Phantom Categories. **13** Subhendu M. 21/11/14 (1)

- X = smooth proj var
  $D(X) = D^b(coh(X))$
  $\mathcal{A} \subseteq D(X)$ Δ-subcat is <u>admissible</u> if it is full and $\mathcal{A} \hookrightarrow D(X)$ has left and right adjoints.
  If X is as above, and $D(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorth. decomp
  ($\mathcal{B} = {}^{\perp}\mathcal{A}$ & generates $\Leftrightarrow$ $\mathcal{B} = {}^{\perp}\mathcal{A}$ & $\underset{\in \mathcal{B}}{b} \to x \to \underset{\in \mathcal{A}}{a} \to b[1]$
  $\Rightarrow$ $\mathcal{A}$ is left admissible.)

  By a result of Bondal-Kapranov $\Rightarrow$ $\mathcal{A}, \mathcal{B}$ are admissible.

- For an admissible subcategory $\mathcal{A} \subseteq D(X)$ we have that

$$\left.\begin{array}{l} K_0(\mathcal{A}) \subseteq K_0(X) \\ HH_*(\mathcal{A}) \subseteq HH_*(X) \end{array}\right\} \text{are direct summands}$$

<u>Belief</u>: If $\mathcal{A}$ is admissible $\Rightarrow K_0(\mathcal{A}) = HH_*(\mathcal{A}) = (0)$
$\Rightarrow \mathcal{A}$ is trivial.

<u>Surprise</u>: "Phantoms" exists

- Böhning – Graf von Bothmer – Katzarkov – Sosna.
  $\exists$ s.o. decomp. of Barlow surface $\langle \mathcal{A}, L_1, \dots, L_n \rangle$
  with $K_0(\mathcal{A}) = HH_*(\mathcal{A}) = 0$
- Gorchinskiy–Orlov: $\exists$ phantom in $D(X \times Y)$, X = Burniat Y = Godeaux

Phantom: $K_0(\mathcal{A}) = 0$, $HH_*(\mathcal{A}) = 0$ quasi-Phantom: $K_0(\mathcal{A})$ = finite, $HH_*(\mathcal{A}) = 0$

- Let $\mathcal{A} \subseteq D(X)$ be right (left) admissible.

I. $K(\mathcal{A}) = \{\text{iso classes of obj. in } \mathcal{A}\} / \text{exact } \Delta\text{'s}$

$\mathcal{A} \xrightarrow{i} X \xrightarrow{R} \mathcal{A}$ Ex:

$$\begin{tikzcd} K(\mathcal{A}) \arrow[r] \arrow[rr, bend right, "1"'] & K(X) \arrow[r] & K(\mathcal{A}) \end{tikzcd}$$

$\Rightarrow K(A)$ is a direct summand of $K(X)$ ②

II.- $HH_*(X)$, $HH_*(A)$

X = smooth proj $\Rightarrow$ $HH^*(X) = \mathrm{Ext}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$ $\left(\begin{array}{c}\simeq \text{natural} \\ (1_{D(X)}, 1_{D(X)})\end{array}\right)$

Hochschild coh

where $X \xrightarrow{\Delta} X\times X$ is diag. embedding

$$HH_*(X) = H^*(X\times X, \Delta_*\mathcal{O}_X \overset{L}{\otimes} \Delta_*\mathcal{O}_X)$$
$$= \mathrm{Ext}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X[\dim X]), \quad \omega_X = \text{can bun}.$$

Comes from A = (comm.) alg Then $HH^*(X) := \mathrm{Ext}^\bullet_{A\otimes A}(A, A)$

and $HH_*(X) = \mathrm{Tor}_{*\,A\otimes A}(A, A)$

- T = Δ-cat (~~with~~ which splits idempotents ie $A \xrightarrow{i} A \in \mathrm{Mor}(T)$ s.t. $i^2 = i \Rightarrow \exists\, B, C \in T$ st. $A = B\oplus C$ and i is proj. to B)

$D(X)$

$E \in T$ is a <u>split generator</u> if the smallest Δ'ed subcat. of T cont. E and closed under taking direct summand is T itself.
Suppose T has a <u>split generator</u> E. Set $A = R\mathrm{Hom}_X(E, E)$, a dg algebra.

There is a Δ-equivalence $\mathcal{D}(A\text{-mod}) \xrightarrow{\sim} T$
$$M \longmapsto M \otimes_A E$$

Ex 1.- $X = \mathbb{P}^1$, $T = \mathcal{D}(X) = \langle \mathcal{O}(-1), \mathcal{O} \rangle$, $E = \mathcal{O}(-1)\oplus\mathcal{O}$
$R\mathrm{Hom}(E, E) \cong \mathbb{C}^4$ as a vector space
Ex. "Path alg of the quiver $\underset{\mathcal{O}(-1)}{e_1} \overset{a}{\underset{b}{\rightrightarrows}} \underset{\mathcal{O}}{e_2}$"

$$D(X) \cong \mathcal{D}(A\text{-mod})$$

<u>Thm</u>: X = smooth proj $\Rightarrow$ X is split generated.

~~Pf~~ Pr L = very ample line bundle. Then, $\mathcal{O}_\Delta$ has a resol:

$$\cdots \to V_2 \otimes (L^{-n_2} \boxtimes L^{-n_2}) \to V_1 \otimes (L^{-n_1} \boxtimes L^{-n_1}) \to \mathcal{O} \to \mathcal{O}_\Delta \to 0$$

<u>know</u>: If $k > 2\dim X$, then $\mathrm{Ext}^k(M,N) = 0$ $\forall M, N \in \mathrm{Coh}(X \times X)$.

$\Downarrow$

$$0 \to P \to (\cdots) \to \mathcal{O}_\Delta \to 0$$

$\Rightarrow$ some truncation, $C = (V_k \otimes (L^{-n_k} \boxtimes L^{-n_k}) \to \cdots \to \mathcal{O})$ with $k > 2\dim X$, we get $C \simeq \mathcal{O}_\Delta \oplus P \in \mathcal{D}(X \times X)$.

Regarding as kernels of integral transforms, since $\mathcal{O}_\Delta$ is the kernel of $1\!\!1_X$, we see that $L^{-n_k}, \ldots, L^{-n_1}, \mathcal{O}$ split generate ~~the~~ $\mathcal{D}(X)$. ■

<u>Thm</u>: X = smooth proj, E = split gen, $A = R\mathrm{Hom}(E,E)$ $\Rightarrow$ $HH_*(X) = HH_*(A)$, etc!

Finally if $\mathcal{D}(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ then define $HH_*(\mathcal{A}) = HH_*(A')$ where $A' = R\mathrm{Hom}(E_{\mathcal{A}}, E_{\mathcal{A}})$ where $E_{\mathcal{A}}$ = proj of E to $\mathcal{A}$.

<u>Finally</u>: we have the HKR isom $\Delta^* \Delta_* \mathcal{O}_X \xrightarrow{\sim} \bigoplus_{p=0}^{n} \Omega^p_X[p]$ $/\mathbb{C}$

$$\Rightarrow \quad HH^N(X) \cong \bigoplus_{p+q=N} H^p(X, \Lambda^q T_X) \qquad HH_N(X) \cong \bigoplus_{q-p=N} H^p(X, \Omega^q_X)$$

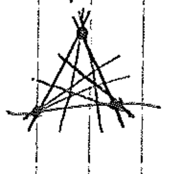
Burniat Surfaces

Ex of $p_g = q = 0$ general type surfaces with $2 \le K_X^2 \le 6$

Explicit description of a component of mod. space with $K_X^2 = 6$: X is a $(\mathbb{Z}/2\mathbb{Z})^2$-cover of $Y = Bl_3 \mathbb{P}^2$

$$X \xrightarrow{\pi} Y \longrightarrow \mathbb{P}^2$$

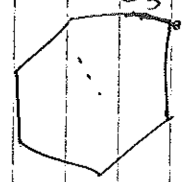
$Y = Bl_3 \mathbb{P}^2$



$$\overline{L}_1^{\,2} = \mathcal{O}_Y(\overline{B} + \overline{C})$$
$$\overline{L}_2^{\,2} = \mathcal{O}_Y(\overline{A} + \overline{C})$$
$$\overline{L}_3^{\,2} = \mathcal{O}_Y(\overline{A} + \overline{B})$$

$\overline{A}, \overline{B}, \overline{C}$

$\overline{C}_3$



Set $\mathcal{A} = \mathcal{O}_Y \oplus \bigoplus_{i=1}^{3} \overline{L}_i^{-1}$

<u>Fact</u> : This describes a full 4-dim'l comp. of mod. space.

Let f_1 div. class of $\overline{A}_1 \sim \overline{A}_2 \sim \overline{A}_0 + \overline{C}_3 \sim \overline{A}_3 + \overline{C}_0$ pencil

f_2 — $\overline{B}_1$ — , f_3 — $\overline{C}_1$ —

The following is an exc. collection on $Y = Bl_3(\mathbb{P}^2)$

$$\Sigma = \langle \mathcal{O}_Y, \underbrace{\mathcal{O}_Y(f_1), \mathcal{O}_Y(f_2), \mathcal{O}_Y(f_3)}_{\text{p.w. orth}}, \underbrace{\mathcal{O}_Y(h_1), \mathcal{O}_Y(h_2)}_{\text{p.w. orth.}} \rangle$$

In particular $K_0(Y) = \mathbb{Z}^6$

•) $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y) = 0 \Rightarrow \chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 1$

$\Rightarrow X, Y$ have same Betti numbers, Pic nos.

$\Rightarrow \underset{\mathbb{Z}^4}{\mathrm{Pic}(Y)} = \mathrm{Pic}(X)/_{\mathrm{Tor}}$

•) Peters $\mathrm{Tor} = (\mathbb{Z}/2\mathbb{Z})^6$

It would be exceptional

$H^0(L_i^{-1} \otimes L_j) = H^2(L_i^{-1} \otimes L_j) = 0$

choose carefully so that

$D = \text{eff} + \text{torsion}$

$\frac{1}{2}\pi^*$

<u>Lemma</u> : $\exists$ isom of integral lattices $\mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(X)/_{\mathrm{tor}}$

<u>Lemma</u> : If $\overline{L}_1, \overline{L}_2$ are line bdles on Y L_1, L_2 are lifts.

$\mathrm{Pic}(X) \to \mathrm{Pic}(X)/_{\mathrm{tor}} = \mathrm{Pic}(Y)$ ... $\chi(L_1 \otimes L_2^{-1}) = \chi(Y, \overline{L}_1 \otimes \overline{L}_2^{-1})$

$\Rightarrow \langle L_1, L_2, \dots, L_6 \rangle$ is a numerical exc. coll. , i.e.

$$\chi(L_i^{-1} \otimes L_j) = 0 \text{ if } i > j.$$

betti # are the same (add vertically at Hodge) $\Longleftarrow$ $H_*(Y) = \mathbb{Z}^6 = H_*(X) = H_*(\langle L_1, \dots, L_6 \rangle) \oplus H_*(\mathcal{A})$, $\langle L_1, \dots, L_6 \rangle$: $\mathbb{Z}^6$