Derived categories I : Sukhendu Mehrotra 22/08/2014 (1)

§ Motivation & basics .

→ X = manifold ⇝ $C^\bullet(X)$ = singular chain complex or simplicial complex ⇝ $H^\bullet(X)$ (co)homology
¡Information lost!
Poincare : ∃ manifolds X, Y s.t. $X \underset{\text{not homot.}}{\not\simeq} Y$ but $H^i(X) \simeq H^i(Y)$.

<u>Thm</u>: (Whitehead) Suppose X, Y simplicial complexes. Suppose ∃ simplicial cx. Z and maps

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

s.t. f, g induce "quasi-isomorphisms" $f_*^\bullet : C_\bullet^\bullet(X) \to C_\bullet^\bullet(Z)$ $g_*^\bullet : C_\bullet^\bullet(Y) \to C_\bullet^\bullet(Z)$.
Then, X homotopic to Y .

→ Declare homotopic maps to be equivalent : If $f, g : X \to Y$ are homotopic
$h : f \sim g \Rightarrow \exists\, h_{i+1} : C_i(X) \to C_{i+1}(Y)$ s.t. $f_i - g_i = h_{i-1} \circ d_i + d_{i+1} \circ h_i$.

→ Suppose now X is an alg. var . X ⇝ coherent / quasi-coh $\quad \begin{matrix} Coh(X) \\ QCoh(X) \\ Mod(X) \end{matrix} \rightsquigarrow H^i(X, \mathcal{F})$.

<u>Defn</u> (1) The homotopy category of X , $K(X)$, is the category whose objects are : complexes

$$\cdots \longrightarrow A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \longrightarrow \cdots \quad \text{where } A^i \in \left.\begin{matrix} Coh(X) \\ QCoh(X) \\ Mod(X) \end{matrix}\right\} \text{ or abelian}$$
$$d^{i+1} \circ d^i = 0$$

morphisms ; morphis of complexes up to homotopy
i.e. $\mathrm{Hom}_{K(X)}(A^\bullet, B^\bullet) = \mathrm{Hom}_{Cx}(A^\bullet, B^\bullet) \big/ \{ f^\bullet \sim g^\bullet \text{ if } \exists h \text{ s.t. } f^\bullet - g^\bullet = h^\bullet \circ d^\bullet + d^\bullet \circ h^\bullet \}$

(2) The derived category $D(X)$ has objects : $ob(K(X))$,
morphisms : morphisms $Mor(K(X))$ with "q'isms inverted".
(if we have iso. in cohom ⇒ is invertible) ie

$$\mathrm{Hom}_{D(X)}(A^\bullet, B^\bullet) = \left\{ \begin{tikzcd} & C^\bullet \arrow[dl, "q"'] \arrow[dr, "t"] & \\ A^\bullet \arrow[rr, dashed] & & B^\bullet \end{tikzcd} \Big/ \begin{matrix} q \in \mathrm{Hom}_{K(X)}(C^\bullet, A^\bullet) \\ t \in \mathrm{Hom}_{K(X)}(C^\bullet, B^\bullet) \\ \text{st } q = q'\text{ism} \end{matrix} \right\}$$

Ex : $\mathbb{P}^1$ $\quad D^b(\text{Coh}(\mathbb{P}^1))$ $\quad$ ob: $\oplus V_i[i]$, where V_i is a v.b. on $\mathbb{P}^1$

$\cdots \xrightarrow{0} 0 \xrightarrow{0} V_i \xrightarrow{0} 0 \xrightarrow{0} \cdots$

Rmk: $i : \mathcal{A} \longrightarrow D(X)$, $\quad A \mapsto \cdots \to 0 \to \underset{0}{A} \to 0 \to \cdots$

① This is a full and faithful embedding $\text{Hom}_{\mathcal{A}}(A,B) = \text{Hom}_{D(X)}(iA, iB)$

② q'isms are not invertible in $K(X)$

$$\begin{tikzcd} \to 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\times n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

③ $D(X)$ no longer has kernels or cokernels.

<u>$D(X)$ is triangulated</u>:

① $\exists$ shift: $A^\bullet \in D(X) \Rightarrow A^\bullet[1] \in D(X)$ s.t. $(A^\bullet[1])^i = A^{i+1}$

$d^i_{A[1]} = -d^{i+1}_A$

② Exact Δ's: First consider a map of Cx's $f : A^\bullet \to B^\bullet$ in $K(X)$

The cone of f, $\text{cone}(f) = A^\bullet[1] \oplus B^\bullet$ $\quad d^i_{\text{cone}(f)} = \begin{pmatrix} -d^{i+1} & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$

We then have a seq.: $\quad A \xrightarrow{f} B \xrightarrow{g} \underbrace{\text{cone}(f) \xrightarrow{h} A[1]}_{\text{really homotopy cok!}} \quad (*)$

This structure passes on to $D(X)$ after inverting q'isms. Any seq in $D(X)$ isom. to the image of a seq $(*)$ is called a ~~distinguished~~ exact Δ.

Ex 1 - Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of sheaves. claim: This induces an exact Δ

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & C \arrow[r] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] \arrow[u] \arrow[d] & A \arrow[r, "f"] \arrow[u] \arrow[d] & B \arrow[r] \arrow[u, "g"] \arrow[d] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r] & 0 \\ & & \text{deg } -1 & \text{deg } 0 & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d] \\ A[1] \end{tikzcd}$$

Ex] Let $H^i : D(X) \to (A)$

$$A^\bullet \longmapsto H^i(A^\bullet)$$

$\Rightarrow$ any exact Δ $\quad A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} A[1] \xrightarrow{f[1]} B[1] \xrightarrow{g[1]} C[1] \to \dots$

Def] A functor $F : D(X) \to D(Y)$ is said to be exact if it preserves (additive preserv direct sums) shifts and exact Δs.

There is an important construction that says: If X, Y are varieties $f : X \to Y$ is a morphism, then the functors f_*, f^*, $Hom_X(-,-)$, $\_\otimes\_$ (which are exact only one side) extend to exact functors.

$$Rf_* : D(X) \to D(Y) \qquad Lf^* : D(Y) \to D(X)$$

$$RHom_{D(X)}(-,-), \quad \_ \overset{L}{\otimes} \_$$

Fact: $H^i(R f_* A) = R^i f_* A$, for $A \in Coh(X)$.

Idea: Let $A \in D(X)$, say A is bounded

$$\begin{array}{ccccccccccc}
\dots \to 0 \to 0 \to & A^0 & \to & A^1 & \to & \dots & \to & A^n & \to 0 \to \dots \\
& \downarrow & & \downarrow & & & & \downarrow & \\
& I^{0,1} & \to & I^{1,1} & \to & \dots & \to & I^{n,1} & \\
& \downarrow & & \downarrow & & & & \downarrow & \\
& I^{0,2} & \to & I^{1,2} & \to & \dots & \to & I^{n,2} & \\
& \downarrow & & \downarrow & & & & \downarrow & \\
& & \ddots & & & \ddots & & &
\end{array}
\qquad
\begin{array}{c} A^\bullet \\ \downarrow \\ I^\bullet \end{array}$$

$$R(f_* A^\bullet) := \quad \dots \to f_* I^0 \to f_* I^1 \to f_* I^2 \to \dots$$