~~Algebra commutative (4 clases)~~ 17-11-2014 G. URZÚA 12

~~Atiyah-Macdonald "Intro to commut. algebra" capítulo 1~~

~~Meta: presentar la geometría detrás del álgebra conmutativa.~~

→ Comparison via blow-up

$Y \subset X$ both smooth projective varieties, defined by $0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$.

$\therefore \quad \mathcal{S} = \bigoplus_{d \geq 0} \mathcal{I}^d$, $\mathcal{I}^d$ = d power of $\mathcal{I}$ & $\mathcal{I}^0 = \mathcal{O}_X$

$$\Rightarrow \quad \begin{array}{ccc} \tilde{X} = \mathrm{Proj}(\mathcal{S}) & \xrightarrow{\pi} & X \\ \uparrow j & & \uparrow i \\ \tilde{Y} & \xrightarrow{p} & Y \end{array} \quad \text{the blow-up along } Y$$

such that $\tilde{Y}$ is now a divisor on $\tilde{X}$, $\pi|_{\tilde{X} \setminus \tilde{Y}}$ is isomorphism with $X \setminus Y$, $\tilde{Y}$ has ideal sheaf $\mathcal{O}(1)$, $\tilde{X}$ and $\tilde{Y}$ are smooth proj varieties ($\tilde{Y} = \mathrm{Proj}_Y(\mathcal{I}/\mathcal{I}^2) \xrightarrow{p'} Y$ is a proj bundle with $\mathcal{E} = \mathcal{I}/\mathcal{I}^2$ rank $= \dim X - \dim Y$, π is birat, proper (surjective).

→ The functor to look at: $L\pi^* : \mathcal{D}(X) \to \mathcal{D}(\tilde{X})$

Lema: $\forall A, B \in \mathcal{D}(X)$ $\exists$ isomorphism $R^i\mathrm{Hom}(L\pi^*A, L\pi^*B) = R^i\mathrm{Hom}(A,B)$. In this way, the functor $L\pi^*$ is faithfull and full embedding.

Dem/.- Standar as before, using loc. free resolution, projection formula & $R^i\pi_* \mathcal{O}_{\tilde{X}} = \begin{cases} 0 & i \neq 0 \\ \mathcal{O}_X & i = 0 \end{cases}$ ∎

→ Again denote by $\mathcal{D}(X)_0$ the image of $\mathcal{D}(X)$ in $\mathcal{D}(\tilde{X})$

We have complete semi-orthogonal $(\mathcal{D}(Y)_{-r+1}, \ldots, \mathcal{D}(Y_0))$ of $\mathcal{D}(\tilde{Y})$, where $r = \mathrm{codim}(Y,X)$ and $\mathcal{D}(Y)_0 = p^*\mathcal{D}(Y)$.

It turns out that $j_* : \mathcal{D}(\tilde{Y}) \to \mathcal{D}(\tilde{X})$ is not full but

(Del Pezzo (due to Karleshov, Orlov) any full except. collection is obtained via mutations from the obvious in (d)) ← be specific

Teo: (a) the functor $j_{*k} : \mathcal{D}(Y)_k \to \mathcal{D}(\tilde{X})$ (restriction j_*) is full and faithful embedding.

$Bl_n(\mathbb{P}^2)$, $n \le 8$, gen. position (or $\mathbb{P}^1\times\mathbb{P}^1$ but this we know!)

(b) If $\tilde{\mathcal{D}}(Y)_k \subset \mathcal{D}(\tilde{X})$ is the image of $\mathcal{D}(Y)_k$ under j_{*k} $\Rightarrow$ $(\tilde{\mathcal{D}}(Y)_{-r+1}, \dots, \mathcal{D}(Y)_{-1}, \mathcal{D}(X_0))$ is semiortogonal.

(c) the set in (b) is complete, ie, generates $\mathcal{D}(\tilde{X})$

(d) If $\mathcal{D}(X)$ & $\mathcal{D}(Y)$ have complete exceptional sets $\Rightarrow$ $\mathcal{D}(\tilde{X})$ has.

Proof: see the paper (D.O. Orlov 1992 "Proj bundles, monoidal trans, and Der cat coh sheaves")

→ How to get exceptional sets on varieties? what bounds the length? When complete? Questions will be taken on surfaces.

obs/ For curves X of genus $g \geq 1$, any semiorthogonal complete set is trivial (S. OKAWA "Semi-orthog. decomp. of d.c. of a curve")
(he proved that: if $D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle \Rightarrow$ every $E \in Coh(X)$,
$0 \to b \to E \to a \to 0$, $b\in\mathcal{B}$, $a\in\mathcal{A}$ but also any coherent sheaf <u>in $\mathcal{B}$ must be torsion</u> $\Rightarrow$ $\mathcal{A}$ contains all torsion free and they generate $D^b(X)$)

But in surfaces we have a lot going on! (Check arXiv for Der. cat. + surface)

obs/ Grothendieck group $= K_0(X) =$ free abelian generated by all coherent $\big/ \mathcal{F}-\mathcal{F}'-\mathcal{F}''$ where $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$

For X nonsingular, one can compute $K_0(X)$ with locally free sheaves. (via finite resolutions of coherent sheaves). This allows to extend Chern classes to $K_0(X)$ via $0\to\mathcal{E}_1\to\mathcal{E}_2\to\mathcal{E}_3\to 0$ $\quad c_t(\mathcal{E}_2) = c_t(\mathcal{E}_1)\cdot c_t(\mathcal{E}_3)$

$\Rightarrow$ There is a map $c_t : K_0(X) \to A(X)[t]$, $A(X) =$ Chow-ring and a ring isomorphism $ch : K_0(X)\otimes\mathbb{Q} \to A(X)\otimes\mathbb{Q}$ ($K_0(X)$ has mult via vector bundles using $\otimes$) $ch(\mathcal{E}) = \sum_{i=1}^{r} e^{a_i}$ ($r=$ rank) (see Hart. p. 432) and $c_t(\mathcal{E}) = \prod_{i=1}^{r}(1+a_i t)$, (where $rank(\mathcal{E}) = \dim_{K(X)} \mathcal{E}_\xi$, ξ = generic point)

... This ends up together with $f^!$, $f_!$ for $f: X\to Y$ proper in Grothendieck-Riemann-Roch ...

(9)

In our case, X = smooth proj surface, $A(X) = \underset{\mathbb{Z}}{\overset{\shortparallel}{A^0(X)}} \oplus \underset{\mathrm{Pic}(X)}{A^1(X)} \oplus \overset{\shortparallel}{A^2(X)}$

zero cycles up to rat. equiv. (does not need to be finitely generated)

→ Connection with $\mathcal{D}(X)$: there is a way to talk about $K_0(\mathcal{D}(X))$ and $K_0(X) \simeq K_0(\mathcal{D}(X))$ (Using exact Δ instead of short exact sequences)
It can be proved that given a semiorthogonal decomposition $\mathcal{D}(X) = \langle \mathcal{A}, \mathcal{B} \rangle \Rightarrow K_0(X) = K_0(\mathcal{A}) \oplus K_0(\mathcal{B})$. Moreover, if $\mathcal{A} = (E_1, \dots, E_n)$ ~~[illegible]~~ exceptional set $\Rightarrow K_0(\mathcal{A}) \simeq \mathbb{Z}^n$

Nothing to do with ranks ...

→ Putting this together with the isomorphism ch, we have that the length of $(E_1, \dots, E_n)$ complete exceptional set in $\mathcal{D}(X)$ (or exceptional collection) is $\dim_{\mathbb{Q}} A(X) \otimes \mathbb{Q}$.
If X is a surface with $q=0$, $n \leq \rho + 1 + \mathrm{rank}(A^2(X))$, ~~[illegible]~~

obs. - Given an except. collection $(E_1, \dots, E_n)$ of vector bundles on X $\Rightarrow (E_n^\vee, E_{n-1}^\vee, \dots, E_1^\vee)$ & $(E_2, E_3, \dots, E_n, E_1(-K_X))$ is.

Goal: Construction of exceptional vector bundles & collections. (and How to use them for something!)

→ We will consider mainly $p_g = 0$ surfaces (taste of the crow & Mumford's result & Bloch conjecture: $p_g = 0 = q \Rightarrow A^2(X) \simeq \mathbb{Z}$, and so we have a conjectural max for length: $e(X)$ = top. Euler char. $= 2 + b_2(X)$. (The conj. is for General type) (Mumford proved that $A^2(X) \neq \mathbb{Z}$ if $p_g \neq 0$) (in fact ∞ dimension)

Lemma: Let X be smooth proj. surface with $q = p_g = 0$ and $\Gamma_1 \; \Gamma_2 \cdots \Gamma_{r-1}$ with $\Gamma_i \simeq \mathbb{P}^1$. Then $(\mathcal{O}_X, \mathcal{O}_X(\Gamma_1), \mathcal{O}_X(\Gamma_1 + \Gamma_2), \dots, \mathcal{O}_X(\Gamma_1 + \dots + \Gamma_{r-1}))$ is exceptional collection.

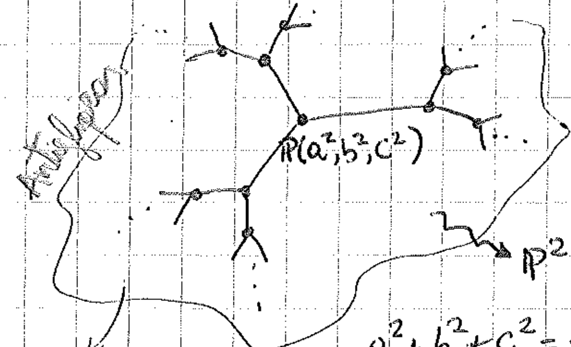
Proof. • Notice that for line bundles: $\mathrm{Ext}^i(\mathcal{L}, \mathcal{L}) = H^i(X, \mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{O}_X)$ so $p_a = p_g = 0 \Rightarrow$ every line bundle is exceptional.

• Notice also that $\underset{\overset{\shortparallel}{0}}{h^0(\mathcal{O}_\Gamma) = h^1(\mathcal{O}_\Gamma)}$ for any Γ = chain rational

(That $B \cup A = \mathbb{P}^1$ has $p_a(B+A)=0$ if $p_a(B)=0$ can be check via adjunction on the surface)

→ How to construct exceptional vector bundles from "moduli degenerations" ($\mathbb{Q}$-Gorenstein smoothing) over a one parameter base?

This is the work of Paul Hacking. It is a very strong constr. when X is rational: Let us see $\mathbb{P}^2$. ("Exep. bundles ass. to deg. of surfaces")



That is a theorem of Hacking-Prokhorov (see older work of Manetti as well)

Ex: $2^2+5^2+29^2 = 3\cdot 2\cdot 5\cdot 29$

$\mathbb{P}(a,b,c)$
$(x,y,z) \mapsto (x^a, y^b, z^c)$
$\mathbb{P}^2/[\ldots]$

$a^2+b^2+c^2 = 3abc$ (Markov equation)

surfaces are determined by singularities.

mutations $(a,b,c) \mapsto (a,b,c' = 3ab-c)$ starting with $(1,1,1)$.

This is Noether's formula for $\mathbb{P}^2 \to \mathbb{P}(a,b,c)$

(Funny connection with Rademacher's reciprocity law
$s(a;b,c) + s(b;c,a) + s(c;a,b) = \frac{1}{12abc}(a^2+b^2+c^2-3abc)$)

In general, given F exceptional $\Rightarrow F^\vee$ & $F\otimes\mathcal{L}$, $\mathcal{L}$ line bundle, are exceptional. Rudakov proved (based on Drezet-Le Potier) that up to this operations, $\{$Exceptional in $\mathbb{P}^2\} \overset{1-1}{\longleftrightarrow} \{$solut. of Markov equat$\}$

The Theorem of Hacking gives construction of exceptional vector bundle for $\mathbb{Q}_{\frac{1}{n^2}(1,na-1)} \rightsquigarrow \mathbb{P}^2$ (in a more general setting, see below).

Teo: $S = \{$isom. classes of Man. normal ($\mathbb{Q}$-Gorenst.) $\mathbb{Q} \rightsquigarrow \mathbb{P}^2\}$
$T = \{$isom. classes of E exc vect bundles on $\mathbb{P}^2\}$ / dualization & $\otimes$ line bundle
$\Rightarrow$ the Hacking map $S \to T$ is bijective.

Q: How far can we go with this?

Moreover (Hacking) Hacking gives bijection (modulo certain relations) between exceptional collections and collections of singularities above. (11)

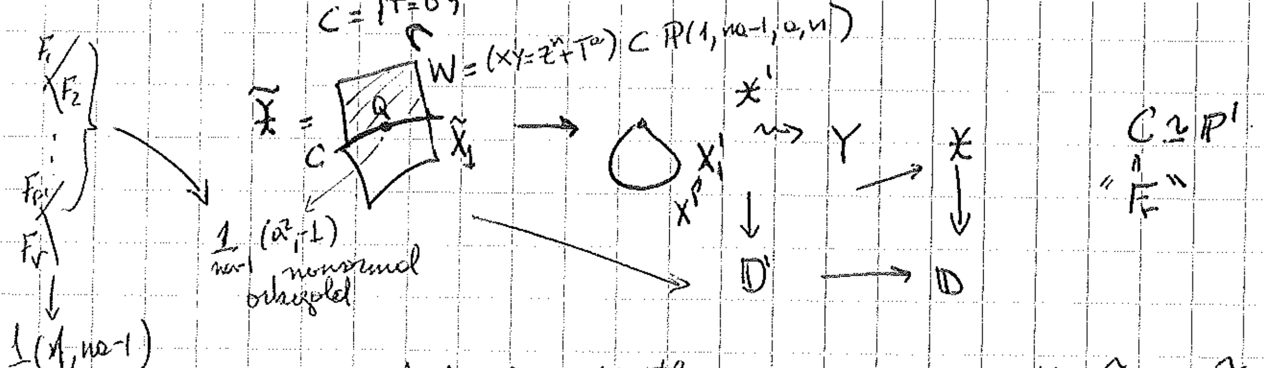
Thm (Paul): ~~Let~~ $X = \bigcirc^{\frac{1}{n^2}(na-1)} \overset{\mathfrak{X}}{\rightsquigarrow} \bigcirc = Y$ $\mathbb{Q}$-Gorenstein smoothing.
Assume $H^2(\mathcal{O}_Y) = 0$ and $H_1(Y,\mathbb{Z})$ finite. Then, after a base change $\mathbb{D}' \to \mathbb{D}$ of degree a, there is $\mathcal{E}$ reflexive sheaf ($\mathcal{E} \xrightarrow{\sim} \mathcal{E}^{\vee\vee}$ or $\mathcal{E}$ torsion free + $\mathcal{U} \overset{i}{\subset} X$ open with complement of codim 2 at least & $i_*(\mathcal{E}|_{\mathcal{U}}) = \mathcal{E}$) on $\mathfrak{X}' = \mathfrak{X} \times_{\mathbb{D}} \mathbb{D}'$ such that:

(a) $F := \mathcal{E}|_Y$ is exceptional vector bundle of rank n on Y.
(b) $E := \mathcal{E}|_X$ torsion free sheaf on X s.t. $E^{\vee\vee} \simeq A^{\oplus n}$ with A = reflexive rank 1 and $E^{\vee\vee}/E$ torsion sup. at $P \in X$.
$= \mathcal{O}(D)$ D Weil div on X'

If $\mathcal{H}$ = line bundle on $\mathfrak{X}/_{\mathbb{D}}$ ample on fibers $\Rightarrow$ F is slope stable wrt $H := \mathcal{H}|_Y$. ($\mu_H(F) := c_1(F)\cdot H/\mathrm{rank}(F)$, F stable wrt to H if $\forall\, W \subset F$ coherent with $0 < \mathrm{rank}(W) < \mathrm{rank}(F)$ we have $\mu_H(W) < \mu_H(F)$)

$c_1(F) = n\,c_1(A) \in H_2(Y,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$, $\quad c_2(F) = \frac{n-1}{2n}\left(c_1(F)^2 + n + 1\right)$

$c_1(F)\cdot K_Y \equiv \pm a \pmod n$ and $H_2(X,\mathbb{Z}) = H_2(Y,\mathbb{Z}) + \mathbb{Z}\left(c_1(F)/n\right)$.

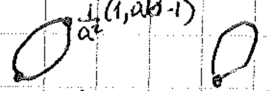


Generalization of this construct. to other singularities

Show it

- Using homol. & cohom hypothesis $\Rightarrow \exists$ line bundle $\hat{A}$ in $\tilde{X}_1$ st $\hat{A}|_C$ has degree 1.
- Then if G excep. bundle of rank n in W s.t. $G|_C \simeq \mathcal{O}_C(1)^n$ $\Rightarrow$ Glue it with $\hat{A}^n$ $\Rightarrow$ get $\tilde{\mathcal{E}}$ excep. bundle on $\tilde{X}$ ($\Rightarrow$ push it to X_1')
- Paul gives constr. of G in W via induction on a:

He proves $\mathbb{P}(1, na-1, a^2) \rightsquigarrow W$



with a very explicit $\mathbb{Q}$-Gorenstein degeneration (smoothing one / trivial on the other)
then there is induction on a but one (he) makes everything to work for this.
(see §5 of his paper)

→ There are many examples of this in general type, due to the technique of Lee & Park (2007). See a list in "KSBA surfaces with elliptic …".
<u>Q</u>: What do we do with this exceptional bundles?

→ There is also a strategy to construct Exceptional collections due to Paul. <u>Explain</u> & explain in our <u>corner situations</u>. Again very effective on del Pezzo surfaces.

→ Anne : X = min. Godeaux surface (ie $K^2 = 1$, gen. type, $p_g = 0$)
$\Rightarrow$ M. Reid $H_1(X, \mathbb{Z}) = 0, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/5$ (last originally from Godeaux).
She wants to know about $\{\text{Wahl deg.}\} \xrightarrow{\text{Paul}} \{\text{except. vect. bundles}\}/\sim$
($\sim$: some rank are equiv dual, $\otimes L$ (it could be done more precisely).)
for rank 2 case, ie, $\frac{1}{4}(1,1)$.

- $\frac{1}{4}(1,1)$ resolves to a proper elliptic surface (one changes this w/ multiple fibers).
- For $\mathbb{Z}/5\mathbb{Z}$ this does not happens (ie deg to $\frac{1}{4}(1,1)$)
(For $X \rightsquigarrow Y$ with $\frac{1}{n^2}(1, na-1)$ $\Rightarrow$ $0 \to H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \to H_1(M, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \to 0$.
(Part of statement of Hacking.) $\quad 0 \cdot \mathbb{Z}/n$)
- For the other cases, there is a degen $\rightsquigarrow$ vect. bundle ($\pm$ something) and $\mathbb{Z}/5\mathbb{Z}$ there is rank 2 bundle for the set wanted, so no surjective (same for some in $\mathbb{Z}/4\mathbb{Z}$).