Derived categories and Bridgeland stability conditions

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1 Derived categories

Let X be a smooth projective variety. The bounded derived category $D(X)$ of X is a category whose objects are finite complexes:

$$\operatorname{Obj} D(X) = \left\{ \begin{array}{l} (E^\bullet, d) : E^\bullet = \bigoplus_{k\in\mathbb{Z}} E^k : \text{ such that } E^k \in \operatorname{Coh} X \text{ and } E^k = 0 \text{ for } |k| >> 0, \\ \text{degree 1 map } d : E^\bullet \to E^\bullet \text{ with } d^2 = 0 \end{array} \right\}$$

Commonly, we represent an object $(E, d) \in \operatorname{Obj} D(X)$ as a complex

$$E : \qquad \dots \xrightarrow{d} E^{-1} \xrightarrow{d} E^0 \xrightarrow{d} E^1 \xrightarrow{d} \dots$$

Morphism in $D(X)$ are constructed in such a way that for every that quasi-isomorphism[1] becomes an isomorphism in $D(X)$. In particular, the complex associated to any resolution of a sheaf is identified with the sheaf itself. As an example, consider a codimension 1 subvariety $Y \subset X$ and its standard sequence:

$$0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0.$$

Then it induces an isomorphism in $D(X)$:

$$(\mathcal{O}_X(-Y) \to \mathcal{O}_X) \xrightarrow{\sim} \mathcal{O}_Y.$$

The derived category is not an abelian category since kernel and cokernel doesn't in general exist but it has the structure of triangulated category. Here we list some of the properties I will use.

(i) $D(X)$ has a shift functor $[1] : D(X) \to D(X)$:

$$\text{degree } k \text{ part of } E[1] = \text{ degree } k+1 \text{ part of } E$$

(ii) It has a collection of exact triangles

$$A \to B \to C$$

which replaces the notion of exact sequence for an abelian category. There is fully faithful embedding $\operatorname{Coh} X \hookrightarrow D(X)$ by sending a sheaf E to the complex $E^\bullet = E$ concentrated in degree zero. This embedding sends s an exact sequence

$$0 \to A \to B \to C \to 0$$

to the exact triangle

$$A \to B \to C.$$

[1] i.e., a morphism of complexes inducing isomorphisms in cohomology.

(iii) Every morphism $A \to B$ fits into an exact triangle

$$A \to B \to C.$$

The complex C is usually called the cone of the morphism $A \to B$ and we write $C = \operatorname{cone}(A \to B)$

1.1 bounded t-structure and hearts

The concept of filtration is in the core of the definition of a Bridgeland stability condition. The notion of a bounded t-structure allows to filtrate any complex $E^\bullet \in D(X)$ by elements in a certain abelian category $\mathcal{A} \subset D(X)$ called the heart. The abelian category $\operatorname{Coh} X \hookrightarrow D(X)$ is an example of the heart of a bounded t-structure and allows to filtrate any complex $E^\bullet$ by coherent sheaves.

As an example, for a two step complex $E : (A^{-1} \xrightarrow{f} A^0)$, then E fits in an exact triangle

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E). \tag{1}$$

which can be seen as a filtration for E 2 factors $\mathcal{H}^{-1}(E)[1]$ and $\mathcal{H}^0(E)$.
In general, every complex $E \in D(X)$ has a (unique) filtration:

$$0 = E_0 \to E_1 \to E_2 \to \ldots \to E_n = E \tag{2}$$

where $\operatorname{cone}(E_{i-1} \to E_i) = F_i[k_i]$ for some sheaf $F_i \in \operatorname{Coh} X$ and integers $k_1 > k_2 > \ldots k_n$. The notion of bounded t-structure formalize this idea.
Definition: The heart of a bounded t-structure is an abelian full subcategory $\mathcal{A} \subset D(X)$ such that:

(i) Every complex $E^\bullet$ has a filtration:

$$0 = E_0 \to E_1 \to E_2 \to \ldots \to E_n = E \tag{3}$$

where $\operatorname{cone}(E_{i-1} \to E_i) = F_i[k_i]$ for some sheaf $F_i \in \mathcal{A}$ and integers $k_1 > k_2 > \ldots k_n$.

(ii) $\operatorname{Hom}(E[k_1], F[k_2]) = 0$ for $E, F \in \mathcal{A}$ and $k_1 > k_2$.

Condition (ii) guaranties that the filtration (3) is unique.
One important property that we will use is the fact that that the embedding $\mathcal{A} \hookrightarrow D(X)$ sends exact sequences in $\mathcal{A}$ to exact triangles in $D(X)$.

2 Bridgeland stability condition

A Bridgeland stability condition on X its a pair $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A} \subset D(X)$ is the heart of a t-bounded structure and Z is a linear map

$$Z : H^*_{\mathrm{alg}}(X, \mathbb{Z}) \to \mathbb{C}$$

called the central charge subject to the following conditions:

(i) (Positive property). For every non-zero $E \in \mathcal{A}$, we have

$$Z(v(A)) \in \mathbb{R}_{>0} e^{i\pi \cdot \phi}, \qquad \phi \in (0,1]$$

The number $\phi = \phi(E)$ is called the phase of E.
An element $E \in \mathcal{A}$ is called σ-stable (or just stable) if for all $F \subsetneq E$ non-zero,

$$\phi(F) < \phi(E).$$

If E satisfies the weaker inequality $\phi(F) \leq \phi(E)$ then E is called semistable

(ii) (Harder-Narasimham filtration) Every element $E \in \mathcal{A}$ has a H-N filtration:

$$0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = E$$

such that $F_i = E_i/E_{i-1} \in \mathcal{A}$ is σ-semistable with decreasing phase:

$$\phi(F_1) > \phi(F_2) > \ldots > \phi(F_n).$$

The elements F_i are called the H-N factors of E.

(iii) (Locally finite property) This is a technical condition whose definition we refer to Bridgeland original paper [Bri08, Page 247]. In particular, this implies that every semistable element $E \in \mathcal{A}$ has a J-H filtration:

$$0 = E_0 \subsetneq E_1 \subsetneq \ldots \subsetneq E_n = E$$

such that $F_i = E_i/E_{i-1} \in \mathcal{A}$ is σ-stable and $\phi(F_i) = \phi(E)$. The elements F_i are called the J-H factors of E.

Stability allows to prove results for Hom's just as in the slope stability case.

Proposition 1. *(i) Let E, F be stable with $\phi(E) < \phi(F)$, then* $\mathrm{Hom}(F, E) = 0$

(ii) If E, F are stable with the same phase, then

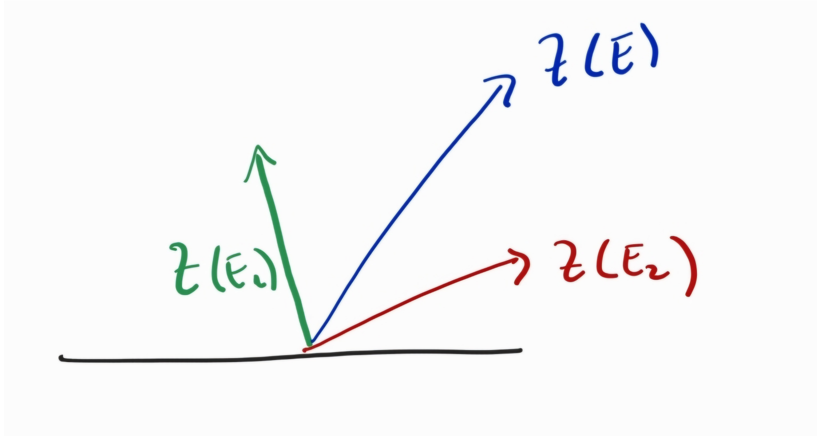
$$E \simeq F \textit{ or } \mathrm{Hom}(F, E).$$

(iii) If E is stable, then $\mathrm{Hom}(E, E) = \mathbb{C}$.

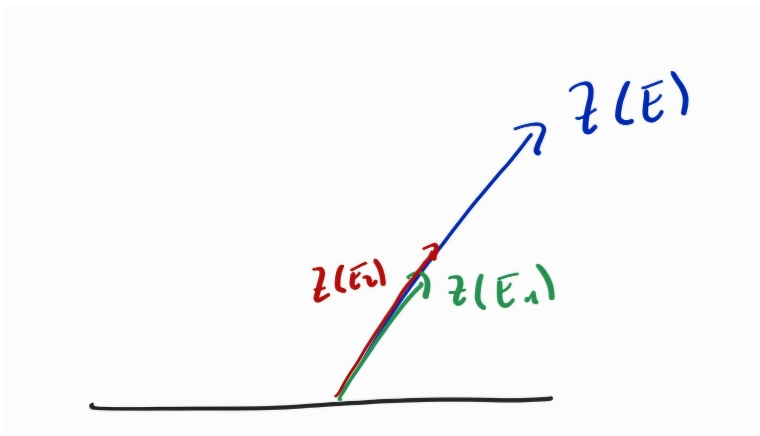
In our examples, we will study the stability of elements E that are non-trivial extension of two stable elements E_1, E_2:

$$0 \to E_1 \to E \to E_2 \to 0$$

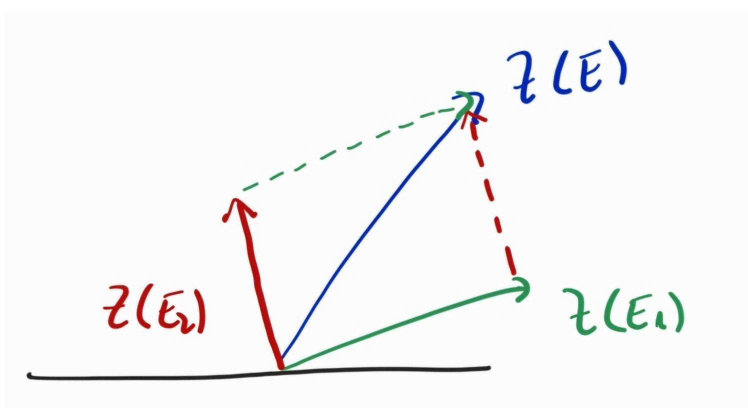
Its stability with respect to σ can be studied using by looking the image of the vectors $Z(E_1), Z(E_2) \in \mathbb{C}$. Let's see some common situations:

(i) 

Then E is not σ-semistable and its H-N factors are E_1 and E_2.

(ii) 

Here E is σ-semistable but not stable and E_1, E_2 are its J-H factors.

(iii) 

Assume also that the interior of the parallelogram doesn't contain points of $Z(\mathcal{A})$. Then E is σ-stable.

Indeed, suppose that E is not σ-stable and let $K \subsetneq E$ be a sub object with $\phi(K) \geq \phi(E)$. We may assume that K is stable. Call $\varphi : K \to E_2$ the composition with $E \to E_2$. Let see that φ is an isomorphism.
If $\varphi = 0$, then $K \to E$ factorizes through a map $K \to E_1$ which must be non-zero, but this

is impossible since $\phi(K) > \phi(E_1)$. Then $\operatorname{Im}\varphi \neq 0$ and by stability of E_2 and K, we have

$$\phi(K) \leq \phi(\operatorname{Im}\varphi) \leq \phi(E_2)$$

Moreover, $\Im Z(\operatorname{Im}\varphi) \leq \Im Z(E_2)$. Since the interior of the parallelogram doesn't contains point of $\operatorname{Im} Z(\mathcal{A})$, then $\phi(\operatorname{Im}\varphi) = \phi(E_2)$. By stability of E_2 we obtain that $\operatorname{Im}\varphi = E_2$ and so φ is surjective. To see that $\operatorname{Ker}\varphi = 0$, we look the exact sequence

$$0 \to \operatorname{Ker}\phi \to K \to E_2 \to 0$$

If $\operatorname{Ker}\varphi \neq 0$, then similarly, we obtain that $\phi(\operatorname{Ker}\varphi) = \phi(E_1)$ and so $E_1 \simeq \operatorname{Ker}\varphi$ but then $Z(E) = Z(K)$ and so $K = E$ which is impossible.
Therefore φ is an isomorphism but this contradicts the fact that the exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

doesn't split.

2.1 Examples of Bridgeland stability conditions

When X has dimension, then $\sigma = (\operatorname{Coh} X, Z)$ given by

$$Z(E) = -\deg E + i \operatorname{rank} E$$

is a Bridgeland stability condition and an object E is σ-(semis)stable iff E is either a slope (semis)stable vector bundle or $E \simeq k(x)$.
If X has dimension at least 2, this example doesn't work anymore since $Z(E) = 0$ when E is a sheaf supported in dimension zero. In fact, when X has dimension 2, then there is no Bridgeland stability condition $\sigma = (\mathcal{A}, Z)$ where $\mathcal{A} = \operatorname{Coh} X$.

Now we restrict to the case when X is a projective K3 surface in order to construct explicit stability conditions although similar construction can be made for any projective surface (see [MS19, Section 6.2])

Let $\omega, B \in \operatorname{NS} X$ with ω be an ample class and write $\beta = (B, \omega) \in \mathbb{R}$. We define the Bridgeland stability condition $\sigma_{\omega,B} = (\operatorname{Coh}^{\omega,B}, Z_{\omega,B})$ where

$$Z(E) = \langle e^{B+i\omega}, v(E)\rangle = -\int e^{B+i\omega} \operatorname{Ch}(E) \operatorname{Tod} X,$$

and $\operatorname{Coh} X^{\omega,B}$ is the category

$$\mathcal{A} = \{E^\bullet : E^{-1} \to E^0 :\ \mathcal{H}^{-1}(E^\bullet) \in \mathcal{F}^{\omega,\beta}, \mathcal{H}^0(E^\bullet) \in \mathcal{T}^{\omega,\beta}$$

where $\mathcal{F}^{\omega,B}$ and $\mathcal{T}^{\omega,B}$ are the subcategories of $\operatorname{Coh} X$ generated by extension:

$$\begin{aligned} \mathcal{T}^{\omega,B} &= \langle \text{torsion sheaves and } E\ \mu_\omega\text{-stable with slope } \mu(E) > \beta\rangle \\ \mathcal{F}^{\omega,B} &= \langle\ E\ \mu\text{-stable with } \mu(E) > \beta\ \rangle \end{aligned} \tag{4}$$

We have the following proposition

Proposition 2. *$\sigma_{\omega,\beta}$ is an stability condition iff $Z_{\omega,B}(\delta) \notin \mathbb{R}_{\leq 0}$ for all $\delta \in H^*_{\text{alg}}(X, \mathbb{Z})$ with $\langle \delta, \delta \rangle = -2$*

The fact that $\operatorname{Coh} X^{\omega,B}$ is the heart of a t-bounded structure is a consequence of the more general fact on the construction of hearts via tilting (see [MS19, Section 6]).

The condition on the classes $\delta \in H^*_{\text{alg}}(X, \mathbb{Z})$ is only used to prove that $\sigma_{\omega,B}$ satisfies the positive property (see [Bri08, Lemma 6.2]).

References

[Bri08] Tom Bridgeland. Stability conditions on $K3$ surfaces. *Duke Mathematical Journal*, 141(2):241 – 291, 2008.

[MS19] Emanuele Macrì and Benjamin Schmidt. Lectures on bridgeland stability, 2019.