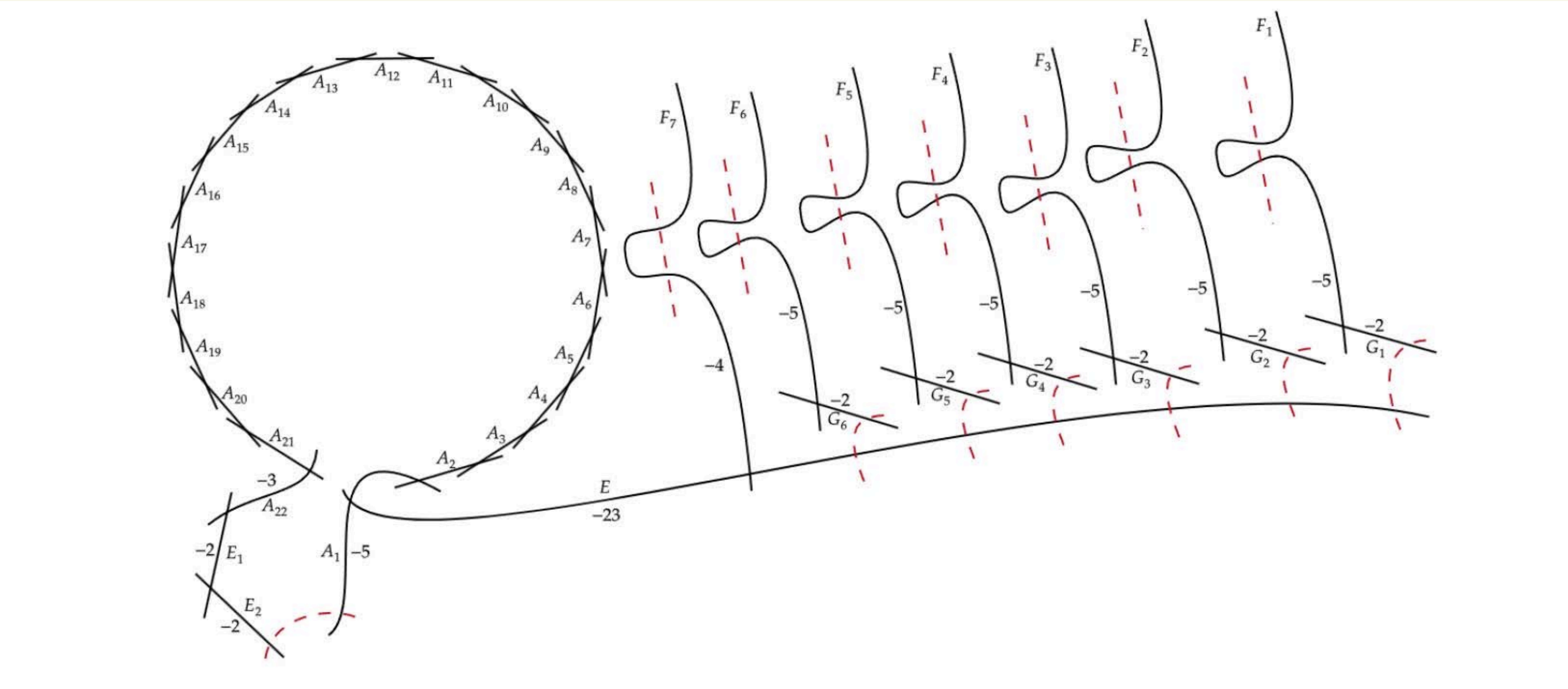
Theorem [MNU 2024] For $p_g \geq 10$, the only KSBA degenerations of Horikawa surfaces with only T-singularities (not all Du Val!) are Lee–Park examples, and only for the nonspin component.

[QHD Horikawa surfaces will be classified soon: [Conedo-U, 25]]

For $p_g = 3$, with J. Evans and A. Simonetti 2024 we classify all.

NEWS! AKAIKE-ENOKIZONO-HATTORI-KOTO classify all log-Con. $\mathbb{Q}$-Gorenstein smoothable (June 25).

One of the new **singular Horikawa surfaces** $K^2=16$ $(p_g=10)$ [MNU24]:



The classification for $K_W^2=2p_g(W)-3$ will appear in [MNU25-26].

NEWS! Ciliberto - Pardini 2025 studied the case of $K_W^2=2p_g-3$ with minimal resolution of general type [New Horikawa problem!]

(2) Surfaces with $p_g = 0$ (and so $1 \leq K_W^2 \leq 8$).

Here we have works by J. Park, Y. Lee (2007), D. Shin, H. Park, etc and more recently joint with Javier Reyes (2021 and 2022). There is even a computer program to search for them, but **No** classification yet. By K^2:

K^2	1	2	3	4	5	6	7	8
In terms of Wahl singularities at least	44	508	2104	1246	20			

44, 508, 2104, 1246: smoothable examples

20: **open**: Does there exist a smoothable W?

6, 7, 8: **open**: Does there exist any W?

It has to do with exotic Blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ at $9-K_W^2$ points.

(R) $-K_W$ big.

- We have no local-to-global obstructions to deformations of W. (not true when K_W big in general)

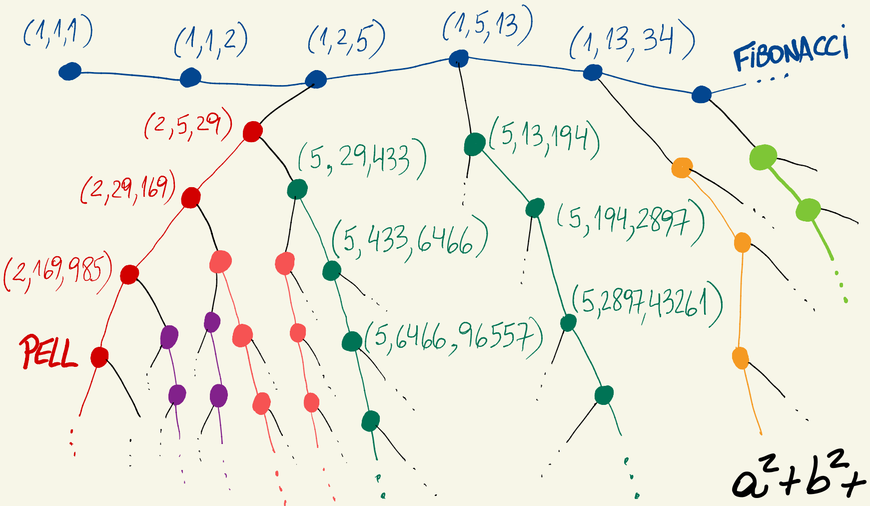
- So, we can think of $W_t \rightsquigarrow W$ $\mathbb{Q}$-Gorenstein smoothing and W_t is a deformation of a del Pezzo surface of degree $\ell = K_W^2 > 0$.

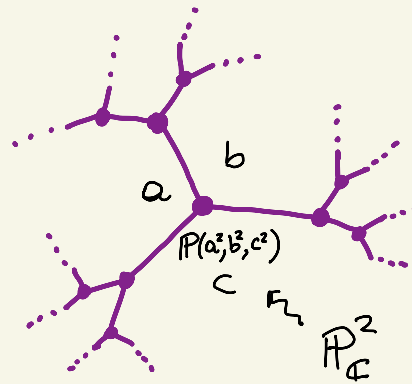
- We can assume that $W = W_*$ has one singularity and ask:

Question: Is there a classification of all possible W_* for a given ℓ?
Are there any constraints on the Wahl singularity in W_* for a given ℓ?

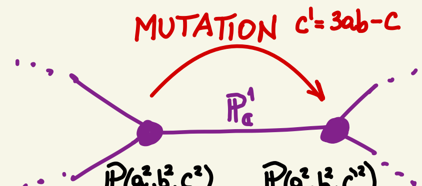
After the work of Bădescu (1986), Manetti (1991), Hacking (2001), we have Hacking–Prokhorov's Theorem (2010):

<u>Theorem</u>: Let $\mathbb{P}^2_{\mathbb{C}} \rightsquigarrow W$ be a $\mathbb{Q}$-Gorenstein smoothing of W. Then W is a partial $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(a^2, b^2, c^2)$ for some Markov triple (a, b, c).





$$a^2+b^2+c^2=3abc$$



Zero continued fractions are Hirzebruch-Jung cont. frac. of 0.

(we now need to introduce 1s in the e_is ...)

$[1,1]$

$[1,2,1]$, $[2,1,2]$

$[1,2,2,1]$, $[2,1,3,1]$, $[1,3,1,2]$, $[3,1,2,2]$, $[2,2,1,3]$

$\vdots$

etc using blow-up $\quad u - \frac{1}{v} = u+1 - \cfrac{1}{1 - \cfrac{1}{v+1}}$.

<u>Definition</u>: We say that $[f_1,\ldots,f_r]$ with $f_i \geq 2$ admits a zero continued fraction of weight λ if

$$[\ldots, f_{i_1}-d_{i_1}, \ldots, f_{i_2}-d_{i_2}, \ldots, f_{i_v}-d_{i_v}, \ldots] = 0$$

for some $d_{i_1}, d_{i_2}, \ldots, d_{i_v} \in \mathbb{Z}_{>0}$ and $\lambda + 1 = \sum_{j=1}^{v} d_{i_j}$.

Example: $[4,3,3,2]$ admits $[4-3,3,3-2,2]=0$ and $\lambda = 3+2-1 = 4$.

Observation: $[f_1,\ldots,f_r]$ admits a zero continued fraction of weight $0 \iff$ It is a dual Wahl chain.

<u>Theorem</u> : [Hacking–Prokhorov combinatorial version [UZ23]]

The Wahl chain $[e_1, \dots, e_r]$ corresponds to a degeneration of $\mathbb{P}^2_{\mathbb{C}}$

$$\Updownarrow$$

There is $i \in \{1, \dots, r\}$ such that $[e_1, \dots, e_{i-1}]$ and $[e_{i+1}, \dots, e_r]$ admit zero continued fractions of weight 0.

observation : $[e_1, \dots, e_{i-1}]$ and/or $[e_{i+1}, \dots, e_r]$ may be empty.

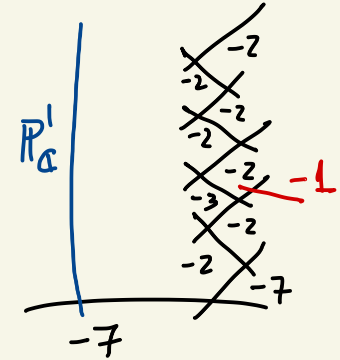
(i) If both are not empty $\Rightarrow e_i = 10$.

(ii) If one is empty $\Rightarrow e_i = 7$.

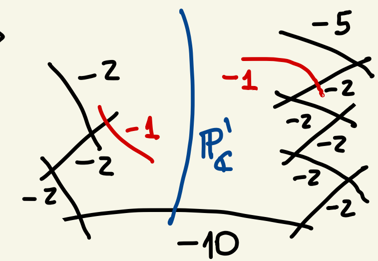
(iii) If both are empty $\Rightarrow e_i = 4$.

[UZ23] = "The birational geometry of Markov numbers"

$$\frac{34^2}{34\cdot 5-1} = [\underset{\substack{\uparrow\\ e_i}}{7}, \underbrace{7, 2, \bar{2}, 3, 2, 2, 2, 2, 2}_{[7,2,1,3,2,2,2,2,2]=0}] \Rightarrow$$



$$\frac{29^2}{29\cdot 22-1} = [\underbrace{2, \bar{2}, 2}_{[2,1,2]=0}, \underset{\substack{\uparrow\\ e_i}}{10}, \underbrace{2, 2, 2, 2, \bar{2}, 5}_{[2,2,2,2,1,5]=0}] \Rightarrow$$



$$\frac{433^2}{433\cdot 104-1} = [\underbrace{5, \bar{2}, 2, 2, 2, 2}_{[5,1,2,2,2,2]=0}, \underset{\substack{\uparrow\\ e_i}}{10}, \underbrace{5, 2, 2, 2, 2, 2, 2, \bar{2}, 8, 2, 2, 2}_{[5,2,2,2,2,2,2,1,8,2,2,2]=0}]$$

Definition: A Wahl chain $[e_1, \dots, e_r]$ is del Pezzo if

(I) $[e_1, \dots, e_{r-1}]$ or $[e_2, \dots, e_r]$ admit zero continued fraction of weight $\lambda \leq 8$.

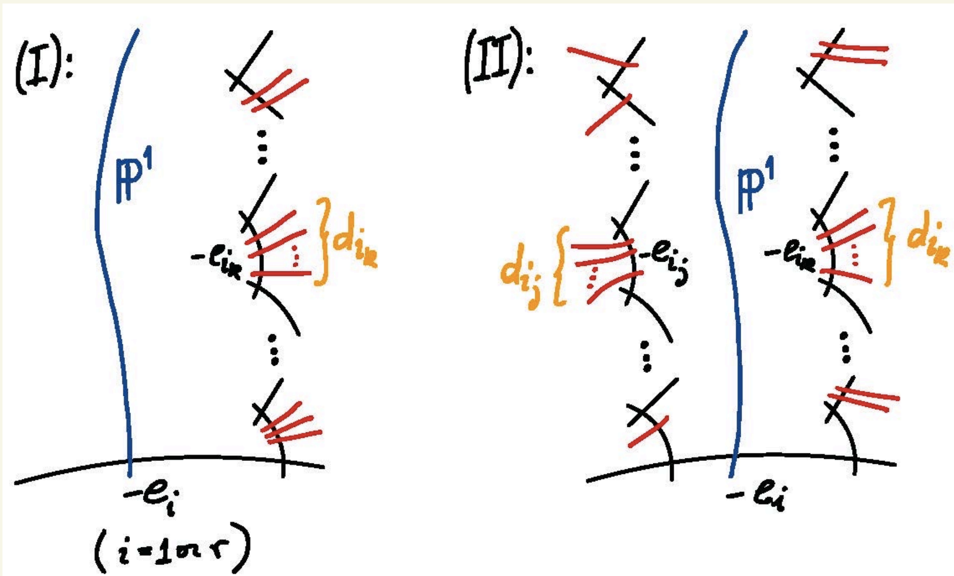
(II) $[e_1, \dots, e_{i-1}]$ and $[e_{i+1}, \dots, e_r]$ admit zero continued fractions of weights λ_1, λ_2 with $\lambda_1 + \lambda_2 \leq 8$.

Degree : $9 - \lambda$ or $9 - \lambda_1 - \lambda_2$.

Marking : $[\underbrace{k_1, \dots, k_{i-1}}, \underline{e_i}, \underbrace{k_{i+1}, \dots, k_r}]$.

corresponding zero continued fractions

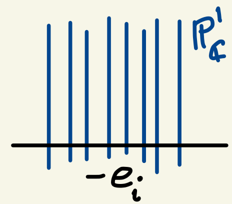
Del Pezzo Wahl chains define Del Pezzo singular surfaces



$\xrightarrow[\text{Contraction of the Wahl chain}]{}$ W_{*m}

$\downarrow$

$\mathbb{F}_{e_i}$ =



= Hirzebruch surface

Theorem [U-Zúñiga 25]

A Wahl singularity appears in a $\mathbb{Q}$-Gorenstein degeneration of del Pezzo surfaces of degree ℓ

$$\Updownarrow$$

The corresponding Wahl chain is del Pezzo of degree $\ell' \geq \ell$.

- This is exactly Hacking-Prokhorov when $\ell = 9$.
- $\ell' \geq \ell$ is needed since for $\ell \leq 8$ we may have blow-ups at nonsingular points (Floating curves).
- A given Wahl chain may be del Pezzo for many distinct markings and various degrees (computer program).

Some ideas in [U-Zúñiga 25]:

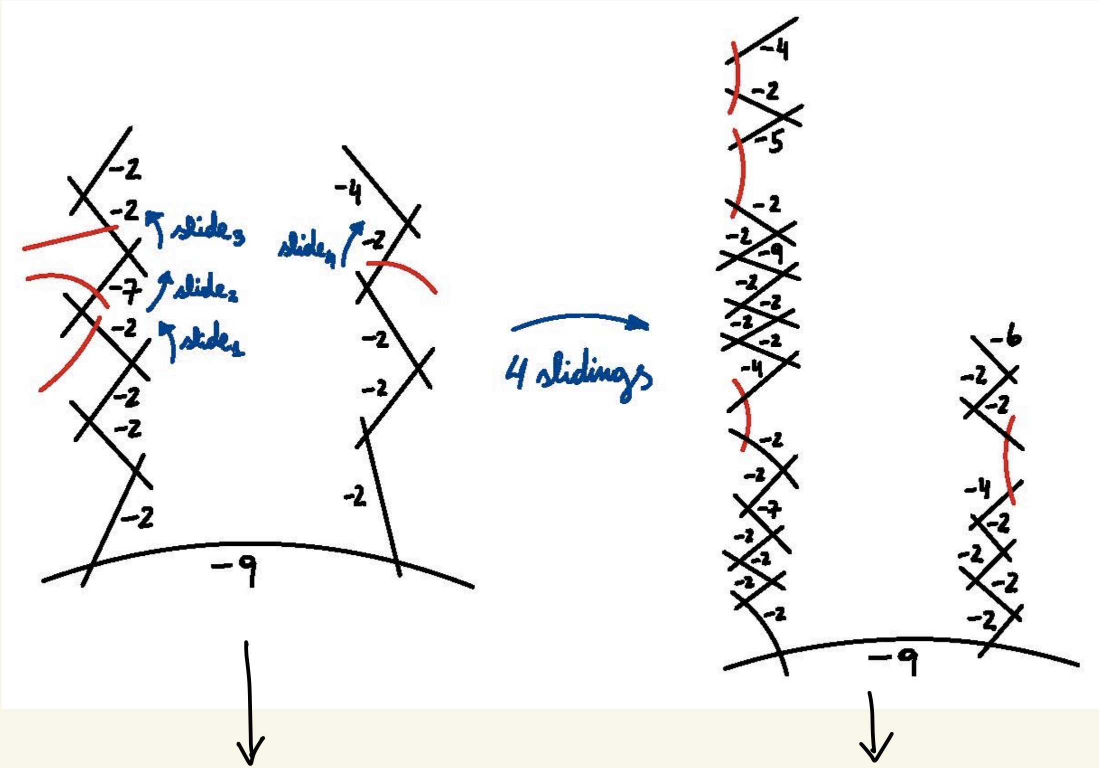
1. Control of the exceptional divisor of $X \xrightarrow[\text{min res}]{} W$, where W has Wahl singularities, $-K_W$ big and $K_W^2 > 0 \Rightarrow h^0(-K_X) \geq 2$.
[This is along the lines of Monetti's approach]

2. It uses some birational geometry of degenerations of surfaces to identify the most general degeneration W_{*m}. [Hacking-Tevelev-U, 2017]

3. To find a certain toric "Canonical" degeneration of W_{*m} we use $\underbrace{\text{slidings}}$ of Markov and Mori type:

$$W_{*m} \rightsquigarrow \hat{W}_{*m} \longrightarrow W_{*m}^{T} \longrightarrow \mathbb{P}(n^2, m_1, m_2)$$

Toric Wahl sing — Toric T-sing — Fake weighted projective plane

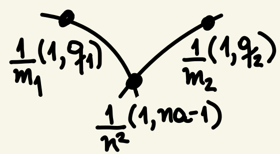


$$\frac{1}{99^2}(1, 99\cdot 68-1) \in W_{*m} \rightsquigarrow \widehat{W}_{*m}$$

del Pezzo of degree 7

Theorem [U-Zúñiga 25]: There is correspondence

$$\left\{ \begin{array}{l} W_{*m} \text{ marked del Pezzo} \\ \text{surface of degree } \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{P}(n^2, m_1, m_2) \text{ Fake Weighted Planes} \\ \cup \end{array} \right\}$$



(1) $q_1 m_2 + q_2 m_1 + n^2 = d m_1 m_2$ some $d \geq 2$

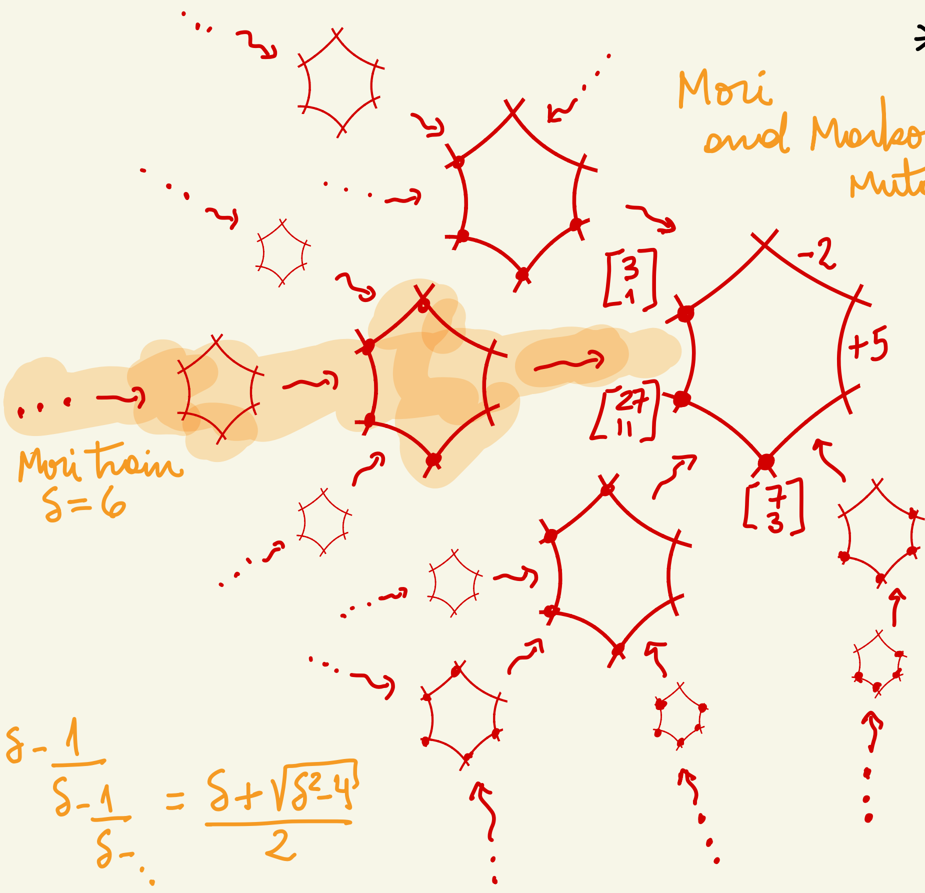
(2) $m_1 + m_2 = n(m_1 a - n q_1^{-1}) = n(m_2(n-a) - n q_2^{-1})$

(3) $\frac{1}{m_1}(1,q_1)$ and $\frac{1}{m_2}(1,q_2)$ admit zero continued fractions $\ell = 9 - \lambda_1 - \lambda_2 > 0$.

For $\ell = 9$ this is:

$$\left\{ \begin{array}{l} W_{*m} \text{ marked del Pezzo} \\ \text{surface of degree } \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{P}(n^2, a^2, b^2) \text{ where } (a, b < n) \\ \text{satisfy } n^2 + a^2 + b^2 = 3abn \end{array} \right\}$$

$$* = \frac{27^2}{27\cdot 11-1} = [\bar{\bar{3}}, \bar{2}, \underline{8}, 2, 2, \bar{2}, 4, \bar{2}]$$



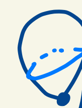
$$\delta - \cfrac{1}{\delta - \cfrac{1}{\delta - \ddots}} = \frac{\delta + \sqrt{\delta^2 - 4}}{2}$$

$\mathbb{P}(27^2, 5, 22)$

CONTRACTION

$=$ $\widehat{W}_{*m}$ = Toric weak del Pezzo surface

SLIDINGS

W_{*m} =  $[(\frac{27}{11})]$ = Marked del Pezzo surface

DEGENERATION

W_t = del Pezzo surface of degree $\ell = 6$

In terms of Wahl singularities, for $\ell = 8$ we show:

<u>Theorem</u>: $\frac{1}{n^2}(1, na-1) \in W_{*m}$ of degree 8. We have:

(0) $\mathbb{F}_{2k} \rightsquigarrow W_{*m}$ for some $k \iff$ n odd and $n\Gamma \cdot K_{W_*}$ even for all marking curve Γ.

(1) $\mathbb{F}_{2k-1} \rightsquigarrow W_{*m}$ for some $k \iff$
- $W_{*m} \xrightarrow{\text{W-blow-up}}$ Markov or
- n is even or
- n is odd and $\exists$ marking curve Γ with $n\Gamma \cdot K_{W_{*m}}$ odd.

2 consequences of our results.

(First) Are there constraints for wall chains in a given degree ℓ ?

<u>Theorem</u> (obs.) [UZ25]: In degrees $\ell \leq 4$, every wall chain is possible.

Reason: There is a canonical marking in degree 4.

<u>Theorem</u> [UZ25]: In degrees $\ell \geq 5$, there are infinite families of wall chains that are not realizable.

(This depends on our classification.)

Examples: $[\underbrace{2, \ldots, 2}_{B}, A+4, \underbrace{2, \ldots, 2, B+2}_{A}]$ for $B > A+3 \geq 5$.

$$\left[\binom{25}{22}\right] = [2,2,2,10,2,2,2,2,2,5]$$

22 15 8 1 2 3 4 5 6 7

```
Input n of a Wahl fraction: 29
Input q of a Wahl fraction: 22
Wahl chain:  [2, 2, 2, 10, 2, 2, 2, 2, 2, 5]
No divisorial contractions on the chain
Type I:
==========================================================
Marking                                 |  K^2
----------------------------------------------------------
Partition:  [2, 2, 2, 10, 2, 2, 2, 2, 2] 5

[2, 2, 1, 4, 2, 2, 2, 2, 1]             |   2
[1, 2, 2, 6, 1, 2, 2, 2, 2]             |   4
[2, 2, 1, 8, 1, 2, 2, 2, 2]             |   6
----------------------------------------------------------
Partition:  2 [2, 2, 10, 2, 2, 2, 2, 2, 5]

[1, 2, 7, 1, 2, 2, 2, 2, 2]             |   2
[2, 1, 8, 1, 2, 2, 2, 2, 2]             |   3
[2, 2, 6, 1, 2, 2, 2, 2, 4]             |   4
Type II:
==========================================================
Markings                                |  K^2
----------------------------------------------------------
Partition:  [2] 2 [2, 10, 2, 2, 2, 2, 2, 5]

[0] [1, 7, 1, 2, 2, 2, 2, 2]            |   1
[0] [2, 6, 1, 2, 2, 2, 2, 3]            |   2
----------------------------------------------------------
Partition:  [2, 2] 2 [10, 2, 2, 2, 2, 2, 5]

[1, 1] [6, 1, 2, 2, 2, 2, 2]            |   1
----------------------------------------------------------
Partition:  [2, 2, 2] 10 [2, 2, 2, 2, 2, 5]

[1, 2, 1] [1, 2, 2, 2, 2, 1]            |   4
[2, 1, 2] [1, 2, 2, 2, 2, 1]            |   5
[1, 2, 1] [2, 2, 2, 2, 1, 5]            |   8
[2, 1, 2] [2, 2, 2, 2, 1, 5]            |   9
----------------------------------------------------------
Partition:  [2, 2, 2, 10] 2 [2, 2, 2, 2, 5]

[2, 2, 1, 3] [2, 2, 2, 1, 4]            |   1
----------------------------------------------------------
Partition:  [2, 2, 2, 10, 2, 2, 2, 2] 2 [5]

[2, 2, 1, 7, 1, 2, 2, 2] [0]            |   1
----------------------------------------------------------
```

Canonical

$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$\mathbb{P}^2_{\mathbb{C}}$

By product on exceptional vector bundles on del Pezzo surfaces.

This was motivated by recent results of Polishchuk and Rains 2024.

__Theorem__ [PR24]: X = del Pezzo nonsingular projective surface of degree 4.

d, r coprime integers with $r > 0$.

$\Rightarrow \exists$ exceptional pair $(E, \mathcal{O}_X)$ on X with E exceptional vector bundle of rank r and degree $= c_1(E) \cdot -K_X = d$.

__Theorem__ [PR24]: Let $\ell \geq 5$, and d, r coprime integers with $r > 0$ such that $-\ell \leq d^2 + \ell r d + \ell r^2 \leq -1$. Let X be del Pezzo degree ℓ.

$\Rightarrow$ (i) for $5 \leq \ell \leq 7$, $\exists$ e. pair $(E, \mathcal{O}_X)$ $\mathrm{rank}(E) = r$, $\deg(E) = d$.

(ii) for $\ell = 8$ and $X = \mathbb{F}_1 \Rightarrow$ %; and $X = \mathbb{F}_0$ & d even $\Rightarrow$ %.

[(iii) for $\ell = 9$ we have the Fibonacci situation; known]

Also they observed that $-\ell \leq d^2 + \ell r d + \ell r^2$ is always true because of Riemann-Roch and Hodge index theorems.

Therefore, for $\ell \geq 5$ ∞ many slopes $\frac{d}{r}$ are impossible.

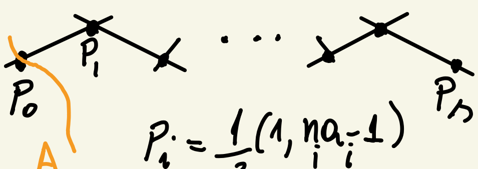
———— o ————

Hacking's philosophy says: X surface with $p_g = q = 0$

$$\left\{ \begin{array}{l} \text{exceptional vector} \\ \text{bundles on } X \\ \text{rank} = n \quad c_1 \cdot K_X \equiv \pm a \, (n) \end{array} \right\} \xleftrightarrow{\text{Hacking v.b.}} \left\{ \begin{array}{l} X \rightsquigarrow W \ni \frac{1}{n^2}(1, na-1) \\ \mathbb{Q}\text{-Gorenstein smoothing} \end{array} \right\}$$

and works perfectly for del Pezzos.

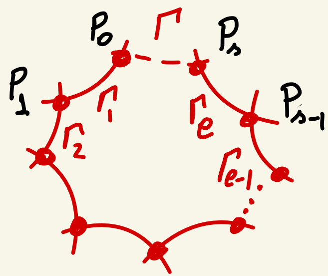
Moreover, certain configurations of rational curves



$\subset W$ produce $\underset{n_s}{E_s}, \underset{n_{s-1}}{E_{s-1}}, \dots, \underset{n_0}{E_0}$ Hacking exceptional coll. on W_t [Tevelev-U, 22]

$P_i = \frac{1}{n_i^2}(1, n_i a_i - 1)$

Corollary 8.9 [U-Zúñiga 25]: Let W_{*m} be a marked del Pezzo and consider $W_t \rightsquigarrow W_{*m}$ with W_t del Pezzo. Then W_t admits a H.e.c. that is full and strong.



$s = 11 - K_W^2$ H.e.c. $E_s, E_{s-1}, \dots, E_0$

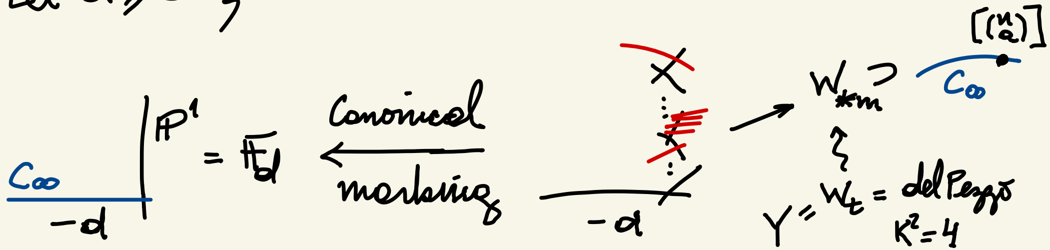
$$Ext^i(E_j, E_k) = 0 = Hom(E_j, E_i)$$
$$\forall i \geq 1,\ j,k \qquad j < i$$

$$hom(E_j, E_i) = rk(E_j) \cdot rk(E_i) \cdot (\Gamma_{i+1} + \dots + \Gamma_j) \cdot (-K_{\hat{W}_{*m}})$$

Theorem 8.11 [U-Zúñiga 25]: Given coprime integers $0 < a < n$, there is a full and strong H.e.c. $E_7, \dots, E_1, \mathcal{O}_Y$ on del Pezzo surfaces Y of degree 4, $rk(E_1) = n$ $c_1(E_1) \cdot -K_Y \equiv \pm a\ (n)$.

Theorem 8.14 [UZúñiga 25]: Let a, n be coprime integers with $n>0$. Then $\exists$ an e.c. $E, \mathcal{O}_Y$ on del Pezzo of degree 4 with $\operatorname{rank}(E) = n$ and $c_1(E)\cdot -K_Y = a$.

Proof 1: Let $d \geq 0$,



$\Rightarrow$ we have H.e.c. $E, \mathcal{O}_Y$ with $c_1(E)\cdot -K_Y = a + (2-d)n$.

For $\left[\binom{n}{n-a}\right]$ get $E', \mathcal{O}_Y$ with $c_1(E)\cdot -K_Y = -a + (3-d)n$.

apply $(E, \mathcal{O}_Y) \mapsto (\mathcal{O}_Y, E^\vee) \mapsto (E^\vee, \mathcal{O}_Y(-K_Y)) \mapsto (E^\vee \otimes \mathcal{O}_Y(K_Y), \mathcal{O}_Y)$ and get degrees $-a + n(d-6)$ and $a + n(d-7)$. $\square$

Theorem 8.15 [U-Zúñiga 25]: There are ∞ many slopes ∂/n that are not realizable by an e.v.b. on del Pezzo surfaces of degree ≥ 5.

We can also reproduce all $(E, \mathcal{O})$ in [PR24] where

$$-\ell \leq \partial^2 + \ell \partial n + \ell n^2 \leq -1 .$$

We can show in general that:

(I) $\ell(an-1) \leq (n+a)^2$.

(II) $\ell n^2 m_1 m_2 \leq (n^2+m_1+m_2)^2$.

The idea of [PR24] was to look at solutions of

$$d^2 + \ell d n + \ell n^2 = -e$$

for $e \in \{1, \dots, \ell\}$ and fixed $\ell \geq 5$.

One can find all possible e and all possible solutions (n, d) via recursions:

$$n_k = (\ell-2) n_{k-1} - n_{k-2} \qquad \partial_k = (\ell-2)\partial_{k-1} - \partial_{k-2}$$

with certain initial conditions. Then

$$\partial_k = -n_k - n_{k-1}$$

and the Wahl singularity is $\frac{1}{n_k^2}(1, n_k n_{k-1} - 1)$.

We classify this geometric situations via their marked Wahl chains:

$(\ell = 8, e = -8)$: For $k \geq 1$,
$[\underbrace{6,\dots,6}_{k}, 5, 3, \underbrace{2,2,2,3,\dots,2,2,2,3}_{(k-1)\text{ blocks } 2,2,2,3}, 2,2,2,2]$; $[\underline{6},\dots,6,5,1,2,2,2,3,\dots,2,2,2,3,2,2,2,2]$

$(\ell = 8, e = -7, j = 0)$: For $k \geq 2$,
$[\underbrace{6,\dots,6}_{k}, 5, \underbrace{2,2,2,3,\dots,2,2,2,3}_{(k-1)\text{ blocks } 2,2,2,3}, 2,2,2,2]$; $[\underline{6},\dots,6,4,1,2,2,3,\dots,2,2,2,3,2,2,2,2]$

$(\ell = 8, e = -7, j = 1)$: For $k \geq 1$,
$[\underbrace{6,\dots,6}_{k}, 6, 2, 3, \underbrace{2,2,2,3,\dots,2,2,2,3}_{(k-1)\text{ blocks } 2,2,2,3}, 2,2,2,2]$; $[\underline{6},\dots,6,1,2,2,2,2,3,\dots,2,2,2,3,2,2,2,2]$

$(\ell = 8, e = -4)$: (Example 3.11) For $k \geq 1$,
$[\underbrace{6,\dots,6}_{k}, 7, 2, 2, 3, \underbrace{2,2,2,3,\dots,2,2,2,3}_{(k-1)\text{ blocks } 2,2,2,3}, 2,2,2,2]$;
$[\underline{6},\dots,6,7,1,2,2,2,2,2,2,3,\dots,2,2,2,3,2,2,2,2]$, $[\underline{6},\dots,6,6,2,1,3,2,2,2,3,\dots,2,2,2,3,2,2,2,2]$
Each marking above produces different general fibers $\mathbb{F}_0$ and $\mathbb{F}_1$ respectively.

$(\ell = 7, e = -5, j = 0)$: For $k \geq 2$,
$[\underbrace{5,\dots,5}_{k}, 5, \underbrace{2,2,3,\dots,2,2,3}_{(k-1)\text{ blocks } 2,2,3}, 2,2,2]$; $[\underline{5},\dots,5,3,1,2,3,\dots,2,2,3,2,2,2]$

$(\ell = 7, e = -5, j = 1)$: For $k \geq 1$,
$[\underbrace{5,\dots,5}_{k}, 5, 3, \underbrace{2,2,3,\dots,2,2,3}_{(k-1)\text{ blocks } 2,2,3}, 2,2,2]$; $[\underline{5},\dots,5,4,1,2,2,3,\dots,2,2,3,2,2,2]$

$(\ell = 7, e = -3)$: For $k \geq 1$,
$[\underbrace{5,\dots,5}_{k}, 6, 2, 3, \underbrace{2,2,3,\dots,2,2,3}_{(k-1)\text{ blocks } 2,2,3}, 2,2,2]$; $[\underline{5},\dots,5,5,1,2,2,2,3,\dots,2,2,3,2,2,2]$

$(\ell = 6, e = -3)$: For $k \geq 2$,
$[\underbrace{4,\dots,4}_{k}, 5, \underbrace{2,3,\dots,2,3}_{(k-1)\text{ blocks } 2,3}, 2,2]$; $[\underline{4},\dots,4,2,1,3,\dots,2,3,2,2]$

$(\ell = 6, e = -2)$: For $k \geq 1$,
$[\underbrace{4,\dots,4}_{k}, 5, 3, \underbrace{2,3,\dots,2,3}_{(k-1)\text{ blocks } 2,3}, 2,2]$; $[\underline{4},\dots,4,3,1,2,3,\dots,2,3,2,2]$

$(\ell = 5, e = -5)$: For $k \geq 2$,
$[\underbrace{3,\dots,3}_{k-1}, 2, 6, \underbrace{3,\dots,3}_{k-2}, 2]$; $[\underline{3},\dots,3,2,1,3,\dots,3,2]$

$(\ell = 5, e = -1)$: For $k \geq 2$,
$[\underbrace{3,\dots,3}_{k}, 5, \underbrace{3,\dots,3}_{k-1}, 2]$; $[\underline{3},\dots,3,1,2,3,\dots,3,2]$

The end