Funciones de Green.

X superficie de Riemann compacta no-singular.

C^ω (resp C^∞) $(1,1)$-forma φ en X es localmente

$$i\, f(z,\bar z)\, dz\wedge d\bar z \quad \text{c/ } f\ C^\omega \text{ (resp } C^\infty).$$

Notar que $\left.\begin{array}{l} dz = dx + i\,dy \\ d\bar z = dx - i\,dy \end{array}\right\}$ $i\, dz\wedge d\bar z = 2\, dx\wedge dy$.

φ es forma de volumen si $f(z,\bar z) > 0\ \forall (z,\bar z)$.

φ es normalizada si $\int_X \varphi = 1$.

X viene c/ una forma canónica de volumen. μ_X o μ

Una ~~[illegible]~~ 1-forma holomorfa en X

es localmente $h(z)\,dz$. c/ h holomorfa

en gral φ, ψ ~~[illegible]~~ $\langle\varphi,\psi\rangle := \frac{i}{2}\int_X \varphi\wedge\bar\psi$ producto Hermitiano

$\varphi_1, \ldots, \varphi_g$ base ortonormal.

$$\Rightarrow \mu = \frac{i}{2g}\sum_{i=1}^{g} \varphi_i\wedge\bar\varphi_i \quad \text{es forma de volumen.}$$

normalizada ∧ es la canónica.

Si $g=0$, se toma como forma de volumen

$$\varphi(z) = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \qquad z \in \mathbb{C}$$

y corresponde a volumen esférico bajo proyección estereográfica

Si $g=1$ ent $X = \mathbb{C}/\langle 1, \tau \rangle$ para cierto $\tau = u + iv$ c/ $v > 0$

$$\varphi(z) = \frac{i}{2v} dz \wedge d\bar{z} = \frac{1}{v} dx \wedge dy.$$

(donde $dz = \pi_* dz$ $\quad \pi : \mathbb{C} \to \mathbb{C}/\langle 1, \tau \rangle$)

$d\bar{z} = \pi_* d\bar{z}$

Cálculo en S.R.

$dz = dx + i\,dy$
$d\bar{z} = dx - i\,dy$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$f : X \to \mathbb{C}$ ó $\omega = u \cdot dz + v\,d\bar{z}$

$$\partial f = \frac{\partial f}{\partial z} dz$$

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

$$\partial \omega = \partial v/\partial z \; dz \wedge d\bar{z}$$

$$\bar{\partial} \omega = -\partial u/\partial \bar{z} \; dz \wedge d\bar{z}$$

$f : X \to \mathbb{C}$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = (\partial + \bar{\partial}) f$$

$$d^c f = \left(\frac{\partial f}{\partial z} dz - \frac{\partial f}{\partial \bar{z}} d\bar{z}\right) \frac{i}{4\pi} = \frac{i}{4\pi}(\partial - \bar{\partial}) f$$

$$df \wedge d^c f = \frac{|\nabla f|^2}{8\pi} dx \wedge dy.$$

$$dd^c f = \frac{i}{2\pi} \partial \bar{\partial} f = \frac{1}{4\pi} \Delta f \, dx \wedge dy.$$

Función de Green para un divisor D c/r a un volumen φ

$g_D : X \setminus \operatorname{supp} D \to \mathbb{R}$ se llama

si ① $\forall (U, f)$ representando a D ie $\operatorname{div}(f) \cap U = D$.

$$g_D(P) = -\log |f(P)|^2 + \alpha(P)$$

donde $\alpha(P)$ es $C^\infty(U)$

② $dd^c g_D = \varphi$ en $X \setminus \operatorname{supp} D$

③ $\displaystyle\int_X g_D \varphi = 0$

obs ① ∧ ② $\Rightarrow$ $\tilde{g}_D - g_D = \alpha$ es suave
$\tilde{g}_D$ tq

y $\Delta \alpha = 0$ $\forall$ en X

$\Rightarrow$ α = harmónica $\Rightarrow$ cte. $\equiv \alpha$.

si además por ③ $\alpha \equiv 0$.

obs $D = \sum [P_i] n_{P_i}$ $\Rightarrow$ $g_D = \sum n_i g_{P_i}$

(linealidad de g en D).

Ejemplo en $\mathbb{P}^1$

$$g_0(z) = \cancel{\log \frac{|z|^2}{1+|z|^2}} - \log \frac{|z|^2}{1+|z|^2} + cte$$

$$dd^c g_0(z) = \underbrace{\frac{-1}{4\pi} \Delta \log |z|^2}_{0} + \underbrace{dd^c \left(\log (1+|z|^2) \right)}_{\varphi}.$$

y $$g_W(z) = -\log \frac{|z-w|^2}{(1+|z|^2)} + \underbrace{cte(w)}_{\varphi}$$

$$dd^c g_W(z) = \underbrace{-\frac{1}{4\pi} \Delta \log |z-w|^2}_{0} + dd^c \log(1+|z|^2)$$
$$- \cancelto{0}{dd^c \log (1+|w|^2)}$$

CALCULAR CTE !

Propiedades

(P1) $\{g_P\}_{P\in X}$ green $\overset{\text{c/r } \varphi}{\Longrightarrow}$ $g_P(Q) = g_Q(P)$ $\forall P\neq Q$ en X.

ie $$G: X\times X\setminus \text{Diag} \longrightarrow \mathbb{R}$$ $$(P,Q) \longmapsto g_P(Q)$$ ⟸ Función green asoc a φ.

es simétrica.

obs ⇒ $\underset{\text{en } \mathbb{P}^1}{\wedge}$ $g_W(z) = g_z(W)$ ∴ $g(z,w) = -\log\dfrac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}$ + cte (indep de z, w))

(P2) φ_1, φ_2 C^∞ (resp C^ω) formas de volumen

$G_1, G_2 : X\times X\setminus \text{Diag} \longrightarrow \mathbb{R}$

$\Rightarrow \exists\, \beta$ C^∞ (resp C^ω) tq

$$G_1(P,Q) - G_2(P,Q) = \beta(P) + \beta(Q)$$

Mas aún. β es la única ~~sol~~ de

$$dd^c\beta = \varphi_1 - \varphi_2$$

$$\text{tq } \int \beta(\varphi_1+\varphi_2) = 0.$$

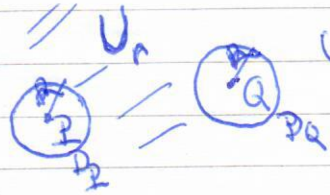
Prop ①

$\boxed{\text{Reescribir}}$

Lema ① ∧ ②

$$\Rightarrow \int_X \overbrace{g_P\, dd^c g_Q - g_Q\, dd^c g_P}^{\mu} = g_P(Q) - g_Q(P)$$

Dem/



$U_R = X \setminus$ discos de radio r en torno a P y Q.

en U_r $\quad d(\underbrace{g_P d^c g_Q - g_Q d^c g_P}_{\omega}) = \mu$

ya que $\quad d\alpha \wedge d^c\beta = -d^c\alpha \wedge d\beta = d\beta \wedge d^c\alpha.$

Stokes $\quad \int_{U_R} \overset{d\omega}{\ddot{\mu}} = \int_{\partial U_R} \omega = -\int_{\partial D_P} \omega - \int_{\partial D_Q} \omega.$

$$\int_{\partial D_P} \omega \simeq \int_{\partial D_P} (\log|z-P|^2 + \alpha(z))\, \underbrace{d^c g_Q}_{\text{en polares } \frac{1}{2} r \frac{\partial g_Q}{\partial r} \frac{d\theta}{2\pi}} - g_Q\, d^c(\log|z-P|^2 + \alpha(P))$$

$$\simeq \underbrace{\int_{\partial D_P} -r \log r \, d\theta}_{\downarrow\, 0} - \underbrace{g_Q \cdot \frac{1}{2} r \left(-\frac{2}{r} + c^0\right) \frac{d\theta}{2\pi}}_{\downarrow\, g_Q(P).}$$

Prop2 $\varphi\ C^\infty(C^\omega)$

⚠ Teo ✓ $\displaystyle\int_X \varphi = 0 \Rightarrow \exists \rho$ tq $dd^c\rho = \varphi$. $C^\infty(C^\omega)$.

$G_2(P,Q) + \rho(Q)$ satisface sing log en P.
$\quad \| $
$\quad g_P(Q)$ y $dd^c g_P = \cancel{\varphi_2} + \varphi_1 - \cancel{\varphi_2}$

$\therefore\ G_2(P,Q) + \rho(Q) = G_1(P,\cdot) + \text{cte}$

PD| $\displaystyle\int_X (G_2(P,Q) + \rho(Q))\,\varphi_1(Q) = -\rho(P)$

$(\int G_2\varphi_2 = 0)$

$$LI = \int_X G_2(P,Q)\underbrace{(\varphi_1(Q) - \varphi_2(Q))}_{dd^c\rho} + \underbrace{\int_X \rho(Q)\varphi_1(Q)}_{\| \\ \left(-\int \rho(Q)\underbrace{\varphi_2(Q)}_{dd^c G_2}\right)}$$

$$= \int_X G_2\, dd^c\rho - \rho\, dd^c G_2 = -\int_X d(\rho\, d^c G_2 - G_2 d^c\rho).$$

$$= \int_{D_P} \rho\, d^c G_2 - G_2 d^c\rho = -\rho(P)$$

Teo $\exists$ ~~una~~ Función de Green G c/r a cualquier $(1,1)$ forma φ C^∞ (resp C^ω). G es C^∞ (resp C^ω) fuera de la diagonal.

Obs G invierte Δ en sentido de distribuciones.

$$dd^c[g_P] : C^\infty(X) \longrightarrow \mathbb{R}$$
$$f \longmapsto \int_X g_P \, dd^c f.$$

Similarmente

$$[\varphi] : C^\infty(X) \to \mathbb{R}$$
$$f \mapsto \int_X f\varphi.$$

Teo $dd^c[g_P] = [\varphi] - \delta_P$

ie $\int g_P dd^c f = \int f\varphi - f(P) \quad \forall f \in C^\infty$

Def Δ c/r φ es por definición

$$(\Delta_\varphi f)\varphi = -dd^c f$$

Corolario $\int_X f\varphi = 0 \Rightarrow \int_X g_P (\Delta_\varphi f)\varphi = f(P)$

Función de Green $g=1$

$$X = \mathbb{C}/\langle 1,\tau\rangle \qquad \tau = u+iv>0$$

$$q_\tau = \exp(2\pi i\tau)$$

$$q_z = \exp(2\pi i z)$$

$$B_2(y) = y^2 - y + 1/6$$

$P = 0+\langle 1,\tau\rangle$

Sea $\lambda(z) = -\log\left| q_\tau^{B_2(y/v)/2} \cdot (1-q_z)\prod_{n=1}^{\infty}(1-q_\tau^n q_z)(1-q_\tau^n/q_z)\right|$

Notar $|e^{2\pi i n\tau}| = |e^{-2\pi v}|^n$

$\therefore \quad |\log(1-q_\tau^n q_z)| \asymp e^{-2\pi v n}$ $\quad\therefore$ cv abs.

y $\lambda(z)$ es C^ω.

$\lambda(z+1) = \lambda(z)$

$$\lambda(z+\tau) - \lambda(z) = -\log\left|\frac{q_\tau^{B_2(y/v+1)/2}}{q_\tau^{B_2(y/v)/2}} \cdot \frac{(1-q_\tau q_z)}{(1-q_z)}\frac{(1-1/q_z)}{(1-q_\tau q_z)}\right| = -\log|q_\tau^{y/v}/q_z| = 0$$

NVO

desaparece

queremos ~~algo(z+τ) − algo(z) = real~~

~~$\text{algo}(x+iy) = \frac{1}{2}B_2(y/v) = ((y/v)^2 - y/v + 1/6)\frac{1}{2}$~~

$$\frac{B_2(y/v+1) - B_2(y/v)}{2} = 2y/v$$

$$e^{2\pi i(u+iv)\frac{y}{v}} = e^{2\pi i\left(\frac{uy}{v} + -2\pi y\right)}$$

$B_2(y) = y^2 - y + 1/6$

$$\frac{B_2(y/v+1) - B_2(y/v)}{2} = \frac{y}{v}$$

obs ① $g(z) = 2\lambda(z)$ tiene polo $\log|z|^2$ cerca de 0. y trasladados.

~~$2\pi i B_2 + 2\pi i B_2$~~

② $dd^c g = - dd^c \log\left|q_\tau^{B_2(y/v)/2}\right|^2$

$= + dd^c\, \cancel{\log}\, (2\pi v\, B_2(y/v))$

$= \frac{1}{4\pi} \Delta (2\pi v\, B_2(y/v))$

$= \frac{1}{4\pi} \cdot 2\pi v \cdot \frac{2}{v^2}\, dx \wedge dy = dx \wedge dy.$

③ $\int g_I \varphi = 0.$ $\qquad \varphi = dx \wedge dy.$

integramos $\lambda(z)$ en dominio fundamental

Ad.

$$I_1 = \int_0^1\int_0^1 \log \left| q_\tau^{B_2(s)/2} \right| dr\, ds.$$

$\cancel{r = \lambda x}$
$\cancel{s = \lambda y}$
$\begin{cases} x = r + s u \\ y = s v \end{cases}$

$$I_2 = \int_0^1\int_0^1 \log \left| 1 - q_\tau^n q_{r+s\tau} \right| dr\, ds$$

$$\left| q_\tau^n q_{r+s\tau} \right| = e^{-2\pi v (n+s)} < 1 \quad \forall n \geq 0$$

$$\therefore \quad -\log\left(1 - q_\tau^n q_{r+s\tau}\right) = \sum_{m=1}^{\infty} \frac{w^m}{m} \Big|_{w = q_\tau^n q_{r+s\tau}}.$$

$$y \quad \mathrm{Re}\left(+\log\left(1 - q_\tau^n q_{r+s\tau}\right)\right) = \log\left|1 - q_\tau^n q_{r+s\tau}\right|$$

para cada s $\quad \displaystyle\int_0^1 w^m\, dr = \int_0^1 q_\tau^{nm} \cdot e^{2\pi i m (r + s\tau)}\, dr$

$$= c \int_0^1 e^{2\pi i m}\, dr = 0.$$

$$I_3 = \int_0^1\int_0^1 \log\left|1 - q_\tau^n q_{-r-s\tau}\right| dr\, ds = 0 \quad \text{idem}$$