$K =$ cuerpo de números

$\mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$, superficie regular, proyectiva, $\mathcal{X} \otimes K$ geométricamente suave

$\mathcal{X}(\mathbb{C}) = \bigsqcup_{\sigma : K \hookrightarrow \mathbb{C}} X_\sigma$, sup de Riemann comp. conexas.

Fijamos $\mu = (\mu_\sigma)_{\sigma : K \hookrightarrow \mathbb{C}}$ colección formas volumen con $\int_{X_\sigma} \mu_\sigma = 1$
$\Rightarrow \widehat{\operatorname{Pic}}(\mathcal{X}, \mu)$

<u>Tma</u>: Existe una única aplicación bilineal $(\ ,\ ) : \widehat{\operatorname{Pic}}(\mathcal{X}, \mu) \times \widehat{\operatorname{Pic}}(\mathcal{X}, \mu) \to \mathbb{R}$ tal que

(i) $D \subset \mathcal{X}$ es un divisor irred., horizontal y reducido
$\Rightarrow \forall \hat{\mathcal{L}} \in \widehat{\operatorname{Pic}}(\mathcal{X}, \mu)$, se tiene que $(\hat{\mathcal{L}}, \hat{D}) = \widehat{\deg}\, \varepsilon_D^*(\hat{\mathcal{L}})$.

$$\begin{tikzcd} \text{normal.} = \tilde{D} \arrow[d] \arrow[dr, "\varepsilon_D"] & \\ D \arrow[r, hook] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

(ii) $D \subset \mathcal{X}$ divisor vertical $\subset \mathcal{X} \otimes \mathbb{F}_v$; $v \in \operatorname{Spec} \mathcal{O}_K$, $v \nmid \infty$
$(\hat{\mathcal{L}} \cdot \hat{D}) = \underbrace{\mathcal{L}|_D}_{\text{grado geométrico}} \log(\# k(v))$ $\quad \mathbb{F}_v = k(v) =$ cuerpo residual.

(iii) $(.)$ es simétrico.

<u>obs</u>: K'/K, $\mathcal{X}' = \mathcal{X} \otimes \mathcal{O}_{K'}$

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f"] \arrow[d] & \mathcal{X} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{K'} \arrow[r] & \operatorname{Spec} \mathcal{O}_K \end{tikzcd}$$

$\forall \hat{\mathcal{L}} \in \widehat{\operatorname{Pic}}(\mathcal{X}, \mu)$, $\hat{m}$ $\Rightarrow (f^*\hat{\mathcal{L}} \cdot f^*\hat{m})_{\mathcal{X}'} = [K':K] \cdot (\hat{\mathcal{L}}, \hat{m})_{\mathcal{X}}$

<u>En ∞</u>: $\mathcal{X}'(\mathbb{C}) = \bigsqcup_{\tau : K' \hookrightarrow \mathbb{C}} X_\tau = \bigsqcup_{\sigma : K \hookrightarrow \mathbb{C}} \underbrace{\bigsqcup_{\substack{\tau : K' \hookrightarrow \mathbb{C} \\ \tau|_K = \sigma}} X_\tau}_{[K':K] \text{ copias de } X_\sigma}$

<u>Fórmula de adjunción</u>: suponer $\mathcal{X}$ semiestable ($\chi \otimes K \geq 1$)

$\omega = \omega_{\mathcal{X}|\operatorname{spec} O_K}$, μ = forma de Arakelov, X = sup. Riemann compacta $g \geq 1$

$\omega_1, \dots, \omega_g$ base de Ω^1_X, ortonormal (c.r. $(\tau, \eta) \mapsto \int_X \tau \wedge \bar{\eta}$)

$\mu_{AR} = \frac{i}{2g} \sum_{i=1}^{g} \omega_i \wedge \bar{\omega}_i$, $\int_X \mu_{AR} = 1$ En $\mathcal{X}(\mathbb{C})$, ponemos $\mu_{Ar} = (\mu_{Ar,\sigma})_{\sigma : K \hookrightarrow \mathbb{C}}$.

$\Rightarrow \hat{\omega} \in \widehat{\operatorname{Pic}}(\mathcal{X}, \mu_{Ar})$

<u>Teorema</u>: Sea D un divisor horizontal irreducible sobre $\mathcal{X}$.
Escribimos $D(\mathbb{C}) = \sum_\sigma [P_\sigma]$, K' = cuerpo de D.

$$\Rightarrow \quad \hat{D} \cdot \hat{D} + \hat{D} \cdot \hat{\omega} = \log \Big[\underbrace{N_{K|\mathbb{Q}}(\operatorname{disc}(K'|K))}_{\in \mathbb{N}} \Big] + \sum_{\sigma \neq \tau} g(P_\sigma, P_\tau)$$

g = la función de Green asociada a μ_{AR}.

<u>Alturas</u>: $\mathcal{X}$, μ cualquiera, $\hat{\mathcal{L}} \in \widehat{\operatorname{Pic}}(\mathcal{X}, \mu)$ con $\mathcal{L}$ amplio.

Si $P \in \mathcal{X}(K')$ donde $K'|K$ extensión finita

$$\begin{tikzcd} \operatorname{spec} O_{K'} \arrow[r, "s"] \arrow[dr] & \mathcal{X} \arrow[d] \\ & \operatorname{spec} O_K \end{tikzcd}$$

<u>Def</u>: $h_{\hat{\mathcal{L}}}(P) := \dfrac{\widehat{\deg}\, s^*(\hat{\mathcal{L}})}{[K':K]} = \dfrac{\widehat{\operatorname{Imagen}(s)} \cdot \hat{\mathcal{L}}}{[K':K]}$

Si

$$\begin{tikzcd} \operatorname{spec} O_{K''} \arrow[r, "s''"] \arrow[dr] & \mathcal{X}'' \arrow[dr] & \\ & \operatorname{spec} O_{K'} \arrow[r] & \mathcal{X} \end{tikzcd}$$

$$\frac{\widehat{\deg}\, s''(\hat{\mathcal{L}})}{[K'':K]} = \frac{\cancel{[K'':K']}\, \widehat{\deg}\, s(\hat{\mathcal{L}})}{\cancel{[K'':K']} \cdot [K':K]}$$

$\therefore \; h_{\hat{\mathcal{L}}} : \mathcal{X}(\bar{K}) \to \mathbb{R}$

Propiedad: $\mathcal{L}$ amplio $\Rightarrow \forall A, B > 0$
$\{ P \in \mathcal{X}(\bar{K}) : h_{\hat{\mathcal{L}}}(P) \leq A ;\ [\mathbb{Q}(P):\mathbb{Q}] \leq B \}$ es finito.

obs 1.- $\mathcal{L}_K$ amplio $\quad$ asumir $\quad$ $\begin{array}{ccc} \mathcal{L} & & \mathcal{L}' \\ \downarrow & & \downarrow \\ \mathcal{X} & & \mathcal{X}' \\ \downarrow & \swarrow & \\ \operatorname{Spec}\mathcal{O}_K & & \end{array}$ $\quad$ $\mathcal{X}\otimes K \simeq \mathcal{X}\otimes K' \simeq X_K$

$\downarrow$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\mathcal{L}\otimes K \simeq \mathcal{L}'\otimes K \simeq \mathcal{L}_K$

$X_K = \mathcal{X}\otimes K$

$\Rightarrow$ Entonces $\exists\, C > 0$ tq $\forall P \in X_K(\bar{K})$, $|h_{\hat{\mathcal{L}}}(P) - h_{\hat{\mathcal{L}}'}(P)| \leq C$.

Si fijamos $\mathcal{X}$ y cambiamos el fibrado amplio. Similar si solo cambiamos μ.

Ej 1.- $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$, $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(1) = \mathcal{O}_{\mathcal{X}}(\infty)$ $\quad P = [a:b]$; $a, b \in \mathbb{Z}$ $\operatorname{mcd}(a,b) = 1$

$\infty = [1,0]$ $\quad s = x_1$ $\quad \|s\| = \dfrac{|s(a,b)|}{\max\{|a|,|b|\}}$.

$\Rightarrow h_{\mathcal{O}_{\mathcal{X}}(1)}([a:b]) = \log\max\{|a|,|b|\}$.

Altura controla el tamaño de los coeficientes del pol. minimal.

Teo de Ullmo: $\mathcal{X}$ semiestable, $\hat{\mathcal{L}} \in \widehat{\operatorname{Pic}}(\mathcal{X},\mu)$, $s \in \mathcal{L}$

$$\|s\|_{sup} = \max_{P\in\mathcal{X}(\mathbb{C})} \|s(P)\|$$

$\hat{H}^0(\mathcal{X}, \hat{\mathcal{L}}) = \{$secciones efectivas$\} = \{s \in \mathcal{L} : \|s\|_{sup} \leq 1\}$

$(\Leftrightarrow -\log\|s\|_{sup} \geq 0)$

$H^0(\mathcal{X}, \mathcal{L})$ es un $\mathbb{Z}$-módulo.

Def: $\hat{\mathcal{L}}$ se dice numericamente positivo si

i) $(\hat{\mathcal{L}}, \hat{\mathcal{L}}) > 0$ $\quad$ ii) $(\hat{\mathcal{L}}\cdot\hat{D}) > 0$ $\forall D \subset \mathcal{X}$ irred.

Def: $\hat{\mathcal{L}}$ es aritméticamente amplio si $\forall \hat{\mathcal{M}} \in \widehat{\operatorname{Pic}}(\mathcal{X},\mu)$ $\exists n_0$ $\forall n \geq n_0$ $H^0(\mathcal{X}, \mathcal{L}^{\otimes n}\otimes\mathcal{M})$ admite una $\mathbb{Z}$-base formada por secciones efectivas.

Teo (Zhang) $\hat{\mathcal{L}}$ numericamente positivo $\Rightarrow \hat{\mathcal{L}}$ es aritméticamente amplio.

Más precisamente la $\mathbb{Z}$-base de $H^0(\mathcal{X}, \mathcal{L}^{\otimes n} \otimes \mathcal{M})$ se puede escoger tal que $\|\cdot\|_{sup} \leq e^{-cn}$ con algún $c(\mathcal{L}, \mathcal{M}, \cancel{\text{todo}}) > 0$ adecuado.

Teo (Ullmo 1) $\mathcal{X} \to \operatorname{Spec} O_K$ semiestable. Sea $Z \subset \mathcal{X}$ la imagen de una sección. Entonces $\exists n_0$, $\forall n \geq n_0$, $\exists s \in H^0(X, \mathcal{L}^{\otimes n})$ con $s \neq 0$ y $\|s\|_{sup} < 1$ y tal que $\underbrace{|\mathrm{div}(s)| \cap Z = \emptyset}_{\text{no hay intersección finita}}$, donde $\mathcal{L}$ es numericamente positivo.

Teo (Ullmo 2) $\mathcal{X} \to \operatorname{Spec} O_K$, $\hat{\mathcal{L}}$, mismas hipótesis. $Z \subset \mathcal{X}$ una sección y $\mathcal{X}^0 = \mathcal{X} \setminus Z$.
Entonces existe un punto entero $P \in \mathcal{X}^0$ tal que

$$h_{\hat{\mathcal{L}}}(P) \leq \underbrace{\frac{\hat{\mathcal{L}} \cdot \hat{\mathcal{L}}}{(\deg \mathcal{L}) \cdot [K:\mathbb{Q}]}}_{\text{Calculable}} \quad (\checkmark).$$

[Existe punto en $\mathbb{Z}$!

Dem/- Sea $s \in \mathcal{L}^{\otimes n}$ como en el teorema de Ullmo 1 $\Rightarrow \mathrm{div}(s) = \underset{\text{vertical}}{V} + \underset{\text{horizontal sección}}{\sum_{i=1}^{n \deg \mathcal{L}} E_i}$ $(*)$

$E_i \leadsto P_i \in \mathcal{X}(K')$; $K'|_K$ extensión finita
los P_i son "Z-enteros" i.e., $|E_i| \cap |Z| = \emptyset$.

En $(*)$, $(\cdot \hat{\mathcal{L}})$: $(\widehat{\mathrm{div}}(s) \cdot \hat{\mathcal{L}}) = \underbrace{(\hat{V} \cdot \hat{\mathcal{L}})}_{\geq 0} + \sum_{i=1}^{n \deg \mathcal{L}} (\hat{E}_i \cdot \hat{\mathcal{L}})$

$\qquad\qquad \| \qquad\qquad\qquad\qquad$ (ya que es geom. amplio)

$(\hat{\mathcal{L}}^{\otimes n} \cdot \hat{\mathcal{L}})$

$\qquad \|$

$n(\hat{\mathcal{L}} \cdot \hat{\mathcal{L}})$

$\therefore\ n(\hat{\mathcal{L}} \cdot \hat{\mathcal{L}}) \geq n\,(\deg \mathcal{L}) \underbrace{\min(\hat{E}_i \cdot \hat{\mathcal{L}})}_{h_{\hat{\mathcal{L}}}(P_i)\,[K:\mathbb{Q}]} \Rightarrow (\checkmark).$