Derived Category ②.

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①

- $X$ = var. $\mathbb{C}$, $K(\mathcal{A})$ $\left(\mathcal{A} = \mathrm{Coh}(X), \mathrm{QCoh}(X), \mathrm{Mod}X\right)$, $D(\mathcal{A})$ = derived cat.

- Triangles: recall, if $f : A \to B$ is a morphism of complexes. The cone of $f$ is a complex cone$(f)$ s.t. $\mathrm{Cone}(f)^n = (A[1] \oplus B)^n$ and $d^n = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}$ Note have a triangle

$$A \xrightarrow{f} B \xrightarrow{g} \mathrm{Cone}(f) \xrightarrow{h} A[1] \qquad (*)$$

Then: $g \circ f \underset{\text{homotopic}}{\sim} 0 \qquad h \circ g = 0$

In $K(\mathcal{A}), D(\mathcal{A})$ : $g \circ f = 0$, $h \circ g = 0$

Further: $B \xrightarrow{g} \mathrm{cone}(f) \xrightarrow{f} A[1]$ is a short exact seq. of complexes.

$$\Rightarrow \ldots \to \underset{H^n(A)}{H^{n-1}(A[1])} \to H^n(B) \to H^n(\mathrm{Cone} f) \to \underbrace{H^n(A[1]) = H^{n+1}(A) \xrightarrow{\partial} H^{n+1}(B)}_{\text{induced actually by } f} \to \ldots$$

is a long exact sequence.

Defn Any $\Delta$ in $K(\mathcal{A})$ or $D(\mathcal{A})$ which is isomorphic to a "standard" $\Delta$ like $(*)$ is called EXACT $\Delta$.

$$\begin{array}{ccccccc} A & \to & B & \to & C & \to & A[1] \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ D & \xrightarrow{f} & E & \to & C(f) & \to & D[1] \end{array} \qquad (*)$$

- In fact, if $0 \to A \to B \to C \to 0$ is an exact sequence of complexes $\Rightarrow$ $A \to B \to C \underset{\exists}{\to} A[1]$ s.t. exact $\Delta$ (in $D(\mathcal{A})$) in $K(\mathcal{A})$.

Intuition : • Exact $\Delta$'s replace short exact seq in $\mathcal{A}$.

Fact : If $A \to B \to C \to A[1]$ is an exact $\Delta$ in $D(\mathcal{A})$
$\Rightarrow$ there is l.e.s.

$$\cdots \to \mathcal{H}^n(A) \to \mathcal{H}^n(B) \to \mathcal{H}^n(C) \to \mathcal{H}^{n+1}(A) \to \cdots$$

Aside : $X$ = smooth proj $\Rightarrow$ $Coh(X)$ determines $X$.
(P. Gabriel)

→ Example : Suppose $0 \to A \to B \to C \to 0$ is a seq. in ~~Coh(X)~~ $\mathcal{A}$
$\Rightarrow$ $A \to B \to C \xrightarrow{\partial} A[1]$ is an exact $\Delta$ in $D(\mathcal{A})$

$\partial$ can be realized as

$$\begin{array}{ccccccc}
\cdots \to & 0 & \to & C & \to & 0 & \to \cdots \\
 & \uparrow & & \uparrow g & & & \\
\cdots 0 \to & A & \to & B & \to & 0 & \cdots \\
 & \downarrow & & & & & \\
\cdots 0 \to & A & \to & 0 & \cdots & & \\
 & \deg -1 & & \deg 0 & & &
\end{array}$$

In fact, $\mathrm{Ext}^1_{\mathcal{A}}(C,A) \simeq \mathrm{Hom}_{D(\mathcal{A})}(C, A[1])$

More generally, $\mathrm{Ext}^n_{\mathcal{A}}(C,A) \simeq \mathrm{Hom}_{D(\mathcal{A})}(C, A[n])$

Recall : A functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ is exact if it preserves $\Delta$'s, shifts.

Construction : If $X \xrightarrow{f} Y$ is a morphism of varieties $\Rightarrow$ the "half-exact functors" $f_*$, $f^*$, $\mathrm{Hom}(-,-)$, $- \otimes -$ can be extended to ~~adr~~ exact functors between derived categories :
$Rf_*$, $Lf^*$, $R\mathrm{Hom}(-,-)$, $- \overset{L}{\otimes} -$.

<u>Idea</u>: If $A \in D(\mathcal{A})$, $A: \cdots \to A^{n-1} \to A^n \to A^{n+1} \to \cdots$

$\Rightarrow$ $A \to I^{\cdot}$ injective $\Rightarrow$ $Rf_* A := \cdots f_* I^{n-1} \to f_* I^n \to f_* I^{n+1} \to \cdots$

+ bounded below

(it does not matter the inj. resolution)

<u>Example</u>: The structure of $D^b(\mathbb{P}^n)$ = derived cat of bound complexes (b) of coherent sheaves in $\mathbb{P}^n$.

$D^b(\mathbb{P}^n)$ is "generated" by the sheaves $\{\mathcal{O}(-n), \mathcal{O}(-n+1), .., \mathcal{O}(-1), \mathcal{O}\} = S$

Generated = the smallest full subcategory of $D^b(\mathbb{P}^n)$ containing $S$ and all its shifts and cones is $D^b(\mathbb{P}^n)$ itself

<u>n=1</u>: Consider

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, hook, "\Delta"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

$$0 \to p_1^*\mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-1) \to \underset{\overset{\parallel}{p_1^*\mathcal{O}_{\mathbb{P}^1} \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}}}{\mathcal{O}} \to \mathcal{O}_\Delta \to 0$$

Let $\mathcal{F} \in D^b(\mathbb{P}^n)$. As derived functors are exact, we have an exact $\Delta$:

Exact triangle: $Lp_i^* \simeq p_i^*$ since $p_i$ is flat

$\Rightarrow$ $$Rp_{2*}\left(Lp_1^*\mathcal{F} \overset{L}{\otimes} p_1^*\mathcal{O}(-1) \otimes p_2^*\mathcal{O}(-1)\right) \to Rp_{2*}(Lp_1^*\mathcal{F}) \to Rp_{2*}(Lp_1^*\mathcal{F} \overset{L}{\otimes} \mathcal{O}_\Delta) \to$$
is exact.

<u>Facts</u>: The derived functors satisfy relations similar to the usual underived functors:

Eg. $X$ = smooth proj. $D^b(\mathrm{Coh}(X))$

- $$Rf_*(Lf^*\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong \mathcal{F} \overset{L}{\otimes} Rf_*\mathcal{G}$$

- Flat base change:
$$Lf^* \circ Rg_* \simeq Rg'_* \circ Lf'^*$$
$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & X \arrow[d, "g"] \\ Z \arrow[r, "f"', "\text{flat}"] & Y \end{tikzcd}$$

$$Rp_{2*}(Lp_1^*\mathcal{F} \overset{L}{\otimes} \mathcal{O}_\Delta) = Rp_{2*}(Lp_1^*\mathcal{F} \overset{L}{\otimes} R\Delta_*\mathcal{O}_{\mathbb{P}^1})$$

$\Vert$ Proj formula

$$Rp_{2*}R\Delta_*(L\Delta^* Lp_1^*\mathcal{F})$$

$\Vert$ fact

$$R(\underbrace{p_2\circ\Delta}_{\mathrm{id}})_*(L(\underbrace{p_1\circ\Delta}_{\mathrm{id}})^*\mathcal{F})$$

$\Vert$

$$\mathcal{F}$$

$$\begin{tikzcd} \mathbb{P}^1\times\mathbb{P}^1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbb{P}^1 \arrow[d, "g"] \\ \mathbb{P}^1 \arrow[r, "f"'] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$

$Rf_*\mathcal{F} = R\Gamma(\mathcal{F})$ (cohomology of $\mathcal{F}$)

$Lg^*\underbrace{Rf_*\mathcal{F}}_{\mathcal{O}_{\mathbb{P}^1}\otimes_{\mathbb{C}} R\Gamma(\mathcal{F})}$ $R\Gamma(\mathcal{F})$ is a complex of vector spaces $\mathcal{H}^i(R\Gamma(\mathcal{F})) = H^i(\mathbb{P}^1, \mathcal{F})$

$\underbrace{R\Gamma(\mathcal{F}(-1))}_{\text{complex of vector spaces}} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(-1)$

bunch of $\mathcal{O}(-1)$ bunch of $\mathcal{O}$

$\therefore$ $R\Gamma(\mathcal{F}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{s} \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} R\Gamma(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow$ exact triangle

$\Rightarrow$ $\mathcal{F} = \mathrm{Cone}(s)$

$\Rightarrow$ $\mathcal{D}^b(\mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\rangle$

Fact: $\mathcal{A} = \mathrm{Vect}/\mathbb{C}$, $0 \to A \to B \to C \to 0 \Rightarrow$ splits

$\cdots \to A^{n+1} \to A^{n+2} \to \cdots \Rightarrow A^\bullet \simeq H^n(A)[n]$

Obs/r $\{\mathcal{O}(n), \ldots, \mathcal{O}\}$ is semi-orthogonal