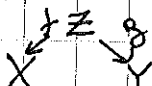


§ Motivation & basics.

→ $X = \text{manifold} \rightsquigarrow C^\bullet(X) = \text{singular chain complex or simplicial complex} \rightsquigarrow H^\bullet(X)$
 (is) information lost! (is) homology

Poincare: \exists manifolds X, Y s.t. $X \not\cong Y$ but $H^i(X) \cong H^i(Y)$.
not homot.

Thm: (Whitehead) Suppose X, Y simplicial complexes. Suppose \exists simplicial z and maps



s.t. f, g induce "quasi-isomorphisms" $f_*: C_\bullet(X) \rightarrow C_\bullet(z)$ $g_*: C_\bullet(Y) \rightarrow C_\bullet(z)$.
 Then, X homotopic to Y .

→ declare homotopic maps to be equivalent: If $f, g: X \rightarrow Y$ are homotopic $h: f \sim g \Rightarrow \exists h_{i+1}: C_{i+1}(X) \rightarrow C_{i+1}(Y)$ s.t. $f_i - g_i = h_{i+1} \circ d_{i+1} + d_i \circ h_i$.

→ Suppose now X is an alg. var. $X \rightsquigarrow$ coherent quasi-coh $\text{Coh}(X)$ $\text{QCoh}(X) \rightsquigarrow H^i(X, \mathcal{F})$.
 $\text{Mod}(X)$

Def - ① The homotopy category of X , $K(X)$, is the category whose objects are: complexes $\dots \rightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \rightarrow \dots$ where $A^i \in \left. \begin{array}{l} \text{Coh}(X) \\ \text{QCoh}(X) \\ \text{Mod}(X) \end{array} \right\} \text{ abelian}$
 $d^{i+1} \circ d^i = 0$

Morphisms: morphisms of complexes up to homotopy
 i.e. $\text{Hom}_{K(X)}(A^\bullet, B^\bullet) = \text{Hom}_{\text{Coh}(X)}(A^\bullet, B^\bullet) / \{ f \sim g \text{ if } \exists h \text{ s.t. } f - g = h \circ d + d \circ h \}$

② The derived category $D(X)$ has objects: $\text{ob}(K(X))$,
 morphisms: morphisms $\text{Mor}(K(X))$ with "q's inverted".
 (if we have iso. in cohom \Rightarrow is invertible) i.e.

$$\text{Hom}_{D(X)}(A^\bullet, B^\bullet) = \left\{ \begin{array}{l} A^\bullet \xrightarrow{g} C^t \xrightarrow{t} B^\bullet \\ \downarrow \\ A^\bullet \xrightarrow{f} B^\bullet \end{array} \middle| \begin{array}{l} g \in \text{Hom}_{K(X)}(C^t, A) \\ t \in \text{Hom}_{K(X)}(C^t, B) \\ \text{s.t. } g = g' \circ \text{im } t \end{array} \right\}$$

Ex: $P^i \rightarrow D^b(\text{Coh}(P^1))$ ob: $\oplus V_i[i]$, where V_i is a v.b. on P^1

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Rule: $i: A \rightarrow D(X)$, $A \mapsto \dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots$

- ① This is a full and faithful embedding $\text{Hom}_A(A, B) = \text{Hom}_{D(X)}(iA, iB)$
- ② g 's are not invertible in $K(X)$

$$\begin{array}{ccccccc} \rightarrow & 0 & \rightarrow & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & \rightarrow & 0 & \rightarrow & \mathbb{Z}/n\mathbb{Z} & \rightarrow 0 \end{array}$$
- ③ $D(X)$ no longer has kernels or cokernels.

$D(X)$ is triangulated:

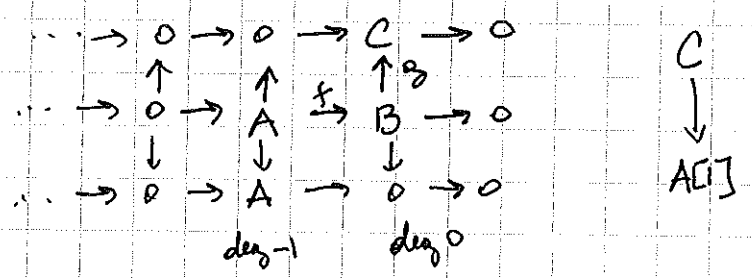
① \exists shift: $A^i \in D(X) \Rightarrow A^i[1] \in D(X)$ st. $(A^i[1])^i = A^{i+1}$
 $d^i_{A[1]} = -d^{i+1}_A$

② Exact Δ 's: First consider a map of G 's $f: A^i \rightarrow B^i$ in $K(X)$
 The cone of f , $\text{cone}(f) = A^i[1] \oplus B^i$ $d^i_{\text{cone}(f)} = \begin{pmatrix} d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$

We then have a seq.: $A \xrightarrow{f} B \xrightarrow{g} \text{cone}(f) \xrightarrow{h} A[1]$ (*)
 really homotopy cat!

This structure passes on to $D(X)$ after inverting g 's. Any seq. in $D(X)$ isom. to the image of a seq. (*) is called a distinguished exact Δ .

Ex 1: Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence of sheaves. claim: This induces an exact Δ
 $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$



Ex) Let $H^i: D(X) \rightarrow (A)$
 $A^\bullet \rightarrow H^i(A^\bullet)$

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\Rightarrow only exact $\Delta \quad A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[i] \xrightarrow{f[i]} B[i] \xrightarrow{g[i]} C[i] \rightarrow \dots$

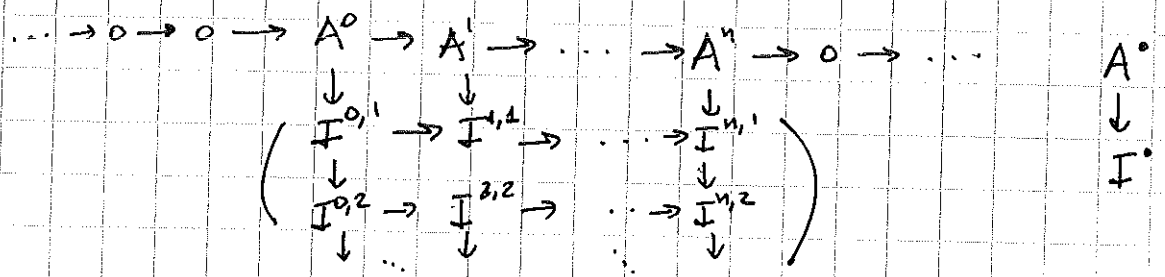
~~Def~~ Def) A functor $F: D(X) \rightarrow D(Y)$ is said to be exact if it preserves (additive preserves direct sums) sheaves and exact Δ s.

There is an important construction that says: If X, Y are varieties $f: X \rightarrow Y$ is a morphism, then the functors f_* , f^* , $\text{Hom}_X(-, -)$, $- \otimes -$ (which are exact only on one side) extend to exact functors.

$Rf_*: D(X) \rightarrow D(Y) \quad Lf^*: D(Y) \rightarrow D(X)$
 $R\text{Hom}_{D(X)}(-, -), - \otimes^L -$

Fact: $H^i(Rf_* A) = R^i f_* A$, for $A \in \text{Coh}(X)$.

Idea: Let $A \in D(X)$, say A is bounded



$R(f_* A) := \dots \rightarrow f_* I^0 \rightarrow f_* I^1 \rightarrow f_* I^2 \rightarrow \dots$