§1 Fundamental concepts.

X = smooth proj variety over $\mathbb{C}$ $\quad \mathcal{D}(X) := \mathcal{D}^b(\mathrm{Coh}\, X)$.

Def 1.- $\mathcal{B}$ = Full subcategory of an additive category (finite products exist, homs are abelian, comp. bilinear) ($\mathcal{D}(A)$ is additive). The RIGHT ORTHOGONAL to $\mathcal{B}$ is a full subcategory $\mathcal{B}^\perp$ s.t. $\mathrm{Hom}(B, C) = 0 \quad \forall C \in \mathcal{B}^\perp, B \in \mathcal{B}$. Similarly there is a ${}^\perp\mathcal{B}$.

Obs 1.- If $\mathcal{B}$ is Δed in a Δed $\Rightarrow$ ${}^\perp\mathcal{B}$ & $\mathcal{B}^\perp$ are Δed.

Obs 1.- For any $\mathcal{E}, \mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$, we have
$$R\mathrm{Hom}_X(\mathcal{E}, \mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong R\mathrm{Hom}_X(\mathcal{E} \overset{L}{\otimes} \mathcal{F}^\vee, \mathcal{G})$$
where $\mathcal{F}^\vee = R\mathcal{H}om_X(\mathcal{F}, \mathcal{O}_X)$. (This is something we have when $\mathcal{E}$ is locally free at level of sheaves.)

Def 1.- $\mathcal{B}$ = full Δed subcat. of a category $\mathcal{A}$ (struct). $\mathcal{B}$ is said RIGHT ADMISSIBLE if $\forall X \in \mathcal{A} \; \exists \; B \to X \to C$ distinguished triangle where $B \in \mathcal{B}$ and $C \in \mathcal{B}^\perp$ (for Left admissible go for $D \to X \to B \quad D \in {}^\perp\mathcal{B}, C \in \mathcal{B}$). Admissible is both (due to Bondal & Kapranov).

Def 1.- An exceptional object in a derived category $\mathcal{A}$ is an object E such that

$$R^i\mathrm{Hom}(E, E) = 0 \quad \forall i \neq 0 \; , \; \mathrm{Hom}(E, E) = \mathbb{C}$$

Obs 1.- For the deformation theory of a pair $(X, \mathcal{F})$ $\mathcal{F}$ = locally free, one has infinitesimal deform in $H^1(X, \mathcal{H}om(\mathcal{F}, \mathcal{F}))$ and obstructions in $H^2(X, \mathcal{H}om(\mathcal{F}, \mathcal{F}))$ (see Hart def §2.7)

But $Ext^i(\mathcal{F},\mathcal{G}) \simeq Ext^i(\mathcal{O}_X, \mathcal{F}^\vee\otimes\mathcal{G}) \simeq Ext^i(\mathcal{O}_X, \mathcal{H}om(\mathcal{F},\mathcal{G})) = H^i(X, \mathcal{H}om(\mathcal{F},\mathcal{G}))$

ie, $Ext^i(\mathcal{F},\mathcal{F})$ is identified with $H^i(X, \mathcal{H}om(\mathcal{F},\mathcal{F}))$

(and this should be true always at the level of $\mathcal{D}(X)$)

check line bundles & B...

& then V/S comp.

So, $\mathcal{F}$ exceptional vector bundle $\Rightarrow$ <u>indecomposible</u>, <u>rigid</u>, <u>unobstructed</u>.

<u>Def</u>/- A <u>complete exceptional set</u> in $\mathcal{D}(A)$ is an ordered set of exceptional objects $(E_0,\dots,E_n)$ satisfying the semi-orthogonality condition

$$R^\circ Hom(E_i,E_j)=0 \quad \forall i>j$$

and generate de category $\mathcal{D}(A)$.

(<u>Def</u>/- Un conj. de subcat. admisibles $(\mathcal{B}_1,\mathcal{B}_2,\dots,\mathcal{B}_n)$ en una categoría derivada $\mathcal{D}(A)$ es <u>SEMIORTOGONAL</u> si $\mathcal{B}_j\subset\mathcal{B}_i^\perp$ cuando $j<i$ para cada $0\le i\le n$, y $\mathcal{B}_j\subset {}^\perp\mathcal{B}_i$ si $j>i$. Es <u>completo</u> si genera)

The goal is to describe in those terms $\mathcal{D}^b_{coh}(X)$, for particular X.

§2 Projective bundles.

$\longrightarrow$ $\mathcal{E}$ = projective vector bundle of rank $n+1$ on X. As in Hartshorne II 7 one glues the corresponding $\mathbb{P}^n_{\mathcal{U}}$ where $\mathcal{U}\subset X$ are affine pieces where $\mathcal{E}|_{\mathcal{U}}$ is free $\mathcal{O}_{\mathcal{U}}$-module and constructs

$$\mathbb{P}_X(\mathcal{E}) \xrightarrow{p} X$$ with line bundle $\mathcal{O}(1)$ on $\mathbb{P}_X(\mathcal{E})$

such that $p_*\mathcal{O}(l) = \begin{cases} 0 & l<0 \\ \mathcal{O}_X & l=0 \\ \mathcal{E} & l=1 \end{cases}$ $\quad S^l(\mathcal{E})$ if $l\ge 2$

and

$$p^*\mathcal{E} \longrightarrow \mathcal{O}(1)$$

Ex/- $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1)) \simeq \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(-1))$ $\leftarrow$ dual $\rightarrow$

$Bl_{pt}(\mathbb{P}^2)$ $=$ $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}\oplus\mathcal{O}(1))$

$\downarrow$ $\mathbb{P}^1$ $\qquad$ $\downarrow$ $\mathbb{P}^1$

$H^0(\mathcal{O}(1))=\mathbb{C}^3$ $\Rightarrow$ $H^0(\mathcal{O}(1))=\mathbb{C}$

a family of sections $\qquad$ one section

$\therefore$ we have canonical $0 \to \mathcal{O}(-1) \to p^*\mathcal{E}^\vee \to Q \to 0$.

<u>obs</u>.- We have functor $Lp^*: \mathcal{D}(X) \to \mathcal{D}(\mathbb{P}(\mathcal{E}))$ which is derived from p^*. But, since p is flat, it is simply p^*.

<u>Lemma</u>: $\forall \mathcal{F}, \mathcal{G} \in \mathcal{D}(X)$; $R\,Hom(p^*\mathcal{F}, p^*\mathcal{G}) = R\,Hom(\mathcal{F}, \mathcal{G})$
and so p^* is a full and faithful embedding. (fully faithful: bijection on morphisms)

<u>proof</u>.- First we have the general: $f: X \to Y$ any morphism, we have $RHom_X(Lf^*A, B) \simeq RHom_Y(A, Rf_*B)$

- Then given $\mathcal{F} \in \mathcal{D}(X)$, $\boxed{R\,Hom(Lp^*\mathcal{F}, \mathcal{H}) = R\,Hom(\mathcal{F}, Rp_*\mathcal{H})}$ General fact
and so $R\,Hom(p^*\mathcal{F}, p^*\mathcal{G}) = R\,Hom(\mathcal{F}, Rp_*p^*\mathcal{G})$

- We have a general projection formula $Rp_*p^*\mathcal{G} = Rp_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \otimes \mathcal{G}$.
Como $R^ip_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = 0 \quad \forall i > 0$ (Take open affine cover $\{\mathcal{U}_i\}$ with $p^{-1}(\mathcal{U}_i) \simeq \mathbb{P}^n_{\mathcal{U}_i}$
$\Rightarrow R^ip_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}|_{\mathcal{U}_i} \simeq h^i(\mathcal{U}_i, p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})})^\sim = 0 \quad \forall i \geq 1$) and $p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_X$
In this way

$$Rp_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})} \overset{L}{\otimes} p^*\mathcal{G}) \simeq Rp_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \overset{L}{\otimes} \mathcal{G}$$

but $Rp_*\mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{O}_X$ by above, and so $\simeq \mathcal{G}$.

(In this way, this applies on any flat $p: X \to Y$ s.t. $R^ip_*\mathcal{O}_X = \begin{cases} \mathcal{O}_Y & i = 0 \\ 0 & i > 0 \end{cases}$) ∎

<u>obs</u>.- Careful with $RHom(A^\bullet, -)$ and $A^\bullet \overset{L}{\otimes} -$ which are defined for complexes in $\mathcal{D}(X)$. When $A^\bullet = \cdots 0 \to A \to 0 \to \cdots$ it is what it should be. Also always

$$Hom^i_X(A^\bullet, B^\bullet) = Hom_{\mathcal{D}(X)}(A^\bullet, B^\bullet[i])$$

→ Let $\mathcal{D}(X)_0$ be the subcategory of $\mathcal{D}(\mathbb{P}(\mathcal{E}))$ that is the image of $\mathcal{D}(X)$ under $p^*:\mathcal{D}(X)\to\mathcal{D}(\mathbb{P}(\mathcal{E}))$.

→ Let $\mathcal{D}(X)_k$ be the subcategory of $\mathcal{D}(\mathbb{P}(\mathcal{E}))$ whose objects have the form $p^*\mathcal{F}\otimes\mathcal{O}(k)$, $k\in\mathbb{Z}$, $\mathcal{F}\in\mathcal{D}(X)$. Note that $\mathcal{D}(X)_k$ is equivalent to $\mathcal{D}(X)_0$ and are fully faithfull copies in $\mathcal{D}(\mathbb{P}(\mathcal{E}))$.

(One can use $R\mathcal{H}om_X(\mathcal{E},\mathcal{F}\overset{L}{\otimes}\mathcal{G})\cong R\mathcal{H}om_X(\mathcal{E}\overset{L}{\otimes}\mathcal{F}^\vee,\mathcal{G})$
where $\mathcal{F}^\vee = R\mathcal{H}om_X(\mathcal{F},\mathcal{O}_X)$, with $\mathcal{F}=\mathcal{O}(k)$)
(Given $\cdots\to 0\to\mathcal{L}\to 0\to\cdots \Rightarrow R\mathcal{H}om_X(\mathcal{L},\mathcal{O}_X)\simeq \cdots 0\to\mathcal{L}^{-1}\to 0\cdots$
this uses that $\mathcal{L}$ locally free + $\mathcal{I}$ injective $\Rightarrow \mathcal{L}\otimes\mathcal{I}$ is injective)

Assertions from Bondal-Kapranov (1989)

1.- For M = smooth proj variety $\Rightarrow D^b_{coh}(M)$ is right & left ~~representable~~ saturated.

2.- For $\mathcal{B}$ right satur (left satur.), $\mathcal{B}\subset\mathcal{A}$ as full Δ category.
$\Rightarrow$ $\mathcal{B}$ is right (left) admissible.

$\therefore$ $\mathcal{D}(X)_k$ are admissible in $\mathcal{D}(\mathbb{P}(\mathcal{E}))$

<u>Lemma</u>: $\forall\mathcal{F}\in\mathcal{D}(X)_k$ and $\forall\mathcal{G}\in\mathcal{D}(X)_m$, we have $R\mathcal{H}om_{\mathbb{P}(\mathcal{E})}(\mathcal{F},\mathcal{G})=0$ when $n\geq k-m>0$ (rank $\mathcal{E}=n+1$)

<u>Proof</u>.- → We prove for $k=0$ and $-n\leq m<0$ (suffices)
→ $R\mathcal{H}om_{\mathbb{P}(\mathcal{E})}(p^*\mathcal{F}',\mathcal{G}) = R\mathcal{H}om_X(\mathcal{F}',Rp_*\mathcal{G})$, $\mathcal{F}=p^*\mathcal{F}'$
→ Say $\mathcal{G}=p^*\mathcal{G}'\otimes\mathcal{O}(m) \Rightarrow Rp_*\mathcal{G}\simeq\mathcal{G}'\otimes Rp_*\mathcal{O}(m)$
→ But $Rp_*\mathcal{O}(m)=0$ when $-n\leq m<0$ $\left(H^i(\mathbb{P}^n,\mathcal{O}(m))\ i\geq 0 \text{ for } -n\leq m<0\right)$ ∎

$\therefore$ $(\mathcal{D}(X)_{-n},\mathcal{D}(X)_{-n+1},\dots,\mathcal{D}(X)_0)$ is semi-orthogonal

To prove: it is complete (ie generates)

Teo : above (special case is $X = pt$, $\mathbb{P}(\mathcal{E}) = \mathbb{P}^n$)

proof :

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E}) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}(\mathcal{E}) \arrow[dr, "p'"'] & & \mathbb{P}(\mathcal{E}) \arrow[dl, "p''"] \\ & X & \end{tikzcd}$$

where $p' = p'' = p$.

Recall :

$$0 \to \mathcal{O}_{\mathcal{E}}(-1) \to p^*(\mathcal{E}^\vee) \to Q \to 0 \quad (*)$$

→ Consider $\mathcal{O}(1) \boxtimes Q := p_1^*\mathcal{O}(1) \otimes p_2^* Q$ in $\mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E})$.

→ From (*), by pushing down we have $p_* p^* \mathcal{E}^\vee = \mathcal{E}^\vee \simeq p_* Q$.

→ Then we have

$$H^0(\mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(1) \boxtimes Q) \simeq H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(1) \otimes p_{1*} p_2^* Q) \simeq H^0(X, \mathcal{E} \otimes \mathcal{E}^\vee)$$

since $p : \mathbb{P}(\mathcal{E}) \to X$ is flat, and so

$$Rp_{1*}(p_2^* \mathcal{F}) \simeq p'^*(Rp''_* \mathcal{F})$$

which gives $p_{1*} p_2^*(Q) \simeq p^* \mathcal{E}^\vee$ + $p_*(\mathcal{O}(1) \otimes p^*(\mathcal{E}^\vee)) \simeq \mathcal{E} \otimes \mathcal{E}^\vee$.

As $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \simeq \mathcal{E}^\vee \otimes \mathcal{E}$, let $s \in H^0(X, \mathcal{E} \otimes \mathcal{E}^\vee)$ the section corresponding to $\mathbb{1}$.

not one equation of course!

→ One can show that $\{s = 0\}$ is the diagonal $\Delta \subset \mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E})$.

(± on $\mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E})$ we have $s(a,b) = (a, f_a(b))$ when evaluated, where f_a is the "dual function to a" $a \leftrightarrow x - a$ & want b with $b - a = 0$)

→ In this way, since Δ is locally complete intersection, we have through s a Koszul resolution of $\mathcal{O}_\Delta$:

(as codim Δ = rank $\mathcal{E}$)

$$0 \to \Lambda^n(\mathcal{O}(-1) \boxtimes Q^\vee) \to \Lambda^{n-1}(\mathcal{O}(-1) \boxtimes Q^\vee) \to \cdots \to \mathcal{O}(-1) \boxtimes Q^\vee \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \mathcal{O}_\Delta \to 0$$

(rank = n+1) This implies by tensoring with $p_2^* \mathcal{F}$ that

$$0 \to \Lambda^n(\mathcal{O}(-1) \boxtimes Q^\vee) \otimes p_2^* \mathcal{F} \to \cdots \to (\mathcal{O}(-1) \boxtimes Q^\vee) \otimes p_2^* \mathcal{F} \to p_2^* \mathcal{F} \to p_2^* \mathcal{F}|_\Delta \to 0$$

is also exact.

→ We will use derived categories to conclude generation, avoiding computations with spectral sequences.

$\longrightarrow$ before notice that :

(1) $\Lambda^k(\mathcal{O}(-1) \boxtimes Q^\vee) \otimes p_2^* \mathcal{F} = \mathcal{O}(-k) \boxtimes (\Lambda^k Q^\vee \otimes \mathcal{F})$

(2)

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d, "\shortparallel"] \arrow[drr, bend left] & \mathbb{P}(\mathcal{E}) \times_X \mathbb{P}(\mathcal{E}) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathbb{P}(\mathcal{E}) & & \mathbb{P}(\mathcal{E}) \end{tikzcd}$$

$$\begin{aligned} Rp_{1*}(Lp_2^* \mathcal{F} \overset{L}{\otimes} \mathcal{O}_\Delta) &= Rp_{1*}(Lp_2^* \mathcal{F} \overset{L}{\otimes} \Delta_*(\mathcal{O}_\Delta)) \\ &\overset{\text{proj}}{=} Rp_{1*} R\Delta_* (L\Delta^* Lp_2^* \mathcal{F}) \\ &= R(p_1 \circ \Delta)_* (L(p_1 \circ \Delta)^* \mathcal{F}) \\ &= \mathcal{F} \end{aligned}$$

In the sense of Derived Categ.

(3) If $0 \to \mathcal{F}_k' \to \mathcal{F}_{k-1}' \to \dots \to \mathcal{F}_1' \to \mathcal{F}' \to 0$ is exact in X and $f: X \to Y$ morphisms $\Rightarrow$ in $\mathcal{D}(Y)$, $Rf_* \mathcal{F}'$ is generated by $Rf_* \mathcal{F}_j'$, $j \in \{1, \dots, k\}$.

$\longrightarrow$ Therefore, we have that $\mathcal{F} \in \mathcal{D}(\mathbb{P}(\mathcal{E}))$ is generated by objects in $\mathcal{D}_{-n}(X), \dots, \mathcal{D}(X_0)$ since

$$\begin{aligned} Rp_{1*}(\mathcal{O}(-k) \otimes p_2^* \text{—}) &= \mathcal{O}(-k) \otimes Rp_{1*} p_2^* \text{—} \\ &= \mathcal{O}(-k) \otimes p^*(p_*(\text{—})). \end{aligned}$$

$\blacksquare$

Cor: If $\mathcal{D}(X)$ has an exceptional complete set $\Rightarrow \mathcal{D}(\mathbb{P}(\mathcal{E}))$ does.

proof: $(E_0, \dots, E_m)$ for $\mathcal{D}(X)$
$\Rightarrow (p^* E_0 \otimes \mathcal{O}(-n), \dots, p^* E_m \otimes \mathcal{O}(-n), \dots, p^* E_0, \dots, p^* E_m)$
is obviously so. (RHom ... with either info from $\underline{X}$ or <u>negative</u>)

$\blacksquare$