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- X = Burniat surface
$p_g = q = 0$ $K_X^2 = 6$ general type $X \xrightarrow{(\mathbb{Z}/2)^2} Y = Bl_3 \mathbb{P}^2$

Showed : $D(X) = \langle \underbrace{L_1, \dots, L_6}_{\text{s.o. exceptional}}, A \rangle$

$HH_*(X) = \mathbb{C}^6 \Rightarrow HH_*(A) = 0$
$K_0(X)_{\mathbb{Q}} = \mathbb{Q}^6 \Rightarrow K_0(A)_{\mathbb{Q}} = 0$ Peters : $K_0(A) = (\mathbb{Z}/2\mathbb{Z})^6$.

- $\underline{Q}$: Does there exist a true phantom?

<u>Notation</u>: $A \subseteq D(X)$ • $A' \subseteq D(Y)$
full Δed subcategories

$A \boxtimes A' \subseteq D(X \times Y)$ the smallest full Δed subcateg. generated by $p_1^*(A) \otimes p_2^*(B)$ ~~and~~, $A \in A$, $B \in A'$, closed under taking direct sums.

<u>Thm</u> [G-O] S, S' = smooth proj. surfaces of gen. type
$q = p_g = 0$
For which $CH^2(S) = CH^2(S') = \mathbb{Z}$ (Inose + Mizukami)

Assume $D(S)$, $D(S')$ admit exceptional collections of max. length (which is the top. Euler Char). If $(\text{Pic}\, S)_{\text{tor}}$, $(\text{Pic}\, S')_{\text{tor}}$ ~~have coprime torsion~~ have coprime order, then
$\Rightarrow$ $A \boxtimes A'$ is a phantom category (where A = right orth of the exc. collection in S, A', etc).

<u>Prop 1</u>- X, Y smooth proj. var. Suppose $D(X) = \langle N_1, \dots, N_n \rangle$
$D(Y) = \langle M_1, \dots, M_m \rangle$ s.o. decomp.
$\Rightarrow$ $D(X \times Y)$ has sem. orth. $\langle N_i \boxtimes M_j \rangle_{i,j}$

Pf: By proj formula, $N_i \boxtimes M_j \subseteq (N_k \boxtimes M_\ell)^\perp$ if $i<k$ or $j<\ell$.

For generation, $D(X\times Y)$ is split generated by objects of the form $p_1^*P \otimes p_2^*Q$. (taking "ample $\boxtimes$ ample"; see last talk). $\blacksquare$

Facts about: $K_0(X)$, $CH^\bullet(X)$, X = var. sm. proj.

$K_0(X)$ comes a decreasing filtration.

$$F^0K_0(X) \supset F^1K_0(X) \supset F^2K_0(X) \supset \cdots$$

where $F^pK_0(X) = \{ \sum_{\text{finite}} [\mathcal{F}] : \mathcal{F} \in coh(X), \operatorname{codim}(\operatorname{supp} \mathcal{F}) \geq p \}$

There is a nat'l hom $\varphi : CH^\bullet(X) \to gr_F^\bullet K_0(X)$
$$Z \mapsto [\mathcal{O}_Z]$$

Facts: ① φ is surjective; $\varphi_{\mathbb{Q}}$ is an isomorphism

② $\_\otimes\_$ on $K_0(X)$ preserves the filtration
$$F^pK \otimes F^qK \subseteq F^{q+p}K$$

$\Rightarrow$ $gr_F K_0(X)$ comes a ring structure

③ $gr_F K_0(X)$ comes a pairing constructed as follows

Euler pairing: $K_0(X) \otimes K_0(X) \to \mathbb{Z}$
$$[E] \otimes [F] \mapsto \chi(X, E\otimes F)$$

Note $\langle F^pK(X), F^{\dim X+1-p}K(X) \rangle = 0$

$\Rightarrow$ Euler pairing descends to give a pairing
$$gr\langle\, ,\, \rangle : gr_F^p K \otimes gr_F^{\dim X-p} K \to \mathbb{Z}$$

The map φ takes the intersection pairing to $gr\langle\, ,\, \rangle$.

proof of thm:

Thm [G-O] S, S' smooth proj surfaces as above, then the external product

$p_g = 0$ with cyclic torsion.

$$CH^0(S) \otimes CH^0(S') \longrightarrow CH^0(S \times S')$$
$$x \otimes y \longmapsto (p_1^* x) \cdot (p_2^* y)$$

is surjective.
(uses Motives ...)

Prop: S, S' as above. Then $\exists$ surjection
$$m : K_0(S) \otimes K_0(S') \to K_0(S \times S')$$
$$[E] \otimes [F] \mapsto [E \boxtimes F]$$
$\longrightarrow E^\vee \otimes F = R\mathcal{H}om(E, F)$

Proof: Use Fact 1, 3 and prev. thm ■

proof thm: $D(S) = \langle \mathcal{D}, A \rangle$, $D(S') = \langle \mathcal{D}', A' \rangle$
where $\mathcal{D}, \mathcal{D}'$ are gen. by max'l length exc.
Then $A \boxtimes A'$ is a s.o. component of $\mathcal{D}(S \times S')$
Also $m : K(S) \otimes K(S') \to K(S \times S')$
preserves s.o. components.

as m is surjective by Prop: $K(A) \otimes_{\mathbb{Z}} K(A') \twoheadrightarrow K(A \boxtimes A')$
But $(|K(A)|, |K(A')|) = 1 \Rightarrow LHS = 0$ ■

An argument to detect phantoms

Recall: $X \xrightarrow{\pi} Y = Bl_3(\mathbb{P}^2)$, and using special line bundles on Y (Burniat) $\langle \mathcal{O}_Y, \mathcal{O}_Y(A_1), \mathcal{O}_Y(B_1), \mathcal{O}_Y(C_1), \mathcal{O}_Y(h_1), \mathcal{O}_Y(h_2) \rangle$

Also, $\exists$ hom $\mathrm{Pic}(Y) \xrightarrow{\frac{1}{2}\pi^*} \mathrm{Pic}(X)/_{\mathrm{Tor}} \leftarrow \mathrm{Pic}(X)$

$\therefore$ appropriate lifts give an exc. coll. $\underbrace{\langle L_1, \dots L_6 \rangle}_{\mathcal{D}} \subseteq \mathcal{D}(X)$

$\mathcal{D}(X) = \langle \mathcal{D}, \mathcal{A} \rangle$, $\mathcal{A} = {}^{\perp}\mathcal{D}$

Prop. The only nonzero Ext groups among the L_i are

(1) $Ext^2(L_i, L_j) \cong k$ $\quad i = 1,2$, $j = 3,4,5$

(2) $Ext^2(L_j, L_6) \cong k^2$ $\quad j = 3,4,5$

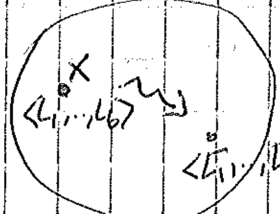
(3) $Ext^2(L_i, L_6) \cong k^3$ $\quad i = 1,2$

<u>Recall</u>:

<u>Thm</u>) X = smooth proj var, $T \subseteq \mathcal{D}(X)$ full admiss. Δ'ed subcat.
Suppose U is a split gen of T.
Set $A = RHom_X(U, U)$ a dg alg
There's a nat'l equiv of Δ'ed cat.

$$\mathcal{D}(A\text{-mod}) \to T$$
$$M \mapsto U \otimes_A M$$

In the case of Burniat (Barlow surface [KBGS])
They take $U = \oplus L_i$, and show that A deforms trivially as a dg algebra as X deforms.



Rmk: Cohomology & base change

$\Rightarrow Ext^*(U, U) = H^*(A)$ remains unchanged

remains unchanged.

[B-O] : $\mathcal{D}(X)$ determines X

$\Rightarrow \mathcal{D}(X) \neq \mathcal{D}(A\text{-modules})$ ie ${}^{\perp}\langle U \rangle \neq 0$

(of course, for X = Burniat, $\mathcal{A} = {}^{\perp}\langle U \rangle$ has nonzero K_0)

__Example__: $B = \mathbb{C}[\varepsilon]/\varepsilon^{n+1}$, $n \geq 2$

$A = R\mathrm{Hom}_B(\mathbb{C},\mathbb{C})$ $\quad H^*A = \mathrm{Ext}^*_B(\mathbb{C},\mathbb{C}) = \bigoplus \mathrm{Ext}^i(\mathbb{C},\mathbb{C})$

__Claim__: $H^*A = \mathbb{C}[u,v]/(v^2)$ $\quad \deg u = 2 \quad \deg v = 1$

$$\cdots \to B \xrightarrow{\varepsilon^2} B \xrightarrow{\varepsilon} B \to \mathbb{C}$$

Apply $\mathrm{Hom}_B(-,\mathbb{C})$ $\quad \mathrm{Hom}(B,\mathbb{C}) \xrightarrow{\varepsilon} \mathrm{Hom}(B,\mathbb{C}) \xrightarrow{\varepsilon^2} \cdots$