Gieseker stability and moduli spaces

1 Gieseker stability

Consider a projective variety X and let $\mathcal{O}(1)$ be an ample line bundle with Chern class $h = c_1(\mathcal{O}(1)) \in \mathrm{NS}(X)$. Gieseker stability allows to extend the notion of slope stability (or μ-stability) for sheaves that are not necessarily torsion free. Let $E \in \operatorname{Coh} X$. The dimension of E is the dimension of its support:

$$\dim E = \dim \operatorname{Supp} E = \dim\{x \in X : E_x \neq 0\}.$$

Definition: A sheaf $E \in \operatorname{Coh} X$ is called pure dimensional if for all non-zero subsheaf $K \subset E$, we have $\dim F = \dim E$.
Examples:

(i) Torsion free sheaves are pure dimensional sheaf of maximal dimension.

(ii) Let $i : Y \hookrightarrow X$ be a integral closed subscheme and $G \in \operatorname{Coh} Y$. Then $F = i_* G$ is pure dimensional iff G is torsion free sheaf.

Recall the Hilbert polynomial of $E \in \operatorname{Coh} X$ with respect to the class h:

$$P(E,m) = \chi(E \otimes \mathcal{O}(m)) = \sum_i \alpha_i(E) \frac{m^i}{i!} \in \mathbb{Q}[m]$$

The degree d of $P(E,m)$ coincides with the dimension of E and $\alpha_d(E)$ is always positive.
Definition: Let $E \in \operatorname{Coh} X$ be a coherent sheaf of dimension d. The reduced Hilbert polynomial $p(E)$ of E is defined by

$$p(E) = p(E,m) := \frac{P(E,m)}{\alpha_d(E)}.$$

If E is pure dimensional, then E is called h-Gieseker stable (or just stable) if for all proper subsheaf $F \subset E$, we have

$$p(F,m) < p(E,m) \qquad m >> 0.$$

If E satisfies the weaker inequality $p(F,m) \leq p(E,m)$, then E is called semistable.

Note that the order in polynomials $p \leq q$ iff $p(m) \leq q(m)$ for $m >>$ is just the usual lexicographic order of polynomials.

The reduced Hilbert polynomial can be computed using H-R-R:

$$\chi(E(m)) = \int \operatorname{Ch}(E \otimes \mathcal{O}(m)) \operatorname{Tod} X.$$

When X is K3 surface, we can use the Mukai pairing from last week to obtain $p(E)$ as follows: Consider a sheaf $E \in \operatorname{Coh} X$ with Mukai vector

$$v(E) = (r, c, d) \in H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) = H^*(X, \mathbb{Z}).$$

Then we have

$$\chi(E(m)) = -\langle \mathcal{O}(-m), v(E) \rangle = -\langle (1, -mh, m^2h^2/2+1), (r, c, d) \rangle = \frac{m^2h^2}{2} r + m(h, c) + d + r$$

and so

$$p(E, m) = \begin{cases} \frac{m^2}{2} + m\frac{(h,c)}{h^2 r} + \frac{d+r}{h^2 r} & \text{if } r \neq 0 \\ m + \frac{d}{(h,c)} & \text{if } r = 0, c \neq 0 \\ 1 & \text{if } r = c = 0. \end{cases}$$

Thus if E torsion free, then E is Gieseker-semistable iff for all non-zero proper $F \subsetneq E$ with Mukai vector $v(F) = (r', c', d')$ we have

$$\mu_h(F) < \mu_h(E) \quad \text{or} \quad \mu_h(F) = \mu(E) \text{ and } \frac{d'}{r'} < \frac{d}{r}.$$

Proposition 1. *Let $E \in \operatorname{Coh} X$ be a torsion free sheaf. Then*

$$E \text{ is } \mu\text{-stable} \implies E \text{ is (Gieseker)-stable} \implies E \text{ (Gieseker)-semistable} \implies E \text{ is } \mu\text{-semistable}$$

Proof. The only non trivial implication is from μ-stable to Gieseker-stable.
Consider a μ-stable sheaf E. Let $F \subset E$ be a non-trivial proper subsheaf. Let $\tilde{F}$ be its saturation. Then there is an exact sequence

$$0 \to K \to \tilde{K} \to \tilde{K}/K \to 0$$

where $\operatorname{rank} \tilde{K} = \operatorname{rank} K$ and $E/\tilde{K}$ is either zero or torsion free. Then

$$p(\tilde{F}, m) = p(F, m) + \chi(\tilde{K}/K(m)) \geq p(F, m), \qquad m >> 0.$$

and the Gieseker-stability follows from μ-stability applied to the subsheaf $\tilde{F} \subset E$. □

The following proposition tell us that semistable sheaves are the building blocks to construct pure-dimensional sheaves:

Proposition 2. *Every pure-dimensional coherent sheaf E has a unique Harder-Narasimham (H-N) filtration, i.e, there is a filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \ldots \subsetneq E_n = E$$

such that each factor $F_i = E_i/E_{i-1}$ is semistable and they satisfies

$$p(F_1) > p(F_2) > \ldots > p(F_n).$$

This filtration is unique and the factors F_i are called the H-N factors of E.

When E is semistable sheaf we have a further filtration:

Proposition 3. *Let E be a semistable coherent sheaf. Then E has a Jordan-Holder (J-H) filtration, i.e, there is a filtration*

$$0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = E$$

such that each factor $F_i = E_i/E_{i-1}$ is stable with reduced Hilbert polynomial $p(E)$. Moreover, the factors F_i are unique up to order and so the sheaf

$$gr(E) = \bigoplus_i F_i$$

doesn't depends on the filtration for E.

A J-H filtration is not necessarily unique, i.e, consider $E = E_1 \oplus E_2$ where E_1, E_2 are two stable sheaves with the same reduced Hilbert but $E_1 \not\simeq E_2$.

Using the this proposition, we obtain the following notion which appears naturally in the context of moduli spaces.

Definition Two semistable sheaf E, E' are called S-equivalent if $gr(E) = gr(E')$.

2 Moduli space

Now we are ready to define moduli space of sheaves on a K3 surface X. We start with the definition of the Moduli functor: Let $v \in H^*(X, \mathbb{Z})$ and $h \in \mathrm{NS}(X)$ be the class of an ample line bundle. We define the functor

$$\mathcal{M}_h(v) : (\mathrm{Sch}\,/\mathbb{C})^o \to (\mathrm{Sets})$$

to be the functor sending an scheme $S \in (\mathrm{Sch}\,/\mathbb{C})$ to the set

$$\mathcal{M}_h(v)(S) = \left\{ \begin{array}{l} \mathcal{E} \in \mathrm{Coh}(X \times S) : \text{flat over } S, \text{ such that} \\ \forall s \in S \text{ close point, } \mathcal{E}_s := \mathcal{E}|_{X \times \{t\}} \text{ is semistable} \\ \text{with Mukai vector } v. \end{array} \right\}$$

A course moduli space for the functor $\mathcal{M}_h(v)$ is a scheme $M \in (\mathrm{Sch}\,/\mathbb{C})$ and a natural transformation of functors:

$$\eta : \mathcal{M}_h(v) \to \mathrm{Hom}(\cdot, M)$$

such that the map

$$\eta_{\mathbb{C}} : \mathcal{M}_h(v)(\mathrm{Spec}(\mathbb{C})) \to M(\mathbb{C}) = \mathrm{Hom}(\mathbb{C}, M)$$

induces a bijection between S-equivalences classes of semistable sheaves and $\mathbb{C}$-points of M. We will write $M = M_h(v)$ and refer to M as a moduli space of h-Gieseker-semistable sheaves with Mukai v on X.

The following proposition tell us that identification of S-equivalents elements is necessary, at least if we want the course moduli space to be separated.

Proposition 4. *Let M be a separated scheme over $\mathbb{C}$ and consider a natural transformation*

$$\eta : \mathcal{M} \to \mathrm{Hom}(\cdot, M)$$

Let $E, F \in \mathcal{M}_h(v)(\mathbb{C})$ be two sheaves S-equivalents. Then

$$\eta_{\mathbb{C}}(E) = \eta_{\mathbb{C}}(F).$$

Proof. For simplicity lets assume that E has two J-H stable factors E_1, E_2 and that $F = E_1 \oplus E_2$. Then E fits in a exact sequence

$$0 \to E_1 \to E \to E_2 \to 0.$$

We claim that there is a flat family $\mathcal{E} \in \mathrm{Coh}(X \times \mathbb{A})$ such that

(i) $\mathcal{E}_t \simeq E$ for $t \neq 0$

(ii) $\mathcal{E}_0 \simeq E_1 \oplus E_2 = F$

If such family exists, then the associated morphism $f = \eta_{\mathbb{A}}(\mathcal{E}) : \mathbb{A} \to M$ satisfies for all $t \neq 0$

$$f(t) = \eta_{\mathbb{C}}(\mathcal{E}_t) = \eta_{\mathbb{C}}(E).$$

Since M is separated, then $\eta_{\mathbb{C}}(\mathcal{E}_0) = f(0) = \eta_{\mathbb{C}}(F)$.

Let $p : X \times \mathbb{A} \to X$ be the projection into X and $i : X = X \times \{0\} \hookrightarrow X \times \mathbb{A}$ be the inclusion. Define $\mathcal{E}$ to be the kernel of the surjective map $p^*E \to i_*E_2$ given by the composition

$$p^*E \to p^*E_2 \to i_*i^*(p^*E_2) = i_*E_2.$$

We claim that $\mathcal{E}$ satisfies the condition (i) and (ii). By definition, $\mathcal{E}$ fits into a short exact sequence in $\mathrm{Coh}(X \times \mathbb{A})$:

$$0 \to \mathcal{E} \to p^*E \to i_*E_2 \to 0$$

Since i_*E_2 is supported in $X \times \{0\}$, part (i) is clear. Now to compute $E_0 = i^*E$, we apply the derived functor Li^* to obtain the long exact sequence

$$0 \to L^1i^*(i_*E_2) \to i^*\mathcal{E} \to E \to i^*i_*E_2 \to 0.$$

Here we used that $L^ki^*(p^*E) = 0$ is zero if $k > 0$ since p^*E is flat over S. Now we have that $L^1i^*(i_*E_2) = E_2$ and $E_2 = i^*i_*E_2$. To see this use the resolution for $\mathbb{C}[t]/(t)$ in $\mathrm{Coh}\,\mathbb{A}$:

$$0 \to \mathbb{C}[t] \xrightarrow{\cdot t} \mathbb{C}[t] \to \mathbb{C}[t]/(t) \to 0$$

Then

$$L^1i^*(i_*E_2) = \mathrm{Ker}(\mathbb{C}[t] \otimes E_2 \overset{\cdot t \otimes \mathrm{id}_{E_2}}{\to} \mathbb{C}[t] \otimes E_2), \quad i^*i_*E_2 = \mathrm{Coker}(\mathbb{C}[t] \otimes E_2 \overset{\cdot t \otimes \mathrm{id}_{E_2}}{\to} \mathbb{C}[t] \otimes E_2),$$

but $t \cdot E_2 = 0$. Therefore $\mathcal{E}|_0$ fits into an exact sequence

$$0 \to E_2 \to \mathcal{E}|_0 \to E_1 \to 0$$

which splits and so $\mathcal{E}_0 = E_1 \oplus E_2$. Finally, the fact that $\mathcal{E}$ is a flat sheaf over S follows from the fact that its restriction to $X \times (\mathbb{A} - \{0\})$ is flat and $L^ki_*(\mathcal{E}) = 0$ for $k > 0$ (see the proof of Lemma [Huy06, Lemma 3.31]). □

The existence of a course Moduli space (under the assumption that $\mathcal{M}_h(v)(\mathbb{C})$ is non-empty) was proven by Gieseker, Maruyama and Simpson and its holds for any projective variety, not necessary for a projective K3 surface. We refer to [HL97, Chapter 4] for the proof of this statement who use GIT.

Proposition 5. *$\mathcal{M}_h(v)$ has always a coarse moduli space $M = M_h(v)$ which is a projective variety.*

Example(Hilbert scheme) Consider the Mukai vector $v = (1, 0, 1-n)$. Then

$$M_h(v) = \mathrm{Hilb}^n(X).$$

Indeed, let $T \in \mathrm{Coh}\, X$ be semistable with Mukai vector $v(T) = (1, 0, 1-n)$. Then T is a torsion free sheaf and we have an exact sequence

$$0 \to T^* \to T^{**} \to Q \to 0,$$

where Q has dimension zero and T^{**} is a line bundle. Since $c_1(T^{**}) = c_1(T) = 0$, we obtain $T^{**} \simeq \mathcal{O}_X$ and Q has length n and so T is the ideal sheaf of a subscheme Z of length n. Conversely, for every such subscheme Z, its ideal sheaf $\mathcal{I}_Z$ is stable (it has rank 1 and so is μ-stable) and it has Mukai vector $(1, 0, 1-n)$. Therefore $M_h(v)$ parametrizes ideal sheaves $\mathcal{I}_Z$ where $Z \subset X$ has length n but this is just the Hilbert-scheme.

References

[HL97] Daniel Huybrechts and Manfred Lehn. The geometry of moduli spaces of sheaves. 1997.

[Huy06] Daniel Huybrechts. Fourier-mukai transforms in algebraic geometry. 2006.