Torelli Theorem for K3 surface, classic and derived version

Referencias:

- Lectures on K3 surfaces (Huybrechts). Page 22-24.
- The Fourier-Mukai transform in Algebraic geometry (Huybrechts) Chapter 10.1-10.2.

1 Torelli theorem for K3 surfaces

1.1 Ample cone of a K3 surface

The ample cone of X is the cone in $\mathrm{NS}(X)_{\mathbb{R}} = \mathrm{NS}(X) \otimes \mathbb{R}$ generated by ample classes:

$$\operatorname{Amp} X = \left\{ \sum_i a_i \mathcal{L}_i : \ \mathcal{L} \in \operatorname{Pic} X \text{ is ample and } a_i \geq 0 \right\}.$$

Since $\mathrm{NS}(X)$ has signature $(1, \rho(X) - 1)$, the set

$$\mathcal{P}_X := \{ w \in \mathrm{NS}(X)_{\mathbb{R}} : w^2 > 0 \}$$

has two connected components. We denote by $\mathcal{C}_X$ the positive cone of X corresponding to the connected component of $\mathcal{P}_X$ containing the ample cone. Note that

$$\mathcal{P}_X = \mathcal{C}_X \sqcup -\mathcal{C}_X$$

We have the following result:

Proposition 1. *Let X be a K3 surface, then*

$$\operatorname{Amp} X = \{ \mathcal{L} \in \mathcal{C}_X : (\mathcal{L}, C) > 0 \textit{ for every curve } C \simeq \mathbb{P}^1 \subset X \}$$

We start with the following short lemma:

Lemma 2. *Let $C \subset X$ be an integral curve. Then $(C)^2 \geq -2$. If C is a (-2)-curve, i.e., $(C)^2 = -2$, then C is a smooth rational curve and so isomorphic to $\mathbb{P}^1$*

Proof. Recall the definition of the arithmetic genus:

$$\rho_a(C) = 1 - \chi(C, \mathcal{O}_C).$$

From the exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O} \to \mathcal{O}_C \to 0,$$

we obtain $\rho_a(C) = 1 + \chi(X, \mathcal{O}(-C)) - \chi(X, \mathcal{O}_X)$. By applying Riemann-Roch twice, we see that

$$2\rho_a(C) - 2 = (C)^2.$$

Let $\nu : \tilde{C} \to C$ be the normalization. From the exact sequence

$$0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\tilde{C}} \to \nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C \to 0$$

we obtain

$$\rho_a(C) = g(\tilde{C}) + h^0(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C).$$

From this we see that $\rho_a(C) \geq 0$ and so $(C)^2 \geq -2$. If $(C)^2 = -2$, then $h^0(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) = 0$ and so C is a smooth rational curve. □

Proof. (Proposition 1) Let $\mathcal{L}$ be a line bundle in $\mathcal{C}_X$. By Nakai-Moishezon-Kleiman criterion, it is enough to prove that $(\mathcal{L}, C) > 0$ for every integral curve $C \subset X$. By the previous Lemma and the hypthesis, it is enough to prove it for the case when $(C)^2 \geq 0$.
Assume that $(C)^2 \geq 0$. Then

$$C \in \overline{\mathcal{P}_X} = \overline{\mathcal{C}_X} \cup \overline{-\mathcal{C}_X}$$

and so C is either in the closure $\mathcal{C}_X$ or $-\mathcal{C}_X$. Now we show that $C \notin \overline{-\mathcal{C}_X}$. Let $\mathcal{V} \in \mathrm{Amp}(X)$. Then we have $(C, \mathcal{V}) > 0$. Now if $C \in \overline{-\mathcal{C}}$, from the fact that $\mathcal{V} \in \mathcal{C}_X$, we may find a $t \in (0, 1)$ such that $((1-t)C + t\mathcal{V})^2 = 0$, but this is impossible since

$$((1-t)C + t\mathcal{V})^2 = (1-t)^2(C)^2 + 2t(1-t)(C, \mathcal{V}) + t^2(\mathcal{V})^2 > 0.$$

Therefore $C \in \overline{\mathcal{C}_X}$. Now consider an element $M \neq 0$ in the boundary of $\overline{\mathcal{C}_X}$. Then $M^2 = 0$. If $(\mathcal{L}, M) < 0$, from the equation

$$((1-t)M + t\mathcal{L})^2 = 2t(1-t)(\mathcal{L}, M) + t^2(\mathcal{L})^2,$$

we can see that by choosing $t > 0$ small enough, we have $((1-t)M + t\mathcal{L})^2 < 0$, but this is impossible since $(1-t)M + t\mathcal{L} \in \mathcal{C}_X$ for $t \in (0, 1)$. Thus $(\mathcal{L}, M) \geq 0$. Moreover, from the fact that $M^\perp = \{v \in \mathrm{NS}(X)_\mathbb{R} : (M, v) = 0\}$ is a plane (the pairing in non-degenerated) and $\mathcal{L} \in \mathcal{C}_X$, we have that $(\mathcal{L}, M) > 0$. Indeed, if $(\mathcal{L}, M) = 0$, then $M^\perp$ would divide the cone $\mathcal{C}_X$ in two non-empty parts, one where the intersection with M is positive and one where it is negative, but this is impossible as $(\mathcal{L}', M) \geq 0$ for every $\mathcal{L}' \in \mathcal{C}_X$ (just as we proved for $\mathcal{L}$). Finally, since very element in $\overline{\mathcal{C}_X}$ is a positive linear combination of elements M in the boundary of $\overline{\mathcal{C}_X}$, we conclude that $(\mathcal{L}, C) > 0$. □

1.2 Torelli Theorem for K3 surfaces

Let X be a K3 surface. As we saw last week, the abelian group

$$H^2(X, \mathbb{Z})$$

endowed with cup product is a lattice abstractly isomorphic to

$$E_8^{\oplus 2} \oplus U^{\oplus 3}$$

where U is the hyperbolic lattice. The complex structure of X induces a Hodge-structure on $H^2(X, \mathbb{Z})$ via the standard Hodge-decomposition on $H^2(X, \mathbb{C})$:

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} = H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

where we have isomorphisms $H^{p,q}(X) \simeq H^q(X, \Omega^q)$, and $H^{p,q}(X)$ and $H^{p',q'}(X)$ are orthogonal if $(p, q) \neq (p', q')$. Let X, Y be two K3 surfaces (or two compact complex surfaces). A Hodge-isometry $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is an isometry of lattices such that its linear extension $\varphi_{\mathbb{C}} : H^2(X, \mathbb{C}) \to H^2(Y, \mathbb{C})$ satisfies:

$$\varphi_{\mathbb{C}}(H^{p,q}(X)) = H^{p,q}(Y).$$

The following theorem says that the lattice $H^2(X, \mathbb{Z})$ and its Hodge-structure determines the K3 surface.

Theorem 3 (Torelli theorem for K3 surfaces)**.** *Let X, Y be two K3 surfaces. Then X is isomorphic to Y iff there is a Hodge-isometry $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$.*

Remarks:

(i) When $X = Y$, there is a natural group morphism

$$\operatorname{Aut} X \to \{\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} H^2(X, \mathbb{Z}) : \varphi \text{ is Hodge-isometry}\}, \qquad f \mapsto f_*.$$

This map is not surjective, i.e, not every Hodge isometry is of the form f_*. One important example is the following: let $C \subset X$ be a (-2)-curve, i,e, its class $\delta = c_1(C)$ satisfies $\delta^2 = -2$ (for example, X could be a Kummer surface and C one of the exceptional divisors). The linear map

$$s_\delta : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}), \qquad s_\delta(v) = v + (v, \delta) \cdot \delta$$

corresponding to the orthogonal reflection along the plane $H_\delta = \delta^\perp$ is a Hodge isometry. Indeed, it is an isometry:

$$(s_\delta(v), s_\delta(w)) = (v + (v, \delta)\delta, w + (w, \delta)\delta) = (v, w) + 2(v, w) + (v, w)\delta^2 = (v, w).$$

It respects the Hodge-structure: its $\mathbb{C}$-linear extension acts as the identity on $H^{2,0}(X)$ and $H^{0,2}(X)$, and preserves $H^{1,1}(X, \mathbb{Z})$.

If $f_* = \varphi$ for some $f \in \operatorname{Aut}(X)$. For an ample line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ we have

$$(s_\delta(c_1(\mathcal{L})), \delta) = -(c_1(\mathcal{L}), \delta) = -\deg \mathcal{L}|_C < 0$$

but $s_\delta(c_1(\mathcal{L}))$ is also the class of an ample line bundle and so it is positive which is impossible.

These isometries are called Lefschetz-Hodge reflections:

(ii) The condition that φ sends an ample class to an ample class in order to be of the from f_* for some isomorphism $f : X \to Y$ is actually sufficient.

Theorem 4. *A Hodge-isometry $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ lifts to an isomorphism iff its $\mathbb{R}$-linear extension $\varphi_{\mathbb{R}}$ preserves the ample cone.*[1].

Using Proposition 1, we can prove that every Hodge isometry φ is of the form

$$\varphi = \pm s_{\delta_1} \circ \ldots \circ \delta s_{\delta_k} \circ f_*$$

for some isomorphism $f : X \to Y$ and a sequence of L-H reflections $\{s_{\delta_i}\}_i$. The basic idea is the following. Let $c = c_1(\mathcal{L})$ be the class of an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X)$. Since φ fixes $\mathcal{P}_X$, then either after multiplying by -1 if necessary, we may assume that $\varphi(c) \in \mathcal{C}_X$. If $\varphi(c)$ is not in the ample cone, then Proposition 1 would imply that there is a curve $C \simeq \mathbb{P}^1$ such that $(\varphi(c), C) < 0$. The L-H reflection associated S_δ to C then change the sign of the intersection. We can find a sequence of such smooth rational curves $C_1, \ldots, C_k$ such that the class

$$c' = \pm \pm s_{\delta_1} \circ \ldots \circ \delta s_{\delta_k} \circ \varphi(c)$$

has positive intersection with each $\mathbb{P}^1$. Proposition 1 then implies that c' is the class of an ample line bundle and the Theorem above implies that the desired form for φ.

2 Derived categories and Derived Torelli Theorem

The derived category $D(X)$ of a variety X was initially created as a formal object to define derived functors and compute cohomology of sheaves on X. But since the work of Bondal and Orlov , the derived category has turned out to be a very interesting variety specially in the context of moduli spaces.

Two smooth varieties X, Y are called derived equivalent if $D(X) \simeq D(Y)$. In general $D(X) \simeq D(Y)$ doesn't imply that $X \simeq Y$, but it stills a very interesting question to relate X and Y.

The derived Torelli Theorem, due to Mukai and Orlov, gives sufficient and necessary conditions for two K3 surfaces to be derived equivalent in terms of the Mukai lattice:

$$\tilde{H}^*(X, \mathbb{Z}) = (H^*(X, \mathbb{Z}), \langle, \rangle)$$

where the bilinear product is given by

$$\langle (r, c, d) \cdot (r', c', d') \rangle = cc' - rd' - r'd,$$

for $(r, c, d), (r', c', d') \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ It has a Hodge-structure of weight 2 given by

$$\tilde{H}^*(X, \mathbb{Z})^{2,0} = H^{2,0}(X), \quad \tilde{H}^*(X, \mathbb{Z})^{0,2} = H^{0,2}(X), \quad \tilde{H}^*(X, \mathbb{Z})^{1,1} = H^0(X, \mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X, \mathbb{C})$$

As in the classic setting, we say that an isometry between the Mukai lattice of two K3 surfaces is a Hodge-isometry if it respects this Hodge-structure.

Theorem 5. *Let X, Y be two K3 surfaces. Then $D(X) \simeq D(Y)$ iff there is a Hodge-isometry $H^*(X, \mathbb{Z}) \simeq H^*(Y, \mathbb{Z})$.*

[1]When X is not projective, we require that $\varphi_{\mathbb{R}}$ preserves the Khaler cone