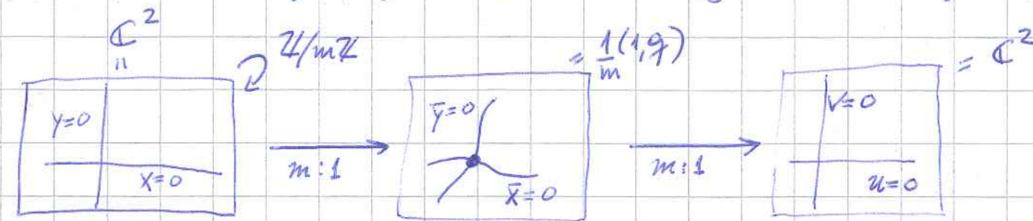
LECTURE 1: T-singularities and boundedness

- A cyclic quot. singularity $\frac{1}{m}(1,q)$ is a germ at the origin of the quotient of $\mathbb{C}^2$ by the action of $\mathbb{Z}/m\mathbb{Z}$: $(x,y) \mapsto (\mu x, \mu^q y)$ where μ is a primitive m-th root of 1, $0<q<m$ integers with $\gcd(q,m)=1$



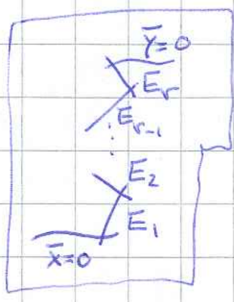
$\mathbb{C}[x,y] \supset \mathbb{C}[x,y]^{\mathbb{Z}/m\mathbb{Z}}$ normalization projection $\mathbb{C}[x^m, y^m]$, $x^m = u$, $y^m = v$

σ min. resol.

$(w^m = u^{m-q}\cdot v) \subset \mathbb{C}^3$

$\mathbb{C}[x^m, y^m, x^{m-q}\cdot y]$, $x^m = u$, $y^m = v$, $x^{m-q}\cdot y = w$

since $\frac{1}{m}(1,q)$ normal and $\frac{u^n v^n}{(u^{n-q}v)^{(n-b)}} = u^a v^b$ with $a+qb \equiv 0\ (m)$

$X =$ 

where $E_i = \mathbb{P}^1$, $E_i^2 = -b_i$ and $\frac{m}{q} = b_1 - \cfrac{1}{b_2 - \ddots \cfrac{1}{b_r}} = [b_1, \dots, b_r]$ $b_i \geq 2$

$K_X \equiv \sigma^*(K_{\frac{1}{m}(1,q)}) + \sum_{i=1}^r \delta_i E_i$, $\delta_i \in (-1,0]$ are the discrepancies of E_i

The *index* of $\frac{1}{m}(1,q)$ is $\frac{m}{\gcd(m,q+1)}$. [Recursion for discr]

- A **T-singularity** is a $\frac{1}{m}(1,q)$ with $q \neq m-1$ and $\left(\sum_{i=1}^r \delta_i E_i\right)^2 \in \mathbb{Z}$.

$\iff \frac{m}{\gcd(m,q+1)} \mid \gcd(m,q+1) \iff m = dn^2,\ q = dna-1$ with $\gcd(n,a) = 1$.

$$\mathbb{C}^2 \circlearrowleft \quad (x,y) \overset{A}{\mapsto} (\mu x, \mu^{dna-1} y) \qquad \mu\ dn^2\text{th primitive root of } 1,\ \gcd(n,a)=1$$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[d, "1:dn"'] \arrow[dr] \arrow[dr, phantom, "\square"] & \\
\mathbb{C}^2/\langle A^n \rangle \arrow[r] & (w^{dn} = uv) \subset \mathbb{C}^3 / \tfrac{1}{n}(1,-1,a) \simeq \mathbb{C}^2/\langle A \rangle
\end{tikzcd}$$

A^n: $(x,y) \mapsto (\mu^n x, \mu^{-n} y)$

$u = x^{dn}$
$v = y^{dn}$
$w = xy$

$\mathbb{C}^2/\langle A^n \rangle = (w^{dn} = uv) \subset \mathbb{C}^3 \circlearrowleft \mathbb{Z}/n\mathbb{Z}$

$u \mapsto \mu^{dn} u$
$v \mapsto \mu^{-dn} v$
$w \mapsto (\mu^{dn})^a w$

If $d=1 \Rightarrow$ we call them Wahl singularities.

Prop: For a T-sing $\frac{1}{dn^2}(1, dna-1)$ we have:

(i) If $n=2$, then they are $[4]$ or $[3,2,\ldots,2,3]$ where $\#2$ is $d-2$. [centers]

(ii) If $[b_1,\ldots,b_r]$ is T-sing $\Rightarrow$ so are $[b_1+1, b_2,\ldots,b_r,2]$ and $[2,b_1,\ldots,b_{r-1},b_r+1]$.

(iii) (i) + (ii) give all of them. In this way, $r-d+2 = \sum_{i=1}^{r}(b_i-2)$.

(iv) If $[b_1,\ldots,b_r]$ has discrep $-1+\frac{t_1}{n},\ldots,-1+\frac{t_r}{n} \Rightarrow [b_1+1,b_2,\ldots,b_r,2]$ has discrep $-1+\frac{t_1}{n+t_1},\ldots,-1+\frac{t_r}{n+t_1}, -1+\frac{t_1+t_r}{n+t_1}$ and for $[2,b_1,\ldots,b_r+1]$ discrep are $-1+\frac{t_1+t_r}{n+t_r}, -1+\frac{t_1}{n+t_r},\ldots,-1+\frac{t_r}{n+t_r}$.

(v) The index is n and if $F_2=1=F_1$ $F_{i+2}=F_{i+1}+F_i$ $i\geq -2$ is the Fibonacci sequence $\Rightarrow$ $n \leq F_{r-d}$ with $=$ $\Leftrightarrow$ Ping-Pong.

(vi) (No center contraction) If $[b_1,\ldots,b_r]-1-[b'_1,\ldots,b'_{r'}]$ with $S_r+S'_1<-1$ $\Rightarrow$ after contr. (-1)curve and all new (-1)-curves, there is no curve in the centers contracted.

(vii) (curiosity) If $0<a<n$ with $\gcd(n,a)=1$. Let $\frac{n}{a}=[a_1,\ldots,a_p]$ and $\frac{n}{n-a}=[b_1,\ldots,b_q] \Rightarrow \frac{n^2}{na-1}=[a_1,\ldots,a_{p-1},a_p+b_q,b_{q-1},\ldots,b_1]$.

- Boundedness for stable surfaces with T-singularities

$$\begin{tikzcd}
 & X \arrow[dl, "\pi"'] \arrow[dr, "\varphi"] & \\
S & & W
\end{tikzcd}$$

blow-down m (-1)-curves $=\pi$

$\varphi=$ minimal resol. of sing

No (-1)-curves $=S$

$W=$ normal proj surface with ℓ T-singularities and K_W ample

$E_i = \pi^*(i\text{th blown-up }(-1)\text{-curve})$ $\quad C=\sum_{i=1}^{\ell} C_i$

$C_i = \sum_{j=1}^{r_i} C_{ij}$ exceptional div of φ over $\frac{1}{d_i n_i^2}(1, d_i n_i a_i - 1)$

$$K_S^2 - m + \sum_{i=1}^{\ell}(r_i-d_i+1) = K_W^2$$

Problem: Give good bounds for the index of T-sing wrt K_W^2.
By Fibonacci's, enough for the r_i-d_i.

Obs.- Consider $m>0$ ($m=0$ is trivial for ineq.). Idea: $\sum E_i\cdot\sum C_j = \sum_{i=1}^{\ell}(r_i-d_i+2)-K_S\,\pi(C)$ and we carefully must bound $\sum E_i \sum C_j$ from below as αm, $\alpha>1$.

obs|.- By log BMY (e.g. see Langer) $\frac{3}{4}\ell \leq \sum_{i=1}^{\ell}\left(d_i - \frac{1}{d_i n_i^2}\right) \leq 12\,\chi(\mathcal{O}_W) - \frac{4}{3}K_W^2$

and Tsunoda-Zhang «Noether's ineq.» $\chi(\mathcal{O}_W) \leq \frac{9}{8}K_W^2 + 3$

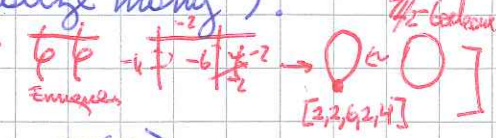
$\therefore$ ℓ and $\sum_{i=1}^{\ell} d_i$ are bounded by K_W^2 linearly.

Thm (Rana-U '17) Let $\kappa(S)$ = Kodaira dim of S. If $\ell = 1$, then

1. $\kappa(S) = 0$, $r - d \leq 4K_W^2$
2. $\kappa(S) = 1$, $r - d \leq 4K_W^2 - 2$
3. $\kappa(S) = 2$, $r - d \leq \max\left(4(K_W^2 - K_S^2) - 4,\ 1\right)$

[under No obstructions and many T-sing ...]

and it is optimal and we can classify = (and realize many).

[e.g. 

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Thm (Figueroa-U '19) Assume $\kappa(S) \geq 0$, then

$$\sum_{i=1}^{\ell}(r_i - d_i) \leq 4\ell(K_W^2 - K_S^2) + \ell - 2\ell K_S \cdot \pi(C).$$

[(2ℓ+2) possible improvement]

obs|.- [Evans-Smith '17] using symplectic topology obtain $\boxed{\text{for } \kappa(S) \geq 0}$ with $p_g > 0$ and $d = 1$, $r \leq 4K_W^2 + 7$. [p_g can be zero by W. Zhang]

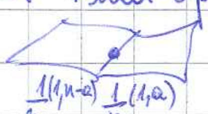
Q1. If K_S is not nef $\Rightarrow$ S must be rational. For $\ell = 1$ we have:

Thm: $r - d \leq 4(K_W^2 - K_S^2) - 2K_S \cdot \pi(C)$.

But we do not know how to bound $-K_S \cdot \pi(C) > 0$ wrt K_W^2. On the other hand, one can fix W and do certain Cremona transf. to increase arbitrarily $-K_S \cdot \pi(C)$. The idea is to find bound for the minimal degree.

Q2. Another problem is to find optimal bounds for the symplectic set up of a $B_{n,a} \subset$ surface with K ample and nonrational symplectic rat. blowup. Is Thm (Rana-U '17) (optimal) valid for sympl. $B_{n,a} \subset W_t$? Maybe Weiyi can do it!

Q3. Last open: Find optimal bounds for n in the case of orbifold normal crossings



. Is it possible to reduce to Wahl sing.?

This may be possible via sympl top as in Evans-Smith ...

Lecture 2 : Deformations and MMP.

- Y = normal surface with only quot. sing, D = small disk. A deformation $(Y \subset \mathcal{Y}) \to (0 \in D)$ of $Y \mapsto 0$ is a smoothing if general fiber is smooth. It is $\mathbb{Q}$-Gorenstein if $K_{\mathcal{Y}}$ is $\mathbb{Q}$-Cartier.

Thm : [KSB] $\underset{\text{quot. sing}}{(0 \in Y)}$ is T-sing or Du Val $\iff$ admits a $\mathbb{Q}$-Gorenstein smoothing.
It was taken as definition by KSB!

obs : $\mathbb{Q}$-Gorenstein smoothing component is unique [see eg below via KSB] and it was described! $(uv = w^{dn} + t_1 w^{(d-1)n} + \dots + t_d) \subset \mathbb{C}^{3+d} / \frac{1}{n}(1,-1,a,\underbrace{0,\dots,0}_{d})$ dim Comp $= d$.

Thm : [KSB] All components of $\mathrm{Def}(\frac{1}{m}(1,q))$ and deformations of $\frac{1}{m}(1,q)$ can be described via $\mathbb{Q}$-Gorenstein deformations of Partial Positive T-Du Val resolutions : P-resolutions.

Def : $X \xrightarrow{f} Y = \frac{1}{m}(1,q)$ is P-resolution if f is proper birational, X only T+Du Val sing, K_X rel. ample.

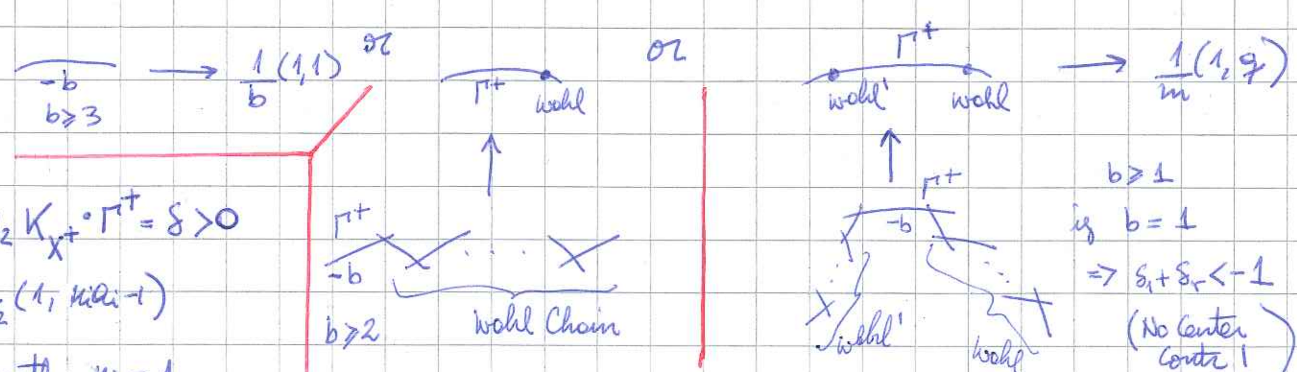
[Wahl] → Deform. of $\frac{1}{m}(1,q)$ are blow-down $\mathbb{Q}$-Gorenstein deform. of X ($X \to \frac{1}{m}(1,q)$ P-res).
→ one can describe all P-resolutions.

$$\left\{ \begin{matrix} \text{P-resol.} \\ \text{of } \frac{1}{m}(1,q) \end{matrix} \right\} \underset{\text{Kollár-Shepherd-Barron}}{\overset{1\text{-}1}{\longleftrightarrow}} \begin{matrix} \text{Components} \\ \text{of } \mathrm{Def}(\frac{1}{m}(1,q)) \end{matrix} \underset{\text{Christophersen-Stevens}}{\overset{1\text{-}1}{\longleftrightarrow}} K\left(\tfrac{m}{m-q}\right) = \{[k_1,\dots,k_s] = 0 : 1 \le k_i \le a_i\}$$
$$\tfrac{m}{m-q} = [a_1,\dots,a_s]$$

If $X \to \frac{1}{m}(1,q)$ is P-resol with α curves and ℓ sing with Milnor numbers μ_i [depend on smoothing, this is $\mathbb{Q}$-Gor] $\Rightarrow$ #(Milnor smoothing of $\frac{1}{m}(1,q)$) $= \alpha + \sum_{i=1}^{\ell} \mu_i = \mu$

$\alpha = 0 \Rightarrow \frac{1}{m}(1,q)$ is T or Du Val sing. ; $\alpha = 1$, $\mu = 1$ $\Rightarrow$ $\boxed{\text{Extremal P-resolution}}$

Classification:

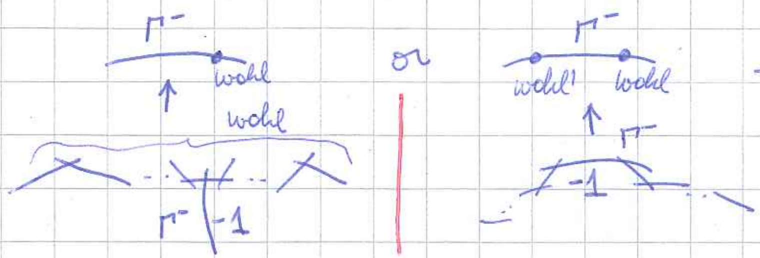


Say : $n_1 n_2 K_{X^+} \cdot \Gamma^+ = \delta > 0$

where $\frac{1}{n_i^2}(1, n_i a_i - 1)$

and smooth $n_i = 1$.

Take now negative Extremal P-resolutions over $\frac{1}{m}(1,q)$



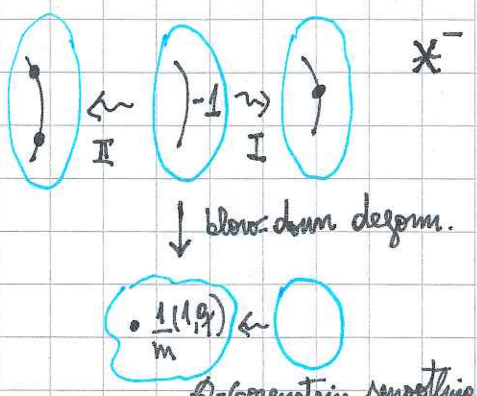
Say: $n_1 n_2 K_{X^-} \cdot \Gamma^- = -\delta < 0$

Take a $\mathbb{Q}$-Gorenstein smoothing $X^- \subset \mathcal{X}^- \longrightarrow 0 \in \mathbb{D}$ of X^- and the blow-down deformation $\boxed{X^- \subset \mathcal{X}^- \longrightarrow \frac{1}{m}(1,q) \subset \mathcal{Y}}$ is a birational morphism with $\Gamma^- \cdot K_{\mathcal{X}^-} < 0$

$\boxed{\Gamma^- \longmapsto 0}$ and with terminal $\mathcal{X}^-$.

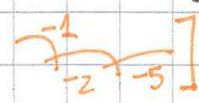
∴ Mori says there are two options: Divisorial contraction or Flip.
to improve this neg. situation

Divisorial contraction

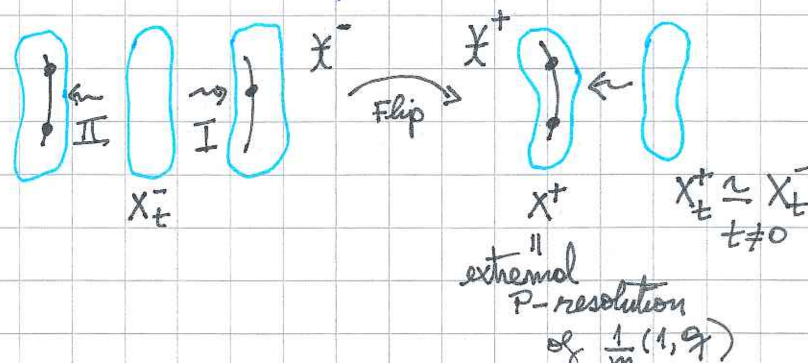


$\Rightarrow$ $\frac{1}{m}(1,q)$ is wahl sing

[$\Leftarrow$ Not true:]



Flip



Questions:

(1) When Div contr or Flip?

(2) How to compute X^+ in case of a flip?

X^- type I

$[b_1, \dots, \bar{b}_i, \dots, b_r]$

X^- type II

$[f_{s_2}, \dots, f_1] - [e_1, \dots, e_{s_1}]$

and they are deformation equivalent! [see HTU]

For X^+

$[f_{s_2}, \dots, f_1] - c - [e_1, \dots, e_{s_1}]$

$\Gamma^{+2} = -c$

There is a flipping family [HTU] $^{\infty}$: $\frac{1}{m}(1,q) = \frac{1}{11}(1,3) = [4,3]$ (2.3)

[the dual $[2,2,3,2]$ admits 1 extremal P-res $[2\bar{2}3\bar{2}]$]

$\phi - [\bar{2},5,3] - [2,3,\bar{2},2,7,3] - [2,3,2,2,2,\bar{2},5,7,3] - \ldots$

$n_1=1$ $n_2=5$ $n_3=14$ $\delta = 3$

$[4] - [2,\bar{2},5,4] - [2,2,3,\bar{2},2,7,4] - \ldots$

$n_1=2$ $n_2=7$ $n_3=19$

one has that $\delta n_1 - n_2 < 0 \Rightarrow$ it is a flip and the last wahl sing. is one of the sing. of the P-res: the other then is determined.

For $\frac{1}{4}(1,1) = \frac{1}{m}(1,q) = [4]$ we have

$[4] - [2,\bar{2},6] - [2,2,2,\bar{2},8] - \ldots$ $\delta = 2$

$n_1=2$ $n_2=4$

one has that $\delta \cdot n_1 - n_2 = 0 \Rightarrow$ it is a divisorial contraction.

obs:

(1) From n_1, n_2 the n_i, a_i can be computed as $n_{i+1} = \delta n_i - n_{i-1}$, etc. If $\delta \geq 2 \Rightarrow$ it is an infinite sequence and $\frac{n_{i+1}}{n_i} \xrightarrow[i\to\infty]{} [\delta,\delta,\delta,\ldots]$ where $[\delta,\delta,\delta,\ldots] = \frac{\delta + \sqrt{\delta^2-4}}{2}$.

(2) If we know that $X^- \to Y$ is flipping over $\frac{1}{m}(1,q)$, then one can try to guess Extremal P-resol. BUT $\frac{1}{m}(1,q)$ could have more than one!

Teo: $\frac{1}{m}(1,q)$ could have at most 2 P-resolutions and $\delta_1 = \delta_2$.

[$\Leftrightarrow$ $\frac{m}{m-q} = [c_1,\ldots,c_s]$ admits at most 2 zero cont. fractions.
★ Problem: Find a simple proof.]

(3) (Usual flip) Say X^- given by $[e_1,\ldots,\bar{e}_s]$ (here $\delta = n-a$). Say $e_j = 2$ for all $i_0 < j \leq s$ and $e_{i_0} \geq 3$ (or $i_0 = s$)
$\Rightarrow$ It is flipping and X^+ is $e_1 - [e_2,\ldots,e_{i_0}-1]$.

Lecture 3 : Global run of MMP.

Def: A W-surface is X = normal proj surface with only Wahl singularities together with a proper deformation $X \subset \mathcal{X} \longrightarrow 0 \in \mathbb{D}$ such that

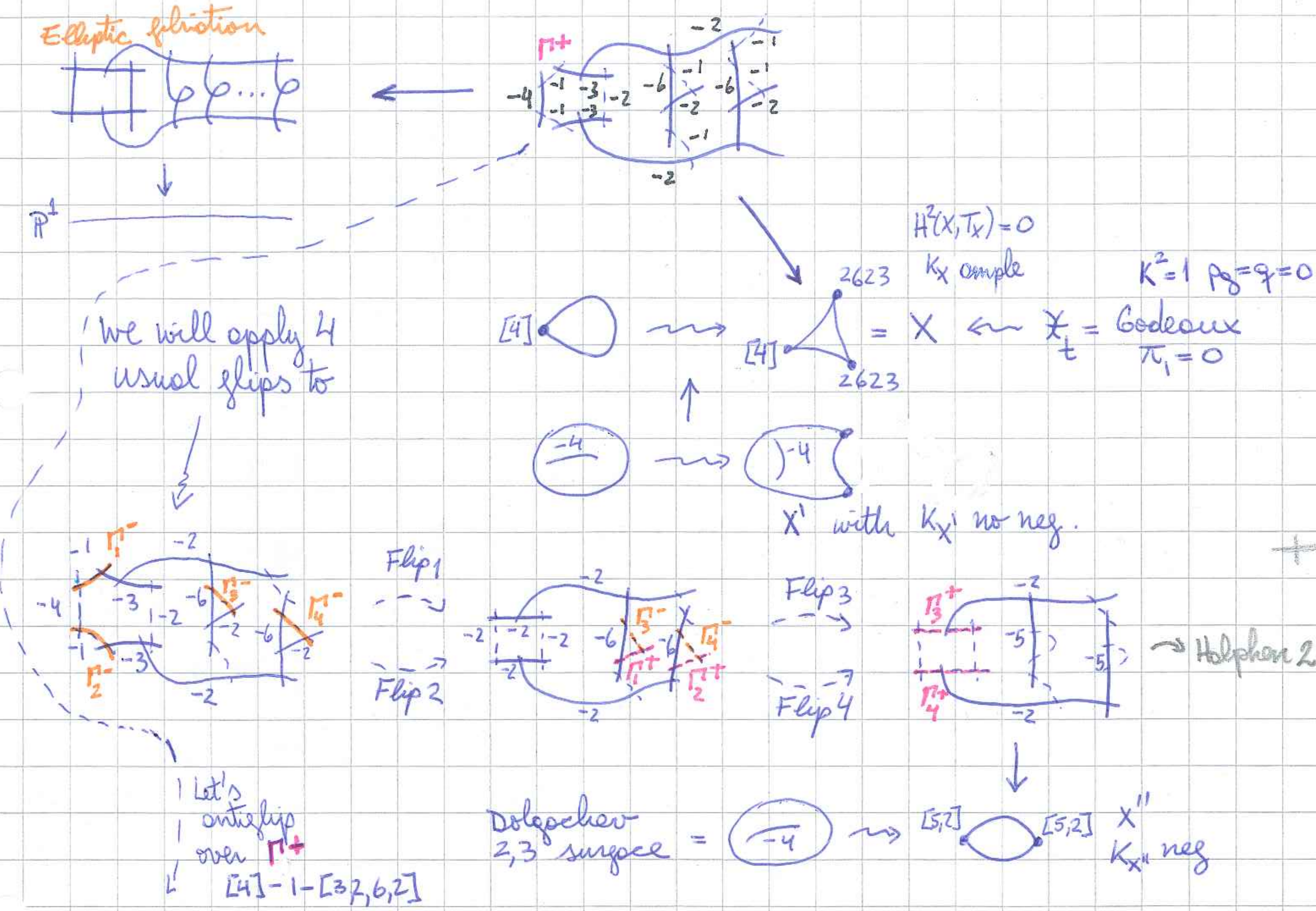
(1) $\mathcal{X}$ normal 3-fold with $K_{\mathcal{X}}$ $\mathbb{Q}$-Cartier
(2) $\mathcal{X}_0 \simeq X$ (in particular reduced fiber)
(3) $\mathcal{X}_t$ nonsingular for $t \neq 0$.

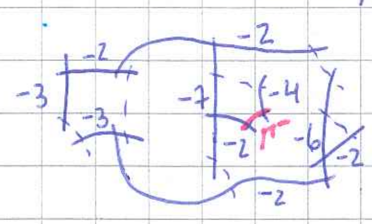
$\therefore$ $\mathcal{X}$ terminal $\mathbb{Q}$-factorial (see [KSB88]). Many examples as in Lee-Park constr.

W-surface (non classic) $\supset$	surface (classic)
Intersection theory $1_{\mathbb{D}}$ constant standard invariants $K^2, \chi_{top}, \chi, p_g, q$	Intersection theory
"$K_{\mathcal{X}_0}$" MMP If $K_{\mathcal{X}_0}$ nef $\Rightarrow$ $K_{\mathcal{X}_t}$ nef $0<\lvert t\rvert \ll \varepsilon$ If $K_{\mathcal{X}_0}$ no nef Flip / div. contr. $\mathcal{X}_0$: «blow-down & up» / blow-down to Wahl sing. $\mathcal{X}_t$: nothing / blow-down (-1)-curve	"K" MMP blowsdown of (-1)-curves to nonsingular points (Castelnuovo's theorem)
MMP result ruled [eg: $\mathbb{F}_1 \rightsquigarrow \mathbb{F}_3$] or unique MM MHP on $\mathbb{P}^2$ or smooth deformation of $\mathbb{P}_C(\mathcal{E})$ 1991 ANETTI ACKING ROKHOROV	MMP result ruled or unique MMP $\mathbb{P}^2$ or $\mathbb{P}_C(\mathcal{E})$
if K nef and $K^2>0$ $\Rightarrow$ there is unique canonical model (KSBA limit) with only T-singularities and Du Val sing.	if K nef and $K^2>0$ $\Rightarrow$ there is unique canonical model with only Du Val sing.

As reference see "$\mathbb{Q}$-Gorenstein smoothings of surfaces and degen. of curves".

Example 1: Around a simply-connected Godeaux surface.

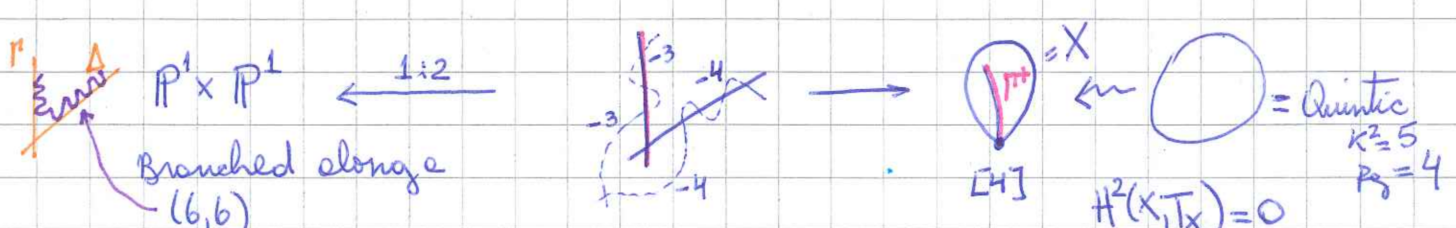




In fact, $\Gamma_1^- \cdot K = \frac{1}{14} \Rightarrow \delta = 1$. The universal flipping is then finite:

$$[4]-[2,\bar{7},2,2,3]-[2,6,2,3]$$

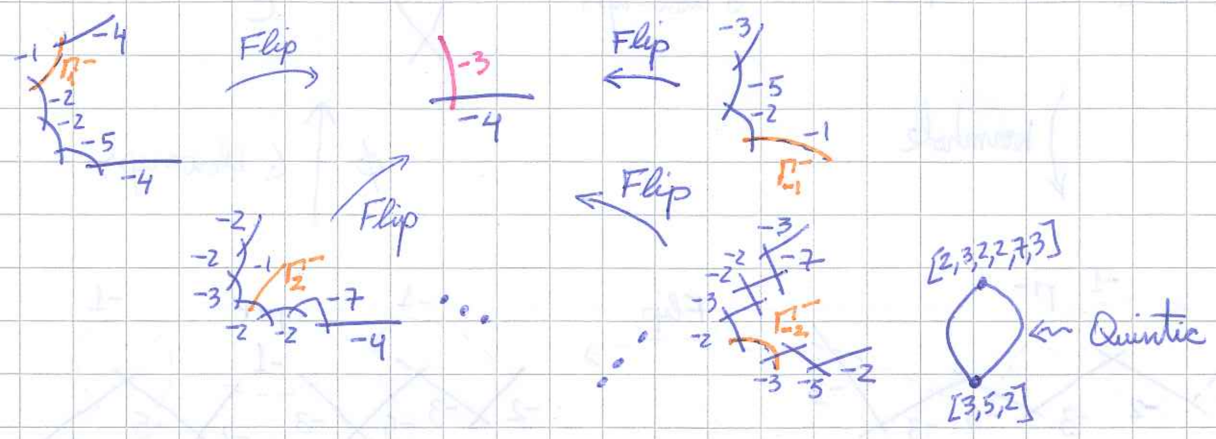
Example 2: (It follows from J. Rana description of stable surfaces for quintics)



Then we can antiflip using the infinite ($\delta=3$) universal family

$\phi - [\bar{2},5,3] - [2,3,\bar{2},2,7,3] - [2,3,2,2,2,\bar{2},5,7,3] - \cdots$

$[4] - [2,\bar{2},5,4] - [2,2,3,\bar{2},2,7,4] - \cdots$

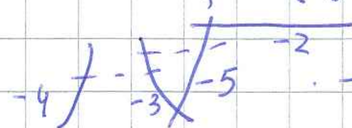


Things :

(1) Via MMP one can prove existence $X_1 = $ [5,2] $D_{2,3}$ ⟵ RAT or Godeoux $\pi_1=0$ / Godeoux $\mathbb{Z}/2$ ⇝ [5,2] $D_{2,4}$ $= X_2$, $D_{2,3}$

but minimal res. X_1 is blow-up of Enriques surface and min. res. X_2 is blow-up of Bolgachev 2,4 surface. If Moduli space of $\mathbb{Z}/2$-Godeoux is connected ⇒ there are two very different ways to put a $B_{3,1}$ in a $\mathbb{Z}/2$-Godeoux. What are these two ways? (It is expected that this moduli space is connected.)

(2) One can describe the nonrational KSBA boundary for singularity in the case of $\mathbb{Z}/2$-Godeoux [Using the Rana U optimal bound & MMP]. There are 11 types and all of them but one resists to exist: We have in an Enriques (joint with C. Rito, E. Dias)

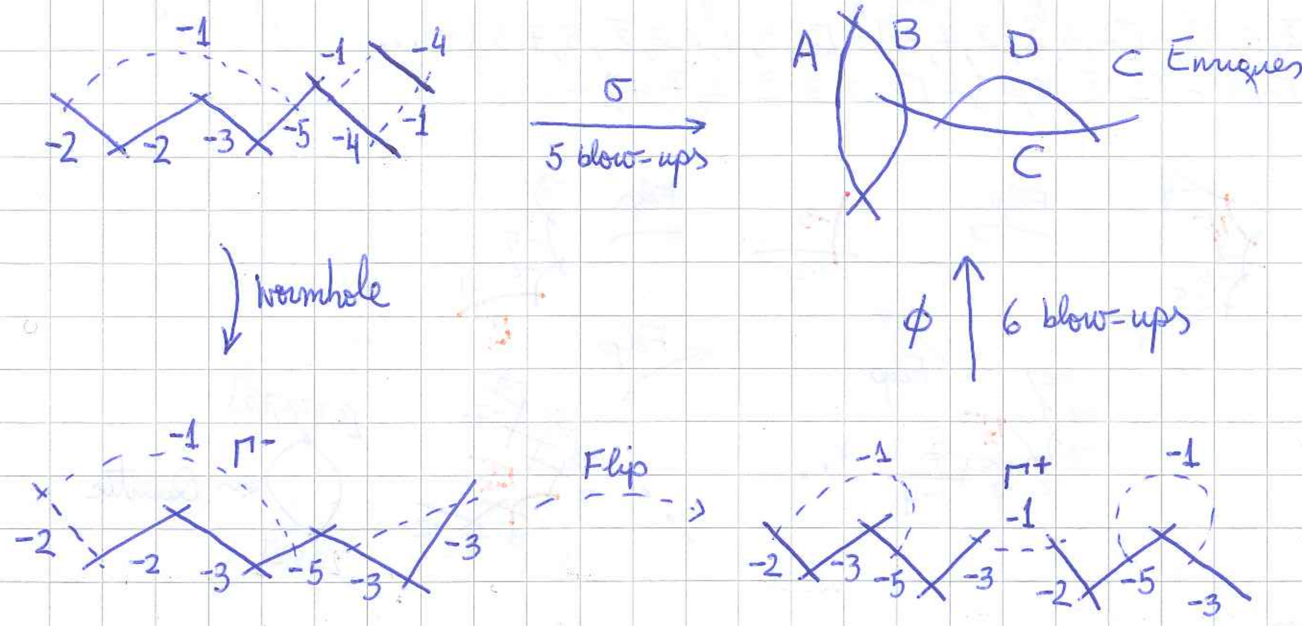
Enriques ⟵  ⟶ [4] ◯ [3,5,2] ⟵ $\mathbb{Z}/2$-Godeoux.

If there is smoothing of [3,5,2] ⇒ $\circ_{[4]}$ has m. res. a $D_{2,6}$ and it exists!

Q: Is it possible to use some sympl. MMP?

(3) It is desirable to understand better the max. 2 ways for $[c_1,\ldots,c_i-1,\ldots,c_j-1,\ldots,c_r]=0$ where $\frac{m}{m-q} = [c_1,\ldots,c_r]$. Is there another proof? ⇝ wormholes ...

work in progress with Jenny ... conjecture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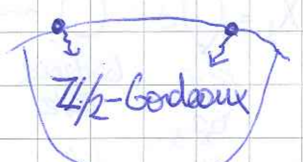
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