- Var tóricas asociadas a $S = \mathbb{N}p + \mathbb{N}q$, $(p,q)=d$
- Familia de conos en $\mathbb{N}^2$

En $\mathbb{N}$ tomar p, q con $(p,q) = d \geq 1$

- $(p,q) = (2,3)$, $S = \mathbb{N}\cdot 2 + \mathbb{N}\cdot 3 = \mathbb{N}\setminus\{1\}$

$X = \operatorname{Spec} k[S] \cong \{u^3 - v^2 = 0\} \subset \mathbb{A}^2_{u,v}$

$T \simeq k^*$

$T \times X \longrightarrow X$, $\lambda\cdot(u,v) \longrightarrow (\lambda^2 u, \lambda^3 v)$

Caso general: $(p,q) = 1$

$X \simeq \{u^q - v^p = 0\}$

⚠ $(p,q) = 2$, $p = 4$, $q = 6$

$u^6 - v^4 = 0$ Reducible

$$S = \mathbb{N}p + \mathbb{N}q, \quad (p,q) = d \geq 1$$

$$\phi : \mathbb{N}^2 \longrightarrow S$$
$$e_1, e_2 \longmapsto p, q$$

$$f : k[u,v] \twoheadrightarrow k[S]$$
$$u = t^{e_1} \longmapsto x^{\phi(e_1)} = x^p$$
$$v = t^{e_2} \qquad x^{\phi(e_2)} = x^q$$

$$\operatorname{Ker} f = (t^a - t^b : a, b \in \mathbb{N}^2, \ \phi(a) = \phi(b))$$
$$t^a - t^b = \underline{u^{a_1} v^{a_2} - u^{b_1} v^{b_2}}$$

$$\phi(a) = \phi(a_1, a_2) = a_1 \phi(e_1) + a_2 \phi(e_2)$$
$$= a_1 p + a_2 q$$

$$\phi(a) = \phi(b) \iff p(a_1 - b_1) = q(b_2 - a_2)$$

$p = dp'$ $\iff p'(a_1 - b_1) = q'(b_2 - a_2) > 0$

$q = dq'$ $\implies a_1 - b_1 = q'k \quad k > 0$

$(p', q') = 1$ $\qquad b_2 - a_2 = p'k$

$$\implies a_1 = q'k + b_1$$
$$b_2 = p'k + a_2$$

$$u^{b_1 + q'k} v^{a_2} - u^{b_1} v^{a_2 + p'k}$$

$$= u^{b_1} v^{a_2} \left((u^{q'})^k - (v^{p'})^k \right)$$

$$= \text{Algo} \, (u^{q'} - v^{p'})$$

$$\Rightarrow \ker f = (u^{q'} - v^{p'})$$

$$X = \operatorname{Spec} k[S] \cong \{ u^{q'} - v^{p'} = 0 \} \subseteq \mathbb{A}^2_{u,v}$$

Fact: En $\mathbb{Z}$, $(p, q) = (d)$

$\exists c, d \in \mathbb{Z}: \; cp + dq = d$

$$\Rightarrow S^{gp} \ni d \Rightarrow S^{gp} = d\mathbb{Z} \simeq \mathbb{Z}$$

$$T = \operatorname{Spec} k[\underline{S^{gp}}] \simeq k^*$$

$$\begin{aligned} k^* \simeq T \times X &\longrightarrow X \\ \lambda \cdot (u, v) &\longmapsto (\lambda^{p'} u, \lambda^{q'} v) \end{aligned}$$

$$k[S^{gp}] = k[x^{\pm p}, x^{\pm q}]$$

$x^p = (x^d)^{p'}$

$x^q = (x^d)^{q'}$

$$= k[x^{\pm d}]$$

$$k[S^{gp}] = \frac{k[u^{\pm 1}, v^{\pm 1}]}{(u^{q'} - v^{p'})}$$

$$\begin{array}{ccccc} k[S] & \longrightarrow & k[S^{gp}] & \xrightarrow{\sim} & k[\lambda^{\pm 1}] \\ u, v & \longmapsto & u, v & \longrightarrow & \lambda^{p'}, \lambda^{q'} \end{array}$$

$$\begin{tikzcd} \underset{\underline{u,\, v}}{k[S]} \arrow[r] \arrow[dr, bend right] & \underset{u,\, v}{k[S^{gp}]} \otimes \underset{u,\, v}{k[S]} \arrow[d, "s"] \\ & \underset{\underline{\lambda^{p'}},\, \underline{\lambda^{q'}}}{k[\lambda^{\pm 1}]} \otimes \underset{\underline{u},\, \underline{v}}{k[S]} \end{tikzcd}$$

- $X = \operatorname{Spec} k[S]$

$S^{sat} = \{\alpha \in S^{gp} : \exists m \geq 1,\ m\alpha \in S\}$

$S^{gp} = d\mathbb{Z}$

- $\alpha \in S^{gp}$, $\alpha < 0 \Rightarrow \nexists m \geq 1 : m\alpha \in S$
- $\alpha \geq 0$, Tomar $m \in \{p, q\}$

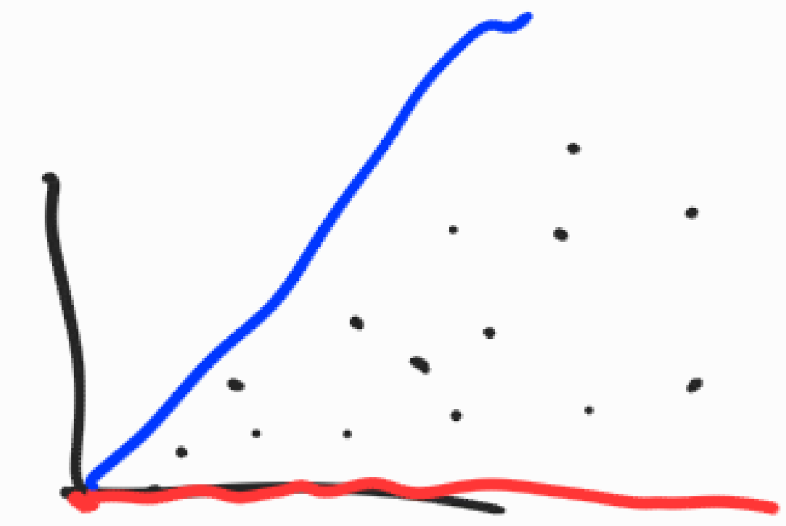
  $\Rightarrow m\alpha \in S$

$\therefore S^{sat} = d\mathbb{N} \simeq \mathbb{N} \Rightarrow \operatorname{Norm} X = \mathbb{A}^1$

- Una familia de conos en $\mathbb{N}^2$

$n \geq 1$, $\quad w_1 = e_1 = (1,0)$

$\qquad\quad w_2 = (n, n+1)$



$$C = \mathbb{R}_{\geq 0} w_1 \oplus \mathbb{R}_{\geq 0} w_2$$

$S = C \cap \mathbb{N}^2$ <u>semigrupo</u>

- S está generado como semigrupo por $w_1, w_2, (1,1)$

$$\phi : \mathbb{N}^3 \longrightarrow S \; ;$$

$$e_1, e_2, e_3 \longmapsto w_1, w_2, (1,1)$$

$$(1,1) = \frac{1}{n+1}(w_1 + w_2)$$

$$f : K[u,v,z] \twoheadrightarrow K[S]$$

$$u = t^{e_1},\ v = t^{e_2},\ z = t^{e_3} \longmapsto \chi^{w_1}, \chi^{w_2}, \chi^{(1,1)}$$

$$\phi(a) = a_1 \phi(e_1) + a_2 \phi(e_2) + a_3 \phi(e_3)$$

$$= (a_1 + n a_2 + a_3,\ (n+1) a_2 + a_3)$$

$$\phi(a) = \phi(b) \Longleftrightarrow$$

$$\begin{cases} (a_1 - b_1) + n(a_2 - b_2) + (a_3 - b_3) = 0 \\ (n+1)(a_2 - b_2) + (a_3 - b_3) = 0 \end{cases}$$

$\Longrightarrow$ • $a_1 - b_1 = a_2 - b_2 = c > 0$

• $a_3 - b_3 = -(n+1)(a_2 - b_2)$

$= -(n+1)\, c$

$\Longrightarrow$ $a_1 = b_1 + c$; $b_3 = a_3 + (n+1)c$

$a_2 = b_2 + c$

$$t^a - t^b = u^{a_1} v^{a_2} z^{a_3} - u^{b_1} v^{b_2} z^{b_3}$$

$$= u^{b_1 + c} v^{b_2 + c} z^{a_3} - u^{b_1} v^{b_2} z^{a_3 + (n+1)c}$$

$$= u^{b_1} v^{b_2} z^{a_3} \left((uv)^c - (z^{n+1})^c \right)$$

$$= Algo\,(uv - z^{n+1})$$

$$\Longrightarrow \ker f = (uv - z^{n+1})$$

$$X = \operatorname{Spec} k[S] \simeq \{ uv - z^{n+1} = 0 \} \subseteq \mathbb{A}^3_{u,v,z}$$

$$S^{gp} \ni (0,1) = (1,1) - (1,0)$$

$$\Rightarrow (0,1) \in S^{gp}$$

$$\Rightarrow S^{gp} = \mathbb{Z}^2$$

$$\Rightarrow T = \operatorname{Spec} k[S^{gp}] \simeq (k^*)^2$$

$$(k^*)^2 \xrightarrow{\sim} T \hookrightarrow X$$

$$k[S^{gp}] = k[x^{\pm\omega_1}, x^{\pm\omega_2}, x^{\pm(1,1)}]$$

$$= k[\underline{x}^{\pm e_1}, x^{\pm e_2}]$$

$$\simeq k[\lambda^{\pm 1}, \eta^{\pm 1}]$$

$\omega_2 = \underline{n}\,e_1 + \underline{(n+1)e_2}$

$\underline{(1,1) = e_1 + e_2}$

$$K[S] \longrightarrow K[S^{gp}] \xrightarrow{\sim} K[\lambda^{\pm 1}, \eta^{\pm 1}]$$

$$u, v, z \longmapsto u, v, z \longmapsto \underline{\lambda, \lambda^n \eta^{n+1}, \lambda\eta}$$

$$(K^*)^2 \times X \longrightarrow X$$

$$(\lambda, \eta) \cdot (u, v, z) \longmapsto (\lambda u, \lambda^n \eta^{n+1} v, \lambda\eta z)$$

- $S^{sat} = \{\alpha \in \mathbb{Z}^2 : \exists\, m \geq 1,\ m\alpha \in S\}$

$\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 < 0$, $\alpha_2 < 0$

$\Longrightarrow \nexists\, m$

$\alpha \in \mathbb{N}^2 \cap S^{sat} \Longrightarrow \alpha \in S$

$\therefore\ S^{sat} = S$

$\therefore\ X$ es normal.

$\bullet \; S \hookrightarrow \mathbb{N}^2$

$w_1 = (1,0) \qquad e_1$

$w_2 = (n, n+1) \longmapsto n e_1 + (n+1) e_2$

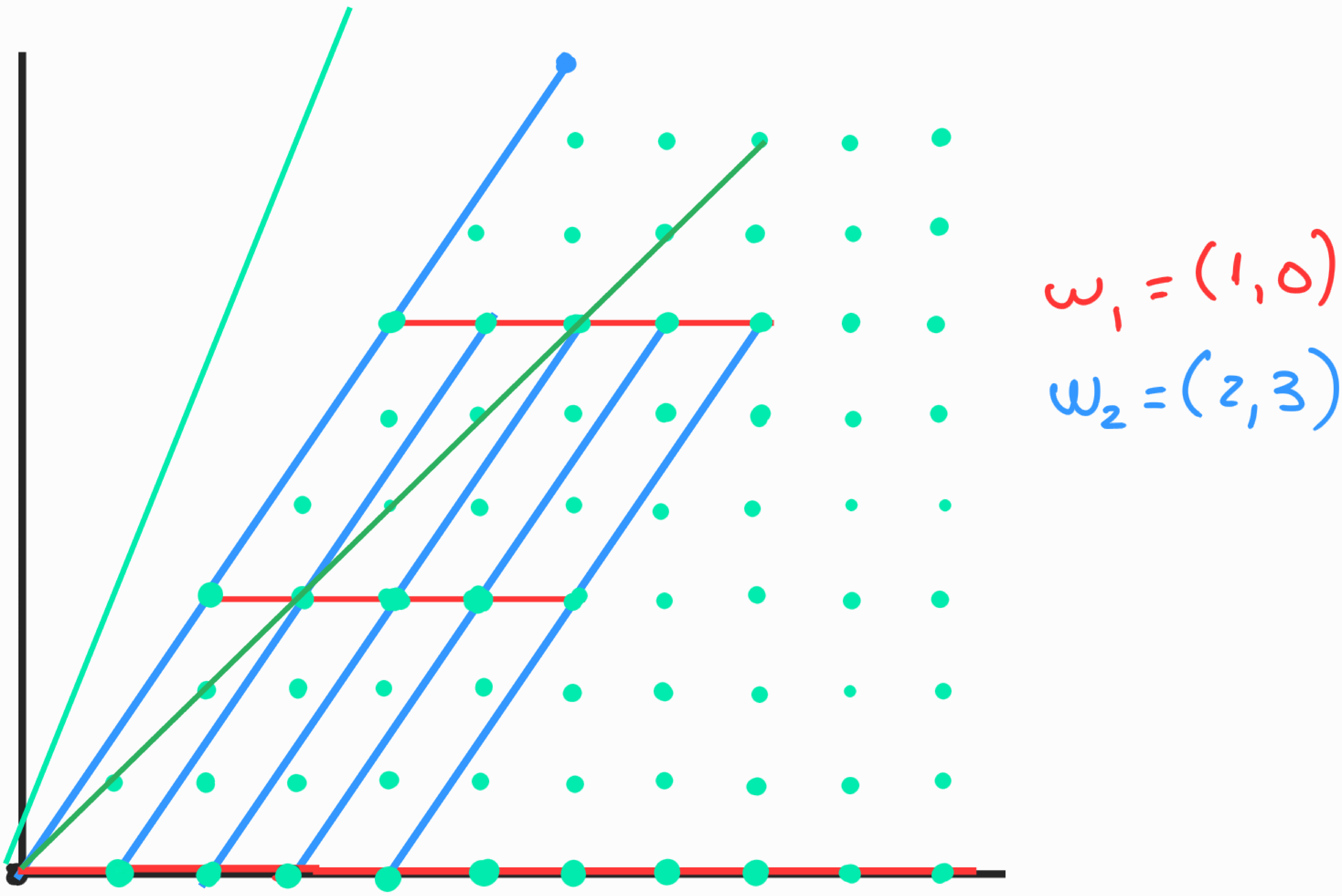
$(1,1) \qquad e_1 + e_2$

$$\mathbb{A}^2 \longrightarrow X$$

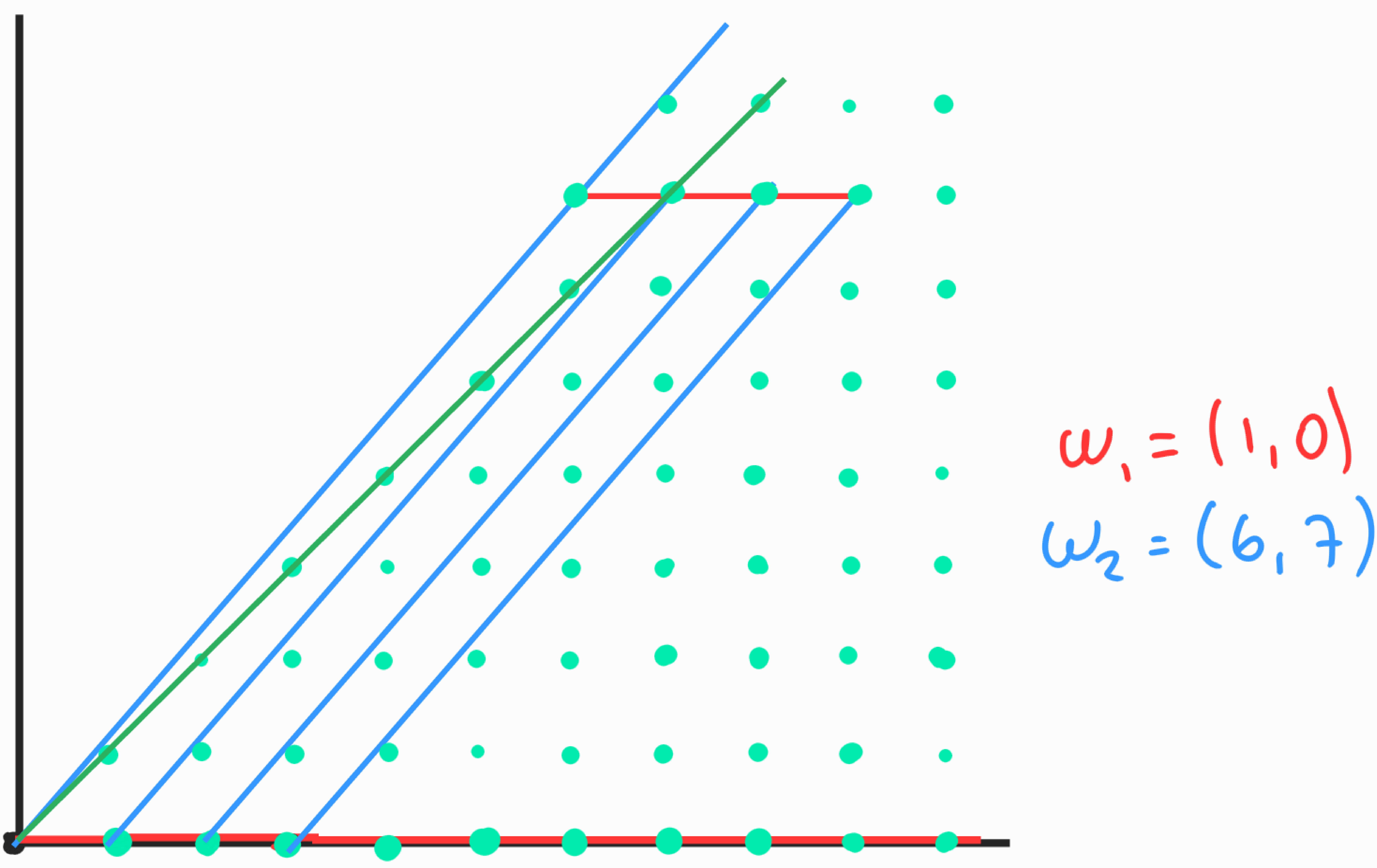
$$(x,y) \longmapsto (x, x^n y^{n+1}, xy)$$

$(0,0,0) \qquad -(u,0,0) \quad -(0,v,0) \quad u \neq 0 \quad v \neq 0$

$$uv - z^{n+1} = 0 \;/\; x^2 + y^2 + z^{n+1} = 0$$



$\omega_1 = (1,0)$
$\omega_2 = (2,3)$



$\omega_1 = (1,0)$
$\omega_2 = (6,7)$