Mukai vector and slope stability

Anibal Aravena

1 The Mukai vector

When X is a K3 surface, it is convenient to use the Mukai vector in order to described the Chern classes of sheaves $E \in \operatorname{Coh} X$.
Definition: Let $E \in \operatorname{Coh}(X)$, the Mukai vector of E is the vector

$$v(E) = (r, c, s) \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) = H^*(X, \mathbb{Z})$$

given by the formula:

$$v(E) = \operatorname{ch}(E)\sqrt{\operatorname{td}(X)} = (\operatorname{rank}(E), c_1(E), \operatorname{ch}_2(E) + \operatorname{rank}(E)).$$

Recall that for a general sheaf $E \in \operatorname{Coh} X$, their Chern classes are computed via resolution of vector bundles, i.e., for an exact sequence

$$0 \to V^r \to \ldots \to V^0 \to E \to 0$$

with V^k vector bundles, we have $\operatorname{ch} E \coloneqq \sum_k (-1)^k \operatorname{ch}(V^k)$.
Remark:
The Mukai vector is additive in short exact sequence, i.e., for all exact sequence of sheaves:

$$0 \to K \to E \to Q \to 0$$

we have $v(E) = v(Q) + v(K)$. Thus the Mukai vector induces a group morphism $v : K(\operatorname{Coh} X) \to H^*(X, \mathbb{Z})$ whose image is

$$H^0(X, \mathbb{Z}) \oplus \operatorname{NS}(X) \oplus H^4(X, \mathbb{Z})$$

Examples

1. $\mathcal{L} \in \operatorname{Pic} X$, then
$$v(\mathcal{L}) = (1, c_1(\mathcal{L}), c_1(\mathcal{L})^2/2 + 1)$$
2. Let $Z \subset \mathcal{O}_X$ be a subscheme of length ℓ, then $v(\mathcal{O}_Z) = (0, 0, \ell)$.
3. Let $C \subset X$ be a smooth curve. By the exact sequence
$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$
we obtain
$$v(\mathcal{O}_C) = v(\mathcal{O}_X) - v(\mathcal{O}_X(-C)) = (0, c_1(\mathcal{O}_X(C)), -(C)^2/2).$$

4. More general consider a coherent sheaf of the form $F = i_*G$ where $i : C \to X$ is the inclusion of a smooth curve. Then the Chern classes of F can be computed by using Grothendieck-Riemann-Roch

$$\operatorname{ch}(f_!G)\operatorname{td}X = f_*(\operatorname{ch}F\operatorname{td}X)$$

Here we consider $f = i$ and the equation becomes

$$\operatorname{ch}(F)\operatorname{td}X = i_*(\operatorname{ch}G\operatorname{td}C)$$

Using the $\operatorname{td}X = (1, 0, 2)$[1] and that $\operatorname{td}C = (1, c_1(T_C)/2)$, we obtain

$$\begin{aligned}(\operatorname{ch}_0(F), \operatorname{ch}_1(F), \operatorname{ch}_2(F)).(1, 0, 2) &= i_*\left((\operatorname{ch}_0(G), \operatorname{ch}_1(G)).(1, c_1(T_C))\right) \\ (\operatorname{ch}_0(F), \operatorname{ch}_1(F), \operatorname{ch}_2(F) + \operatorname{ch}_0(F)) &= i_*(\operatorname{ch}_0(G), \operatorname{ch}_1(G) + c_1(T_C))\end{aligned} \tag{1}$$

Why we use the Mukai pairing? One reason is his relation with the Euler pairing. Recall the definition of the Euler characteristic

$$\chi(F) = \sum_i (-1)^i H^i(X, F).$$

We can generalize this expression as a quadratic form called the Euler pairing as follows: for $E, F \in \operatorname{Coh}X$, define the Euler pairing:

$$\chi(E, F) = \sum_i (-1)^i \dim \operatorname{Ext}^i(E, F)$$

From last week, we have the Mukai lattice

$$\tilde{H}(X, \mathbb{Z}) = (H^*(X, \mathbb{Z}), \langle , \rangle)$$

where the pairing $\langle , \rangle$ is defined as:

$$\langle (r, c, s), (r', c', s') \rangle = cc' - rs' - r's \tag{2}$$

Proposition 1.

$$\chi(E_1, E_2) = -\langle v(E_1), v(E_2) \rangle$$

Proof. For a locally free sheaves E, we have $\chi(E, F) = \chi(E^* \otimes F)$ and by Riemann roch we obtain

$$\chi(E, F) = \int \operatorname{ch}(E^* \otimes F)\operatorname{td}X = \int \operatorname{ch}(E^*)\sqrt{\operatorname{td}X}\operatorname{ch}(F)\sqrt{\operatorname{td}X}$$

But $\operatorname{ch}(E)^* = (-1)^i \operatorname{ch}_i(E)$ and from (2) the formula of the Proposition follows. For a general sheaf E, we resolve E by a complex of vector bundles vector bundle and then use the additive property of the Euler characteristic and the Mukai vector. □

[1] Here we assume that X is a K3

2 Stability

In this section, we will focus on the case when X is a smooth projective surface but the definitions and results can be generalized to any dimension. The notion of stability can be defined for vector bundles on X, but in order to have more flexibility with computations, we will work with the more general notion of torsion free sheaf.

Definition A non-zero coherent sheaf $E \in \operatorname{Coh} X$ is called torsion free sheaf if for every point $p \in X$ E_p torsion free $(\mathcal{O}_X)_p$-module.

Examples. Vector bundles Any sub-sheaf of a vector bundle (or a torsion free sheaf)

Every torsion free sheaf on a smooth curve is a vector bundle. For smooth surfaces, we have the following result:

Proposition 2. *Assume that X is a smooth surface. Let $E \in \operatorname{Coh} X$ be a torsion free sheaf and E^{**} be its double dual. Then E^{**} is a vector bundle and there is an exact sequence*

$$0 \to E \to E^{**} \to Q \to 0$$

where Q is a sheaf supported in dimension zero.

Let $\mathcal{L} \in \operatorname{Pic} X$ be an ample line bundle and $h = c_1(\mathcal{L})$ be its class in $\operatorname{NS}(X)$. The slope (with respect to h) of a torsion free sheaf E is given by the formula

$$\mu(E) = \mu_h(E) := \frac{(h, c_1(E))}{\operatorname{rank} E}$$

A torsion free sheaf E is called stable if for all non-zero sub sheaf $K \subset E$ with $\operatorname{rank} K < \operatorname{rank} E$, we have

$$\mu(K) < \mu(E)$$

If E satisfied the weaker inequality $\mu(K) \leq \mu(E)$, then E is called semistable. The following proposition is clear.

Proposition 3. *For a short exact sequence*

$$0 \to F \to E \to G \to 0$$

we have that

$$\mu(F) < \mu(E) \iff \mu(E) < \mu(G)$$

The slope stability condition allows to prove result on Hom's in a ease way

Proposition 4. *(i) Consider two semistable objects F and E such that $\mu(F) > \mu(E)$. Then $\operatorname{Hom}(F, E) = 0$.*

(ii) If E, F are stable with $\mu(E) = \mu(F)$, then

$$\textit{either } F \simeq E \textit{ or } \quad \operatorname{Hom}(F, E) = 0$$

(iii) If E is stable, then $\operatorname{Hom}(E, E) = \mathbb{C}$

We start with a useful lemma concerning quotient of torsion free sheaves:

Lemma 5. *Let E be a torsion-free sheaf a consider a non-zero subsheaf $K \subset E$, then there is subsheaf $F \subset \tilde{K} \subset E$ such that $E/\tilde{K}$ is either zero is torsion free, and it satisfies*

$$\mu(\tilde{K}) \geq \mu(K).$$

Proof. For simplicity let's assume that X has dimension 2. For every sheaf $F \in \operatorname{Coh} X$, there is a decomposition

$$0 \to F_1 \to F \to F_2 \to 0$$

where F_1 is torsion part of F and F_2 is torsion free. By applying this decomposition to E/K, we may find a sheaf $\tilde{K}$ containing K such that $\tilde{K}/K$ is isomorphic to the torsion part of E/K. It is then clear that $\operatorname{rank} \tilde{K} = \operatorname{rank} K$ and that $E/\tilde{K}$ is torsion free. Now consider the exact sequence

$$0 \to K \to \tilde{K} \to \tilde{K}/K \to 0$$

Since $\tilde{K}/K$ is torsion, then it is supported in dimension zero or 1
Case 1: $\tilde{K}/K$ is supported in dimension 0:
Then $c_1(\tilde{K}/K) = 0$ and $\mu(K) = \mu(\tilde{K})$.
Case 2:: $\tilde{K}/K$ is supported in dimension 1:
Write $Y = \operatorname{Supp} \tilde{K}/K$, endowed with the annihilator subscheme structure. Write $i : Y \to X$ the inclusion. Then

$$i_* i^* \tilde{K}/K \simeq \tilde{K}/K$$

and compute $c_1(\tilde{K}/K)$ via Grothendieck-Riemann-Roch. For simplicity, let us assume that Y is a smooth curve smooth. From (1), we see that

$$c_1(\tilde{K}/K) = \operatorname{rank}(i^*(\tilde{K}/K))c_1(\mathcal{O}(C)$$

and so

$$(h, c_1(\tilde{K}) > (h, c_1(K))$$

which implies the inequality for the slope. □

It is not hard to see that this $\tilde{K}$ is unique and it is called the saturation of K in E.

Proof. (Proposition 4)

(i) Assume that there is a non-zero morphism $\varphi : F \to E$. Consider the exact sequence

$$0 \to \operatorname{Ker} \varphi \to F \to \operatorname{Im} \varphi \to 0$$

Note $\operatorname{Im} \varphi$ is torsion free and so $\operatorname{rank} \operatorname{Ker} \varphi < \operatorname{rank} F$. Thus semi stability of F we have

$$\mu(\operatorname{Ker} \varphi) \leq \mu(F) \implies \mu(\operatorname{Im} \varphi) \geq \mu(F)$$

By the previous Lemma, we may find sheaf $K \subset E$ with $\mu(K) \geq \mu(\operatorname{Im} \varphi)$ such that E/K is torsion free (and so positive rank) or zero. Since $\mu(K) \geq \mu(F) > \mu(E)$, these two cases are impossible.

(ii) Assume that there is a non-zero morphism $\varphi : F \to E$. We need to show that $F \simeq E$. Let's see first that it is injective. Assume that $\operatorname{Ker}\varphi \neq 0$ and consider the exact sequence

$$0 \to \operatorname{Ker}\varphi \to F \to \operatorname{Im}\varphi \to 0$$

If $\operatorname{rank}\varphi = \operatorname{rank} F$ then $\operatorname{Im}\varphi$ zero which is impossible. Thus since $\operatorname{Ker}\varphi \neq 0$, we may apply stability and obtain that

$$\mu(\operatorname{Ker}\varphi) < \mu(F) \implies \mu(F) < \mu(\operatorname{Im}\varphi).$$

An by a similar argument as before, we obtain a contradiction. The proof that φ is surjective is analogous.

(iii) Let $\varphi : E \to E$ a non-zero morphism. Consider a point $x \in X$ and the induced map on fibers $\varphi_x : E|_x \to E|_x$. Let $\lambda \in \mathbb{C}$ be an eigen-value of φ_x. Then the map $\varphi - \lambda \cdot \mathrm{id}_E : E \to E$ is not-injective and so it must be zero by part (ii). Thus $\operatorname{Hom}(E, E) = \mathbb{C}$

□

Examples:

1. Any line bundle is stable.
2. A direct sum of stable sheaf $E_1 \oplus E_2$ is never stable. It is semisstable iff $\mu(E_1) = \mu(E_2)$.
3. Consider two line bundles E_1, E_2 such that $\mu(E_1) = 0$ and $\mu(E_2) = 1$. Assume furthermore that $\operatorname{Ext}^1(E_1, E_2) \neq 0$ and consider a non-split exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

Then E is stable. Indeed, assume that there is a subsheaf $F \subset E$ such that $0 < \operatorname{rank} F < \operatorname{rank} E = 2$ (and so of rank 1) such that $\mu(F) \geq \mu(E) = 1/2$. Note that F is stable. Consider the composition

$$F \subset E \to E_2.$$

If it is zero, then $F \subset E$ factorize through a map $F \to E_1$ which is non-zero and so $\mu(F) \leq \mu(E_1) = 0$ by part (i) of Proposition 4 which is impossible. Thus $F \to E_2$ is non-zero and so $\mu(F) \leq \mu(E_2) = 1$. The only possibility is that $\mu(F) = 1$ but this would imply that $F \to E_2$ is an isomorphism which is impossible since the exact sequence defining E doesn't split.

Bogomolov-Gieseker inequality Let $E \in \operatorname{Coh} X$ be a stable coherent sheaf. We have that $\dim \operatorname{Hom}(E, E) = 1$. By duality, we obtain that $\dim \operatorname{Ext}^2(E, E) = 1$ and so

$$\langle v(E), v(E) \rangle + 1 = \dim \operatorname{Ext}^1(E, E) \geq 0$$

Expanding the Righ-hand side we obtain the inequality

$$\Delta(E) = 2 \operatorname{rank} E c_2(E) - (\operatorname{rank} E - 1) c_1^2(E) \geq 2((\operatorname{rank} E)^2 - 1).$$

This is the the Bogomolov-Gieseker inequality. We will use this inequality for the construction of Bridgleand stability conditions.