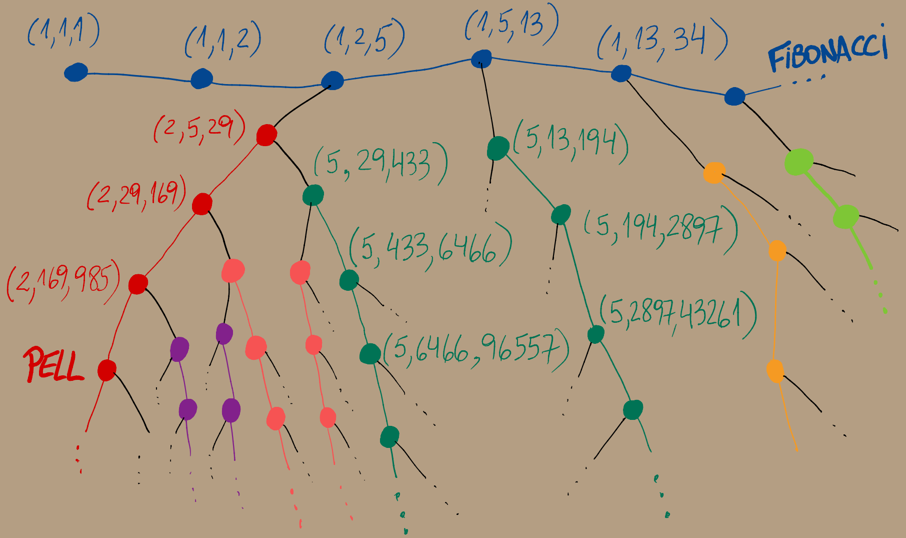
" From Markov to del Pezzo through Wahl "

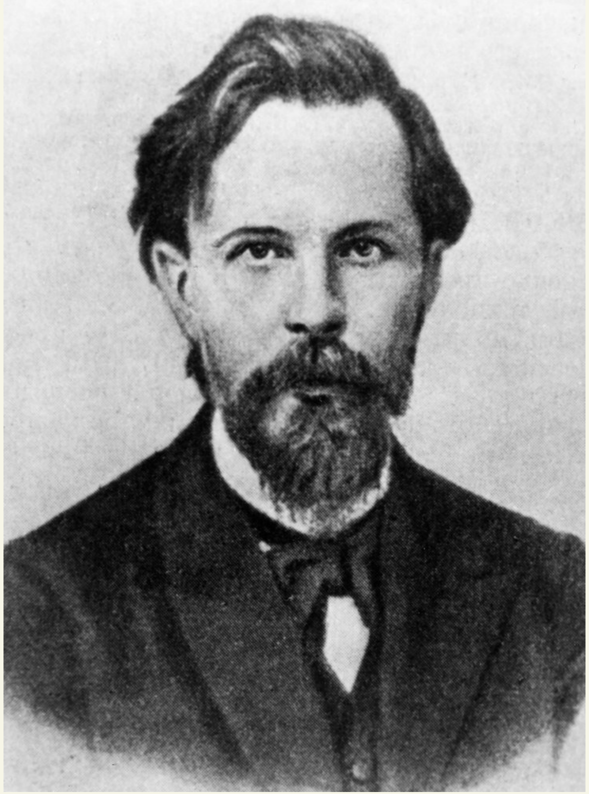
(based on a joint with Juan Pablo Zúñiga, PhD student UC Chile)



Giancarlo Urzúa
UC Chile

July 2025

MCA



Andrey A. Markov (1856 - 1922)

His Master's thesis

« On binary quadratic forms with positive determinant »

(Math. Ann. 15 (1879) and 17 (1880))

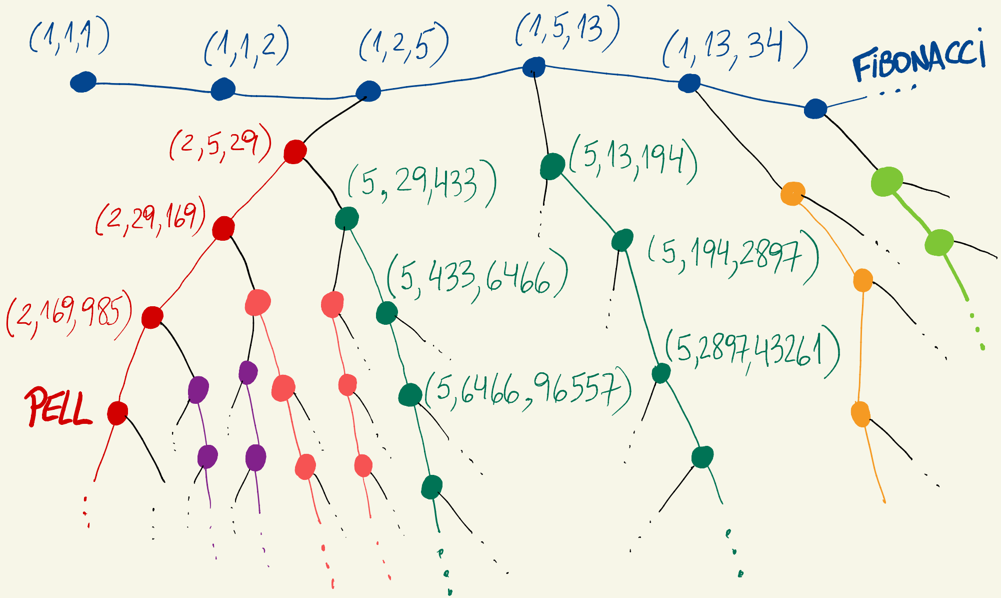
introduces Markov triples

$(a, b, c) \in \mathbb{Z}^3_{>0}$

such that

$$a^2 + b^2 + c^2 = 3abc$$

Symmetries and mutations $(a,b,c)\mapsto(a,b,3ab-c)$ define the Markov tree:

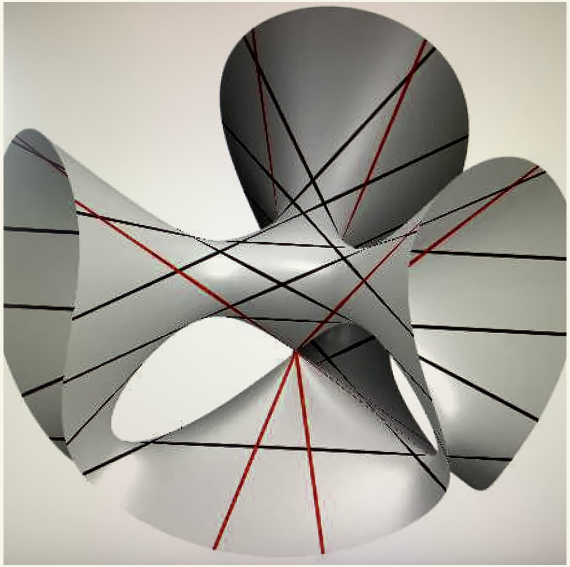


Markov's conjecture: If $(a,b<c)$ and $(a',b'<c)$ are Markov
(due to Frobenius 1913) triples, then $a=a'$ and $b=b'$.

Pascuale del Pezzo (1859-1936)

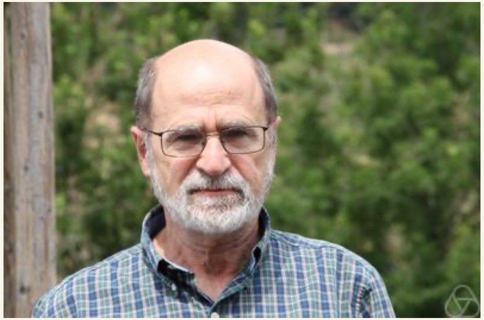
In 1887, he publishes a work studying algebraic surfaces of degree d in $\mathbb{P}^d_{\mathbb{C}}$.

Example : $\{Wx^2+Wy^2+Wz^2=3xyz\} \subset \mathbb{P}^3_{\mathbb{C}}$



If nondegenerated and nonsingular, these projective surfaces of degree d in $\mathbb{P}^d$ $(d \geq 3)$ are the same as nonsingular projective surfaces S with $-K_S$ ample & [AMPLE: some multiple of $-K_S$ is a hyperplane curve on S] $K_S^2 \geq 3$.

Reference: "Clasical Algebraic Geometry" Chapter 8 by Igor Dolgachev



we have that $1 \leq \underbrace{d = K_S^2}_{\text{degree of } S} \leq 9$.

Theorem: The list of del Pezzo surfaces is

$(d=9)$ $\mathbb{P}^2_{\mathbb{C}}$.

$(d=8)$ $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} = \mathbb{F}_0$ and $Bl_p(\mathbb{P}^2_{\mathbb{C}}) = \mathbb{F}_1$.

$(1 \leq d \leq 7)$ $Bl_{p_1,\dots,p_{9-d}}(\mathbb{P}^2_{\mathbb{C}})$ where $p_1,\dots,p_{9-d}$ are points in general position.

No 3 on a line

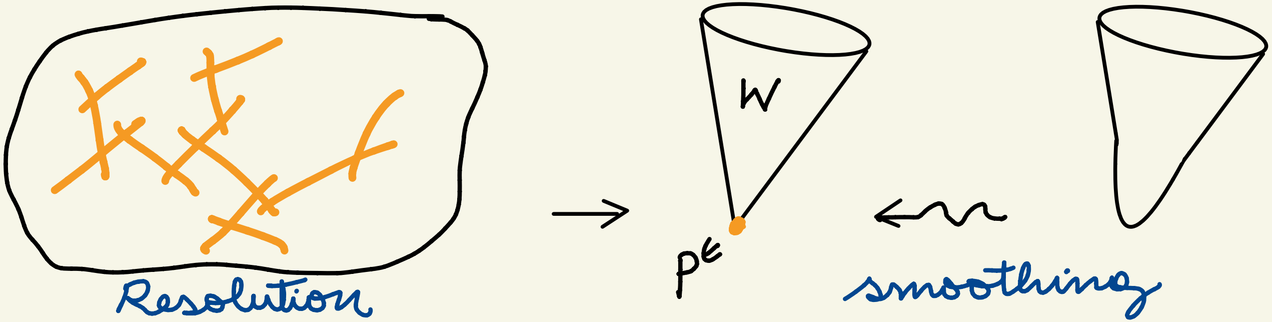
No 6 on a conic

No ∝ or ≺ cubic passes through 8 with one of them being the singular point

A relation between Markov and del Pezzo appears when surfaces get singular ...

Today, a singularity is a complex 2-dimensional germ $(P \in W)$ which is not isomorphic to an open set of $\mathbb{C}^2$ at P.

Two ways to get rid of P:

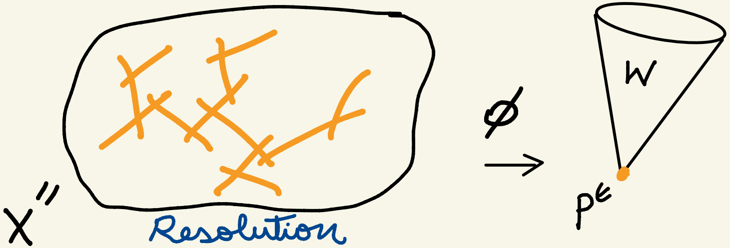


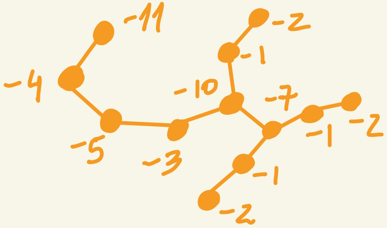
Resolution : It is a birational morphism ϕ :

(1) X is nonsingular surface,

(2) exceptional curves contract to p,

(3) $\phi|_{X \setminus \phi^{-1}(p)}$ is isomorphism.





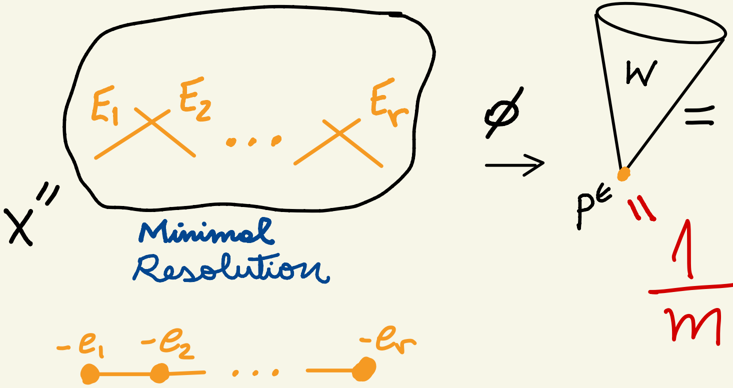
exceptional curves are represented by the Resolution graph of ϕ

Cyclic quotient singularities:

Given $0<q<m$ coprime

$\mathbb{C}^2 \xrightarrow{T} \mathbb{C}^2$, $T(x,y)=(\zeta x, \zeta^q y)$,

where $\zeta^m = 1$, ζ primitive root.



$= (p \in W) = ((0,0) \in \mathbb{C}^2/\langle T\rangle)$

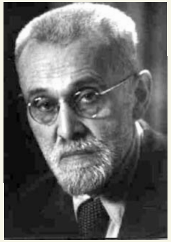
$p \in \; || \; \frac{1}{m}(1,q)$

$E_i \simeq \mathbb{P}^1_{\mathbb{C}}$ $\quad E_i^2 = -e_i \leq -2$

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}}$$

Hirzebruch
(1927–2012)

—



Jung
(1876–1953)

continued fractions are key in all of this!

$$\frac{m}{q} = e_1 - \cfrac{1}{e_2 - \cfrac{1}{\ddots - \cfrac{1}{e_r}}} =: [e_1, \dots, e_r]$$

Examples:

$$\frac{19}{7} = 3 - \frac{2}{7} = 3 - \frac{1}{\frac{7}{2}} = 3 - \cfrac{1}{4 - \cfrac{1}{2}} = [3,4,2] = 3 - \cfrac{1}{5 - \cfrac{1}{1 - \cfrac{1}{3}}} = [3,5,1,3]$$

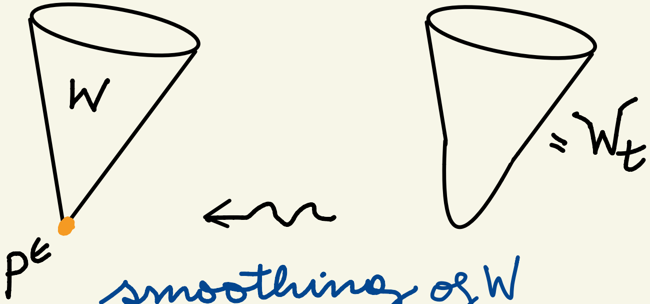
$$\frac{19}{19-7} = \frac{19}{12} = [2,3,2,3] \quad \left(\text{dual to } \frac{19}{7} = [3,4,2]\right)$$

Smoothing:

It is a morphism $\mathcal{W} \longrightarrow \mathbb{D} = \{t \in \mathbb{C} : |t| < \varepsilon\}$ such that

(1) $\mathcal{W}_0 = W$,

(2) $\mathcal{W}_t$ is nonsingular.



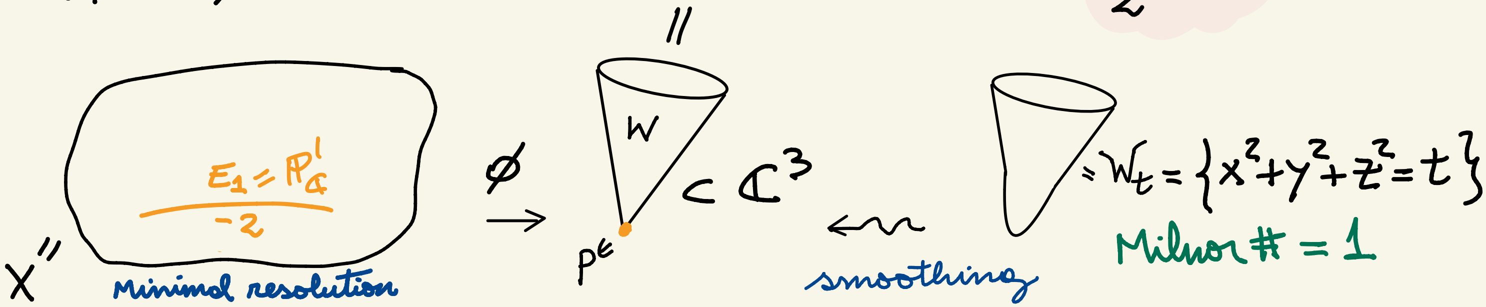
smoothing of W
or degeneration of W_t $(t \neq 0)$

If $K_{\mathcal{W}}$ is $\mathbb{Q}$-Cartier, then we call it $\mathbb{Q}$-Gorenstein smoothing.

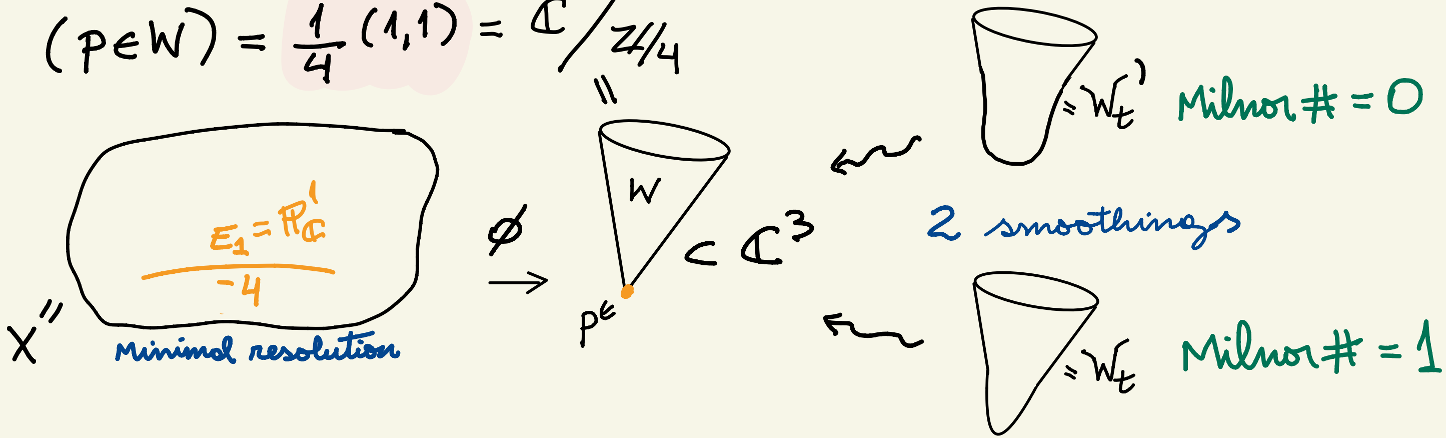
Cyclic quotient singularities:

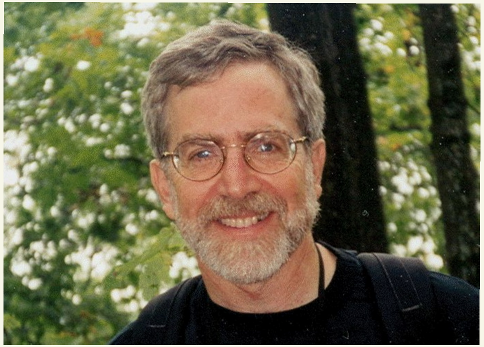
- They have in general many distinct smoothings, in part distinguished by $\operatorname{rank}(H^2(W_t, \mathbb{Z}))$ = Milnor number of the smoothing.
- Kollár–Shepherd-Barron (1988): There is a way to control every smoothing of $\frac{1}{m}(1,q)$ and depends on $\frac{m}{m-q}$ ...
(Christophersen-Stevens 1991)

- $(p \in W) = ((0,0,0) \in \{x^2+y^2+z^2=0\} \subset \mathbb{C}^3) = \frac{1}{2}(1,1) = \mathbb{C}^2/\mathbb{Z}/2$



- $(p \in W) = \frac{1}{4}(1,1) = \mathbb{C}^2/\mathbb{Z}/4$





Jonathan Wahl

In the 1980s, he studied singularities that admit a Milnor# $= 0$ smoothings.

(The mildest degenerations)

we call them QHD singularities.

<u>Example</u>: Among all quotient singularities, the QHD are $\frac{1}{n^2}(1, na-1)$, $\gcd(n,a) = 1$.

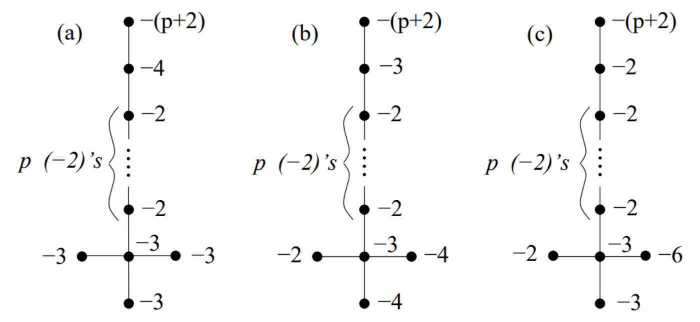
we call them Wahl singularities.

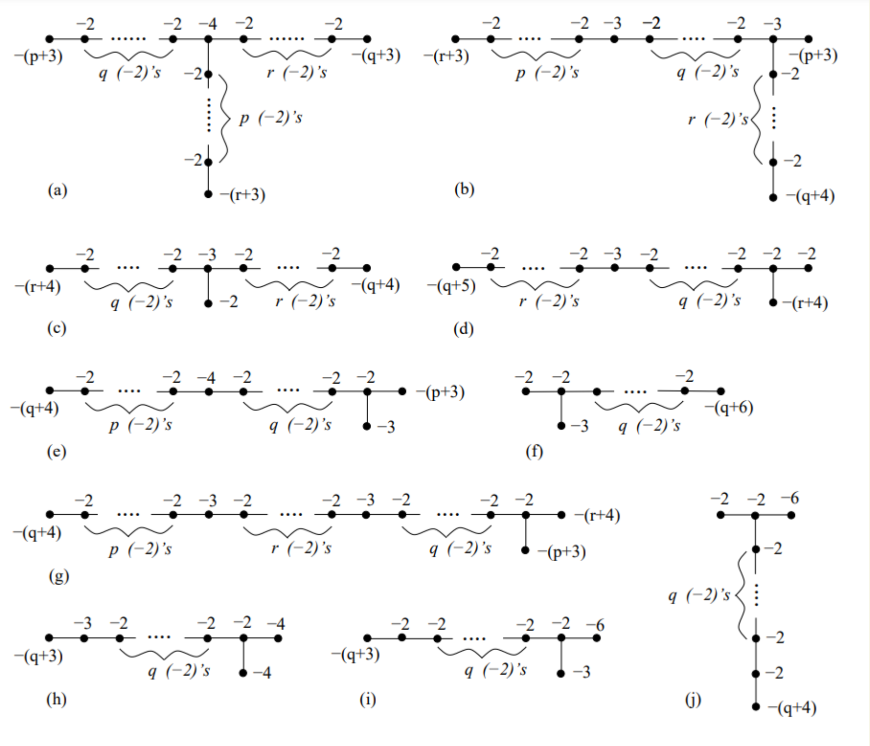
$$\boxed{\begin{array}{c}\text{quot. sing.}\\ \shortparallel \\ \text{log-terminal}\end{array}}$$

List of (minimal resolutions of) known QHD :

(due to Wahl, Stipsicz-Szabó-Wahl, Buphal-Stipsicz)

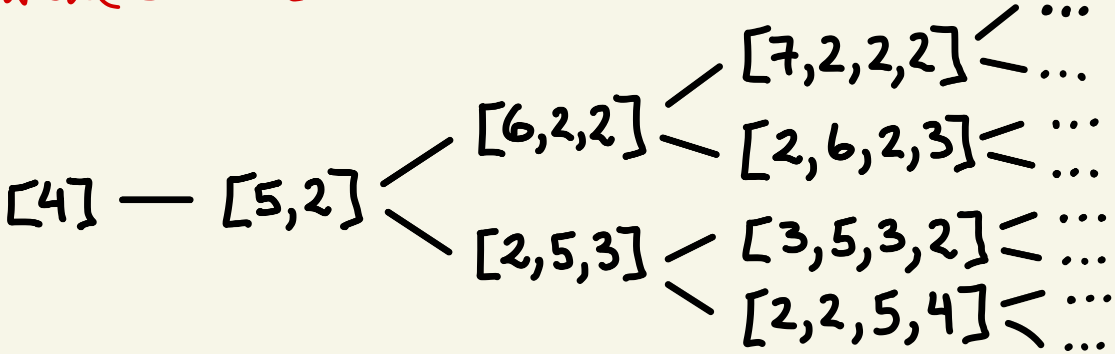
Wahl chains





Wahl conjecture:
This is the complete list of families of QHD.

The wohl chains are



Symmetry: $[e_1, \dots, e_r]$ and $[e_r, \dots, e_1]$ represent the same.

wohl algorithm: $[e_1, e_2, \dots, e_r]$ wohl $\Rightarrow$ $[e_1+1, e_2, \dots, e_r, 2]$ wohl.

Example: $[2,2,2,2,3,3,2,2,7,3,6]$ is wohl; $[2,2,7,6,5]$ is not

So far :

Markov triples (a,b,c) , $a^2+b^2+c^2=3abc$

Del Pezzo surfaces $\mathbb{P}^2_{\mathbb{C}}$, $\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}$, $Bl_{p_1,\dots,p_s}(\mathbb{P}^2_{\mathbb{C}})$ where $1\leqslant s\leqslant 8$.

Wahl singularities $\frac{1}{n^2}(1,na-1) \longleftarrow \underbrace{[e_1,\dots,e_r]}_{\text{Wahl chain}}$

(minimal resolution)

and degenerations (or smoothings!)

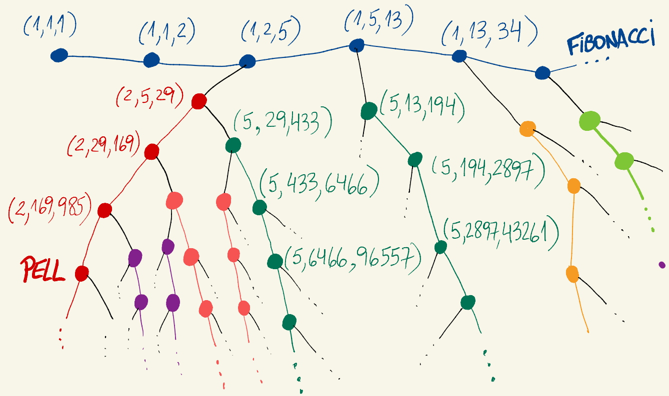
Are there $\overbrace{\text{degenerations}}^{\text{the mild ones}}$ of del Pezzo surfaces with only wahl singularities?

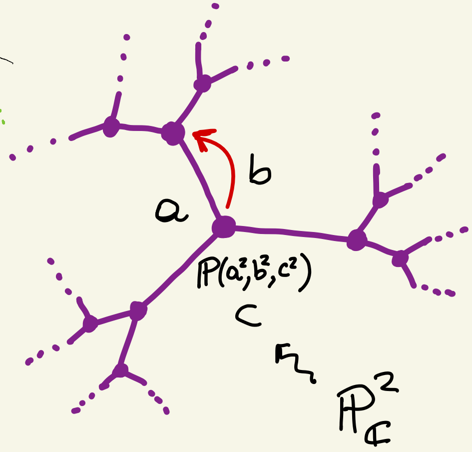
Are there constraints on wahl singularities after we fix the degree of del Pezzo surfaces?

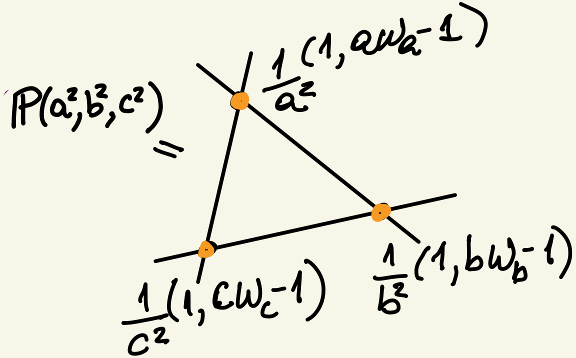
Let us do the case of $\mathbb{P}^2_{\mathbb{C}}$ (degree 9)...

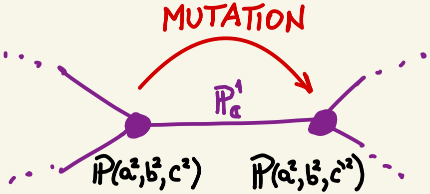
After the work of Bădescu (1986), Manetti (1991), Hacking (2001), we have Hacking–Prokhorov's Theorem (2010):

Theorem [HP10]: Let $\mathbb{P}^2_{\mathbb{C}} \rightsquigarrow W$ be such a degeneration. Then W is a partial $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{P}(a^2,b^2,c^2)$ so that $a^2+b^2+c^2=3abc$.









Observations :

(1) In general, this type of degenerations are the allowed ones in the Kollár–Shepherd-Barron–Alexeev moduli space of surfaces of general type.

(The analogue to $\overline{M_g}$ for curves)

(2) Hacking–Prokhorov's "moduli of $\mathbb{P}^2_{\mathbb{C}}$" is a final output of the birational geometry (MMP) of l.t. degenerations, taking the role of $\mathbb{P}^2_{\mathbb{C}}$ in the classical MMP.

(3) However, there is a rich MMP on these degenerations of $\mathbb{P}^2_{\mathbb{C}}$; see [U-Zúñiga, 2023] :

« The birational geometry of Markov numbers »

(4) one byproduct of [U-Zúñiga, 2023] was to find equivalences to the Markov Conjecture.

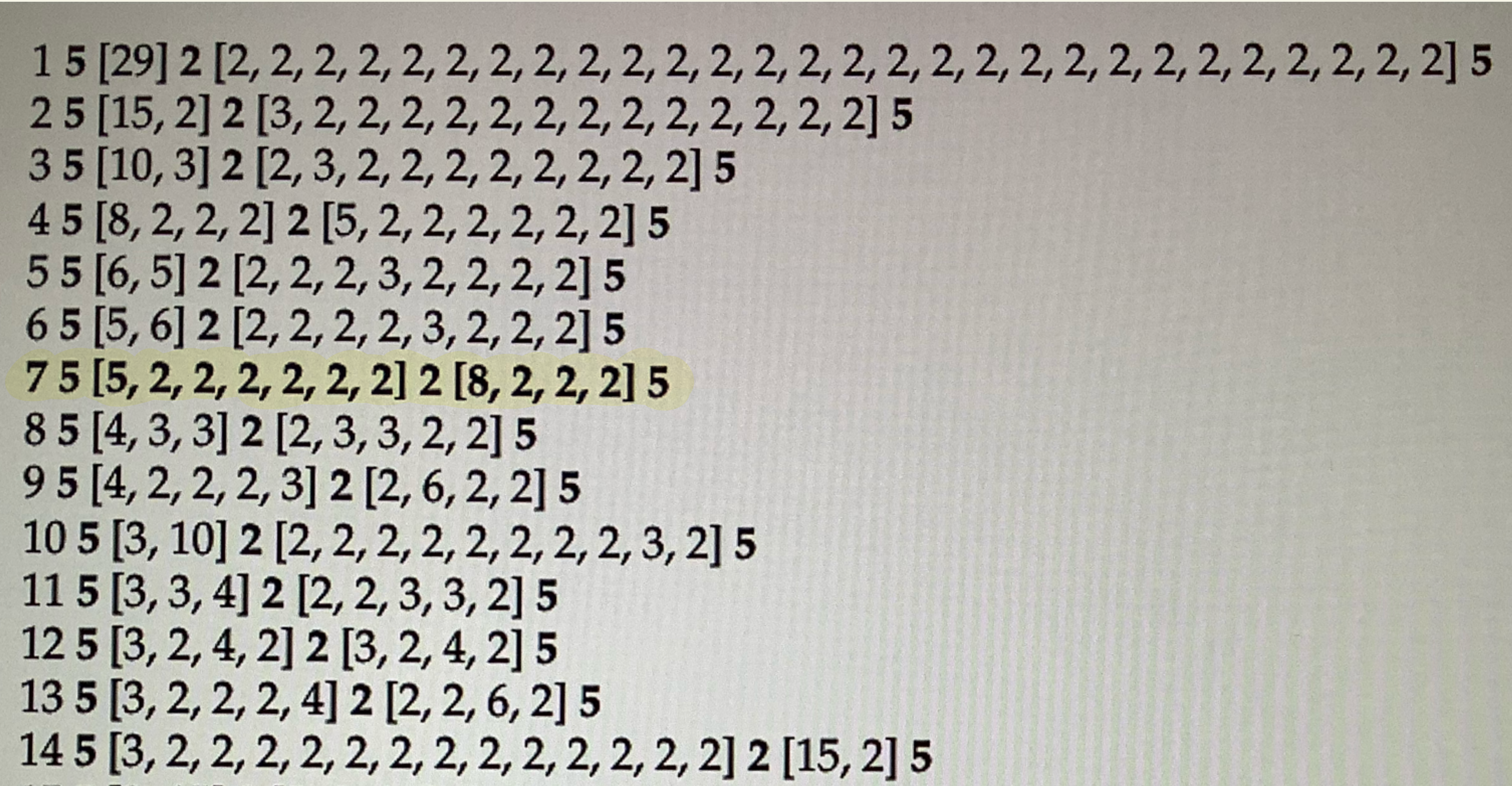
Markov Conjecture: Given an integer $m>0$, there is at most one $0<q<\frac{m}{2}$ coprime to m such that

$$[5, e_1, \dots, e_r, 2, b_s, \dots, b_1, 5] = [\text{Wahl}, 2, \text{Wahl}'],$$

where $\frac{m}{q} = [e_1, \dots, e_r]$ and $\frac{m}{m-q} = [b_1, \dots, b_s]$.

There are more equivalences in the article ...

m = 29



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1 5 [29] 2 [2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] 5
2 5 [15, 2] 2 [3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] 5
3 5 [10, 3] 2 [2, 3, 2, 2, 2, 2, 2, 2, 2, 2] 5
4 5 [8, 2, 2, 2] 2 [5, 2, 2, 2, 2, 2, 2] 5
5 5 [6, 5] 2 [2, 2, 2, 3, 2, 2, 2, 2] 5
6 5 [5, 6] 2 [2, 2, 2, 2, 3, 2, 2, 2] 5
7 5 [5, 2, 2, 2, 2, 2, 2] 2 [8, 2, 2, 2] 5
8 5 [4, 3, 3] 2 [2, 3, 3, 2, 2] 5
9 5 [4, 2, 2, 2, 3] 2 [2, 6, 2, 2] 5
10 5 [3, 10] 2 [2, 2, 2, 2, 2, 2, 2, 2, 3, 2] 5
11 5 [3, 3, 4] 2 [2, 2, 3, 3, 2] 5
12 5 [3, 2, 4, 2] 2 [3, 2, 4, 2] 5
13 5 [3, 2, 2, 2, 4] 2 [2, 2, 6, 2] 5
14 5 [3, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2] 2 [15, 2] 5
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Are there any analogues of Hacking-Prokhorov's theorem for del Pezzo surfaces ?

$$\left(\mathbb{P}^2_{\mathbb{C}} \quad \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}} \quad Bl_{p_1,\dots,p_s}(\mathbb{P}^2_{\mathbb{C}}) \text{ where } 1 \leq s \leq 8 \right)$$

Yes and for any fixed degree ; this is [U-Zúñiga, 2025].

To explain this, I need to define some combinatorial artefacts : Markings on Wahl chains.

Zero continued fractions are Hirzebruch-Jung for 0

(we now need to introduce 1s in the e_is ...)

$[1,1]$

$[1,2,1]$, $[2,1,2]$

$[1,2,2,1]$, $[2,1,3,1]$, $[1,3,1,2]$, $[3,1,2,2]$, $[2,2,1,3]$

$\vdots$

etc using blow-up $u - \frac{1}{v} = u+1 - \cfrac{1}{1 - \cfrac{1}{v+1}}$.

Definition: We say that $[f_1, \dots, f_r]$ with $f_i \geq 2$ admits a zero continued fraction of weight λ if

$$[\dots, f_{i_1} - d_{i_1}, \dots, f_{i_2} - d_{i_2}, \dots, f_{i_v} - d_{i_v}, \dots] = 0$$

for some $d_{i_1}, d_{i_2}, \dots, d_{i_v} \in \mathbb{Z}_{>0}$ and $\lambda + 1 = \sum_{j=1}^{v} d_{i_j}$.

Example: $[4,3,3,2]$ admits $[4-3, 3, 3-2, 2] = 0$ and $\lambda = 3+2-1 = 4$.

Observation: $[f_1, \dots, f_r]$ admits a zero continued fraction of weight 0 $\Longleftrightarrow$ It is a dual Wahl chain.

Theorem : [Hacking-Prokhorov combinatorial version [UZ23]]

The wahl chain $[e_1, \dots, e_r]$ corresponds to a degeneration of $\mathbb{P}^2_{\mathbb{C}}$

$$\Updownarrow$$

There is $i \in \{1, \dots, r\}$ such that $[e_1, \dots, e_{i-1}]$ and $[e_{i+1}, \dots, e_r]$ admit zero continued fractions of weight 0.

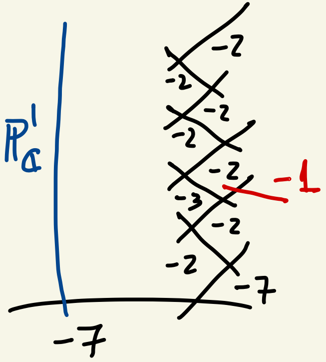
observation : $[e_1, \dots, e_{i-1}]$ and/or $[e_{i+1}, \dots, e_r]$ may be empty.

(i) If both are not empty $\Rightarrow e_i = 10$.

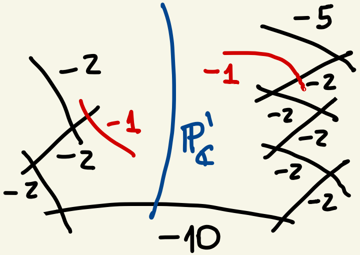
(ii) If one is empty $\Rightarrow e_i = 7$.

(iii) If both are empty $\Rightarrow e_i = 4$.

$$\frac{34^2}{34\cdot 5-1} = [\underset{\substack{\uparrow\\ e_i}}{7}, \underbrace{7,2,\overline{2},3,2,2,2,2,2}_{[7,2,1,3,2,2,2,2,2]=0}] \Rightarrow$$



$$\frac{29^2}{29\cdot 22-1} = [\underbrace{2,\overline{2},2}_{[2,1,2]=0}, \underset{\substack{\uparrow\\ e_i}}{10}, \underbrace{2,2,2,2,\overline{2},5}_{[2,2,2,2,1,5]=0}] \Rightarrow$$



$$\frac{433^2}{433\cdot 104-1} = [\underbrace{5,\overline{2},2,2,2,2}_{[5,1,2,2,2,2]=0}, \underset{\substack{\uparrow\\ e_i}}{10}, \underbrace{5,2,2,2,2,2,2,\overline{2},8,2,2,2}_{[5,2,2,2,2,2,2,1,8,2,2,2]=0}]$$

Enough about Markov and $\mathbb{P}^2_{\mathbb{C}}$, now del Pezzo...

Definition: A Wahl chain $[e_1,\dots,e_r]$ is **del Pezzo** if

(I) $[e_1,\dots,e_{r-1}]$ or $[e_2,\dots,e_r]$ admit zero continued fraction of weight $\lambda \leq 8$.

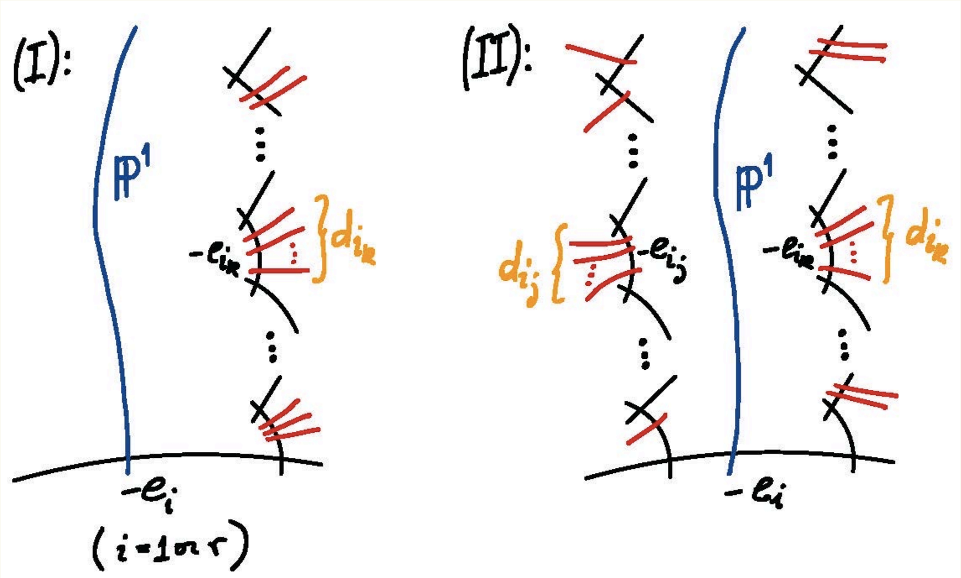
(II) $[e_1,\dots,e_{i-1}]$ and $[e_{i+1},\dots,e_r]$ admit zero continued fractions of weights λ_1, λ_2 with $\lambda_1+\lambda_2 \leq 8$.

Degree: $9-\lambda$ or $9-\lambda_1-\lambda_2$.

Marking: $[\underbrace{k_1,\dots,k_{i-1}}, \underline{e_i}, \underbrace{k_{i+1},\dots,k_r}]$.

corresponding zero continued fractions

Del Pezzo Wahl chains define Del Pezzo singular surfaces



$\longrightarrow$ Contraction of the Wahl chain

W_{*m}

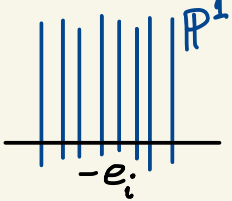
$\|$

$\frac{1}{n^2}(1,na-1)$

Marked surface

$\downarrow$

$\mathbb{F}_{e_i}$ =



= Hirzebruch surface

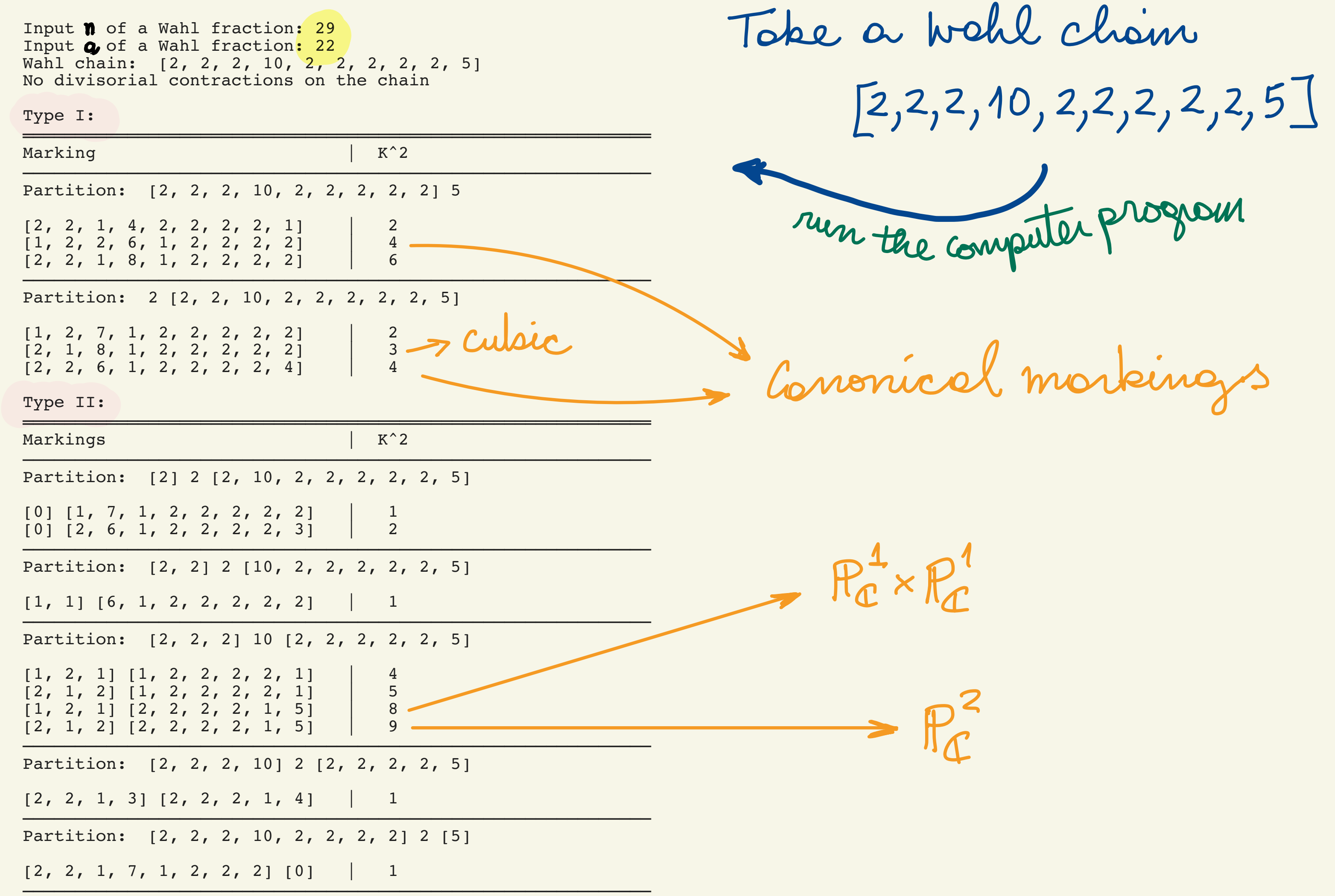
Theorem [U-Zúñiga 25]

A Wahl singularity appears in a $\mathbb{Q}$-Gorenstein degeneration of del Pezzo surfaces of degree ℓ

$$\Updownarrow$$

The corresponding Wahl chain is del Pezzo of degree $\ell' \geq \ell$.

- This is exactly Hacking-Prokhorov when $\ell = 9$.
- $\ell' \geq \ell$ is needed since for $\ell \leq 8$ we may have blow-ups at nonsingular points (Floating curves).
- A given Wahl chain may be del Pezzo for many distinct markings and various degrees (computer program).



```
Input n of a Wahl fraction: 29
Input a of a Wahl fraction: 22
Wahl chain:  [2, 2, 2, 10, 2, 2, 2, 2, 2, 5]
No divisorial contractions on the chain

Type I:
==============================================================
Marking                              |  K^2
--------------------------------------------------------------
Partition:  [2, 2, 2, 10, 2, 2, 2, 2, 2] 5

[2, 2, 1, 4, 2, 2, 2, 2, 1]          |   2
[1, 2, 2, 6, 1, 2, 2, 2, 2]          |   4
[2, 2, 1, 8, 1, 2, 2, 2, 2]          |   6
--------------------------------------------------------------
Partition:  2 [2, 2, 10, 2, 2, 2, 2, 2, 5]

[1, 2, 7, 1, 2, 2, 2, 2, 2]          |   2
[2, 1, 8, 1, 2, 2, 2, 2, 2]          |   3
[2, 2, 6, 1, 2, 2, 2, 2, 4]          |   4
Type II:
==============================================================
Markings                             |  K^2
--------------------------------------------------------------
Partition:  [2] 2 [2, 10, 2, 2, 2, 2, 2, 5]

[0] [1, 7, 1, 2, 2, 2, 2, 2]         |   1
[0] [2, 6, 1, 2, 2, 2, 2, 3]         |   2
--------------------------------------------------------------
Partition:  [2, 2] 2 [10, 2, 2, 2, 2, 2, 5]

[1, 1] [6, 1, 2, 2, 2, 2, 2]         |   1
--------------------------------------------------------------
Partition:  [2, 2, 2] 10 [2, 2, 2, 2, 2, 5]

[1, 2, 1] [1, 2, 2, 2, 2, 1]         |   4
[2, 1, 2] [1, 2, 2, 2, 2, 1]         |   5
[1, 2, 1] [2, 2, 2, 2, 1, 5]         |   8
[2, 1, 2] [2, 2, 2, 2, 1, 5]         |   9
--------------------------------------------------------------
Partition:  [2, 2, 2, 10] 2 [2, 2, 2, 2, 5]

[2, 2, 1, 3] [2, 2, 2, 1, 4]         |   1
--------------------------------------------------------------
Partition:  [2, 2, 2, 10, 2, 2, 2, 2] 2 [5]

[2, 2, 1, 7, 1, 2, 2, 2] [0]         |   1
--------------------------------------------------------------
```

Take a Wahl chain

[2,2,2,10,2,2,2,2,2,5]

run the computer program

cubic

Canonical markings

$\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$

$\mathbb{P}^2_{\mathbb{C}}$

Some ideas around the study:

1. Control of the exceptional divisor of $X \xrightarrow[\text{min res}]{} W$, where W has Wahl singularities, $-K_W$ big and $K_W^2 > 0$.

2. It uses some birational geometry of degenerations of surfaces to identify the most general degeneration W_{*m}. [Hacking-Tevelev-U, 2017]

3. To find a certain toric "canonical" degeneration of W_{*m} we use $\underbrace{\text{slidings}}$ of Markov and Mori type:

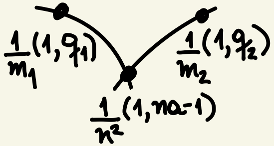
$\frac{1}{n^2}(1,na-1)$

$$\begin{tikzcd} = W_{*m} \arrow[r, rightsquigarrow] & \widehat{W}_{*m} \arrow[r] & W^{T}_{*m} \arrow[r] & P(n^2, m_1, m_2) \\ & \text{Toric wahl sing} & \text{Toric T-sing} & \text{Fake weighted projective plane} \end{tikzcd}$$

Theorem [U-Zúñiga 25]: There is correspondence

$\left\{ \begin{array}{l} W_{*m} \text{ marked del Pezzo} \\ \text{surface of degree } \ell \end{array} \right\} \longleftrightarrow \Big\{ P(n^2, m_1, m_2)$ Fake Weighted Planes

$\cup$



(1) $q_1 m_2 + q_2 m_1 + n^2 = d m_1 m_2$
some $d \geqslant 2$

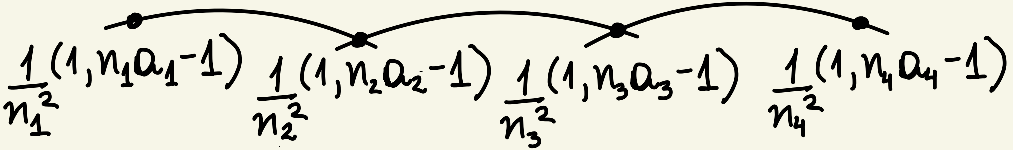
(2) $m_1 + m_2 = n(m_1 a - n q_1^{-1}) = n(m_2(n-a) - n q_2^{-1})$

(3) $\frac{1}{m_1}(1,q_1)$ and $\frac{1}{m_2}(1,q_2)$ admit zero continued fractions $\ell = 9 - \lambda_1 - \lambda_2 > 0$. $\Big\}$

For $\ell = 9$ this is:

$\left\{ \begin{array}{l} W_{*m} \text{ marked del Pezzo} \\ \text{surface of degree } \ell \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathbb{P}(n^2, a^2, b^2) \text{ where } (a, b < n) \\ \text{satisfy } n^2 + a^2 + b^2 = 3abn \end{array} \right\}$

In terms of wahl singularities, for $\ell=8$ we find:



$$n_1^2+n_2^2+n_3^2+n_4^2 = \delta n_1n_2+\delta' n_3n_4$$

$$n_1a_1+n_2a_2+n_3a_3+n_4a_4 = \delta n_1a_2+\delta' a_3n_4$$

where $\delta = n_1a_2-n_2a_1$ and $\delta' = n_3a_4-n_4a_3$.

<u>observation</u>: For $\ell=8$, we have two del Pezzo nonsingular surfaces $\mathbb{F}_0=\mathbb{P}^1\times\mathbb{P}^1$ and $\mathbb{F}_1$. We can distinguish among the W_{*m}.

2 consequences of our results.

(First) Are there constraints for wall chains in a given degree ℓ ?

Theorem (obs.) [UZ25]: In degrees $\ell \leq 4$, every wall chain is possible.

Reason: There is a canonical marking in degree 4.

Theorem [UZ25]: In degrees $\ell \geq 5$, there are infinite families of wall chains that are not realizable.

This really depends on our classification.

Examples: $[\underbrace{2,\dots,2}_{B}, A+4, \underbrace{2,\dots,2}_{A}, B+2]$ for $B > A+3 \geq 5$.

OPen: Is it possible to write down a precise list of Wahl chains for each $\ell = 5, 6, 7, 8$?

There are Markov and Mori mutations which allow to create infinite lists ...

(Second) Exceptional vector bundles on del Pezzo surfaces.

There is a parallel world of exceptional collections on $p_g = q = 0$ surfaces (e.g. del Pezzos), and via our work we can prove geometrically some recent 2024 results of Polishchuk and Rains ...

From Markov $= \mathbb{P}^2_{\mathbb{C}}$ to del Pezzo surfaces through degenerations with only Wahl singularities.

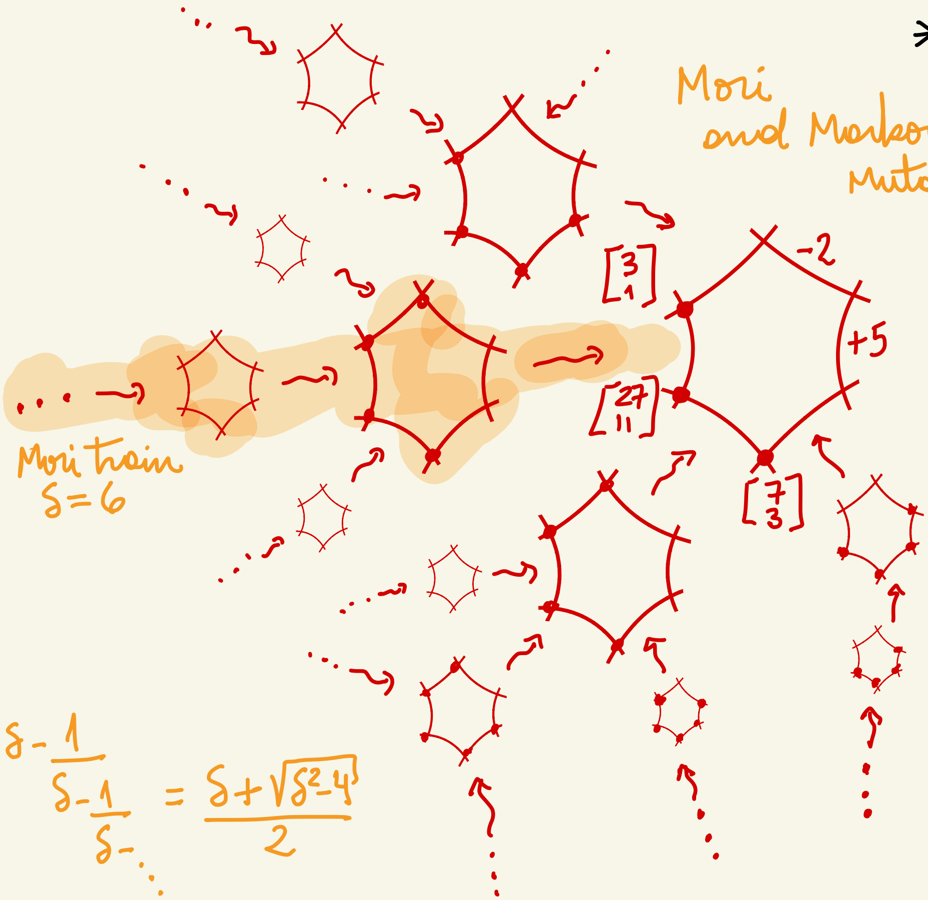
For more see the book/guide

"Negative continued fractions in birational geometry: A guide to degenerations of surfaces with Wahl singularities"

(check my webpage and/or 35 Colóquio Brasileiro de Matemática)

$$* = \frac{27^2}{27\cdot 11-1} = [\bar{\bar{3}}, \bar{2}, \underline{8}, 2, 2, \bar{2}, 4, \bar{2}]$$

Mori and Markov Mutations



$$\delta - \cfrac{1}{\delta - \cfrac{1}{\delta - \ldots}} = \frac{\delta + \sqrt{\delta^2-4}}{2}$$

$\mathbb{P}(27^2, 5, 22)$

CONTRACTION

$= \widehat{W}_{*m}$ = Toric weak del Pezzo surface

SLIDINGS

$W_{*m} =$ 

= Marked del Pezzo surface

DEGENERATION

W_t = del Pezzo surface of degree $\ell = 6$

The end