

Derived Categories (2)

- $X = \text{var. } \mathbb{C}$ ,  $K(A)$  ( $A = \text{coh}(X), \text{Qcoh}(X), \text{Mod } X$ ),  $D(A) = \text{derived cat.}$
- Triangles: recall, if  $f: A \rightarrow B$  is a morphism of complexes, the cone of  $f$  is a complex  $\text{cone}(f)$  s.t.  $\text{cone}(f)^n = (A[n] \oplus B)^n$  and  $d^n = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}$  Note have a triangle

$$A \xrightarrow{f} B \xrightarrow{g} \text{cone}(f) \xrightarrow{h} A[1] \quad (*)$$

Then:  $g \circ f \sim 0$  homotopic,  $h \circ g = 0$   
In  $K(A), D(A)$ :  $g \circ f = 0, h \circ g = 0$

Further:  $B \xrightarrow{g} \text{cone}(f) \xrightarrow{h} A[1]$  is a short exact seq. of complexes.

$$\Rightarrow \dots \rightarrow H^{n-1}(A[1]) \rightarrow H^n(B) \rightarrow H^n(\text{cone}(f)) \rightarrow H^n(A[1]) = H^{n+1}(A) \xrightarrow{f} H^{n+1}(B) \rightarrow \dots$$

$H^n(A)$  induced actually by  $f$

is a long exact sequence.

Defn: Any  $\Delta$  in  $K(A)$  or  $D(A)$  which is isomorphic to a "standard"  $\Delta$  like  $(*)$  is called EXACT  $\Delta$ .

$$\begin{array}{ccccccc} A & \rightarrow & B & \rightarrow & C & \rightarrow & A[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ D & \xrightarrow{f} & E & \rightarrow & C(f) & \rightarrow & D[1] \end{array} \quad (*)$$

- In fact, if  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence of complexes  $\Rightarrow A \rightarrow B \rightarrow C \xrightarrow{f} A[1]$  s.t. exact  $\Delta$  (in  $D(A)$ ) in  $K(A)$ .

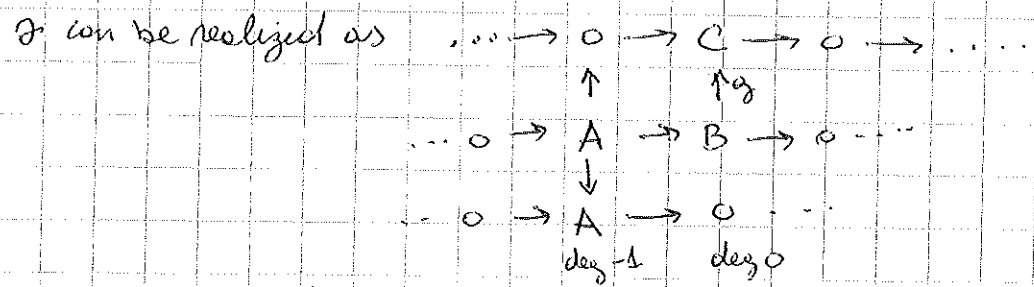
Intuition: • Exact  $\Delta$ 's replace short exact seq. in  $A$ .

Fact: If  $A \rightarrow B \rightarrow C \rightarrow A[n]$  is an exact  $\Delta$  in  $D(A)$   
 $\Rightarrow$  there is l.e.s.

$$\dots \rightarrow H^n(A) \rightarrow H^n(B) \rightarrow H^n(C) \rightarrow H^{n+1}(A) \rightarrow \dots$$

Aside:  $X = \text{smallest proj} \Rightarrow \text{Coh}(X)$  determines  $X$ .  
(P. Gabriel)

Example: Suppose  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a seq. in  ~~$D(A)$~~   $A$   
 $\Rightarrow A \rightarrow B \rightarrow C \xrightarrow{\partial} A[n]$  is an exact  $\Delta$  in  $D(\text{ ~~$C$~~ )$   
 $A$



In fact,  $\text{Ext}_A^1(C, A) \cong \text{Hom}_{D(A)}(C, A[n])$

More generally,  $\text{Ext}_A^n(C, A) \cong \text{Hom}_{D(A)}(C, A[n])$

Recall: A functor  $F: D(A) \rightarrow D(B)$  is exact if it preserves  $\Delta$ 's, shifts.

Construction: If  $X \xrightarrow{f} Y$  is a morphism of varieties  $\Rightarrow$  the "half-exact functors"  $f_*$ ,  $f^*$ ,  $\text{Hom}(-, -)$ ,  $- \otimes -$  can be extended to exact functors between derived categories:  
 $Rf_*$ ,  $Lf^*$ ,  $R\text{Hom}(-, -)$ ,  $- \hat{\otimes} -$ .

Idea: For  $A \in D(A)$ ,  $A: \dots \rightarrow A^{n-1} \rightarrow A^n \rightarrow A^{n+1} \rightarrow \dots$

$\Rightarrow A \rightarrow I^\bullet$  injective  $\Rightarrow Rf_* A := \dots f_* I^{n-1} \rightarrow f_* I^n \rightarrow f_* I^{n+1} \rightarrow \dots$

+ bounded below

(it does not matter defining resolution)

Example: The structure of  $D^b(\mathbb{P}^n) =$  derived cat. of bounded complexes (b) of coherent sheaves in  $\mathbb{P}^n$ .

$D^b(\mathbb{P}^n)$  is "generated" by the sheaves  $\{\mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O}\} = S$   
 Generated = the smallest subcategory of  $D^b(\mathbb{P}^n)$  containing  $S$  and all its <sup>full</sup> shifts and cones in  $D^b(\mathbb{P}^n)$  itself

$n=1$ : Consider  $\mathbb{P}^1 \xrightarrow{\Delta} \mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{array}{ccc}
 \mathbb{P}^1 & \xrightarrow{\Delta} & \mathbb{P}^1 \times \mathbb{P}^1 \\
 & & \swarrow p_1 \quad \searrow p_2 \\
 & & \mathbb{P}^1 \quad \mathbb{P}^1
 \end{array}$$

$$0 \rightarrow p_1^* \mathcal{O}_{\mathbb{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O} \rightarrow 0$$

$\parallel$   
 $p_1^* \mathcal{O}_{\mathbb{P}^1} \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}$

Let  $F \in D^b(\mathbb{P}^n)$ . As derived functors are exact, we have an exact  $\Delta$ :

Exact triangle:  $Lp_i^* \simeq p_i^*$  since  $p_i$  is flat.

$\Rightarrow R_{p_2*} (Lp_1^* F \otimes^L p_1^* \mathcal{O}(-1) \otimes p_2^* \mathcal{O}(-1)) \rightarrow R_{p_2*} (Lp_1^* F) \rightarrow R_{p_2*} (Lp_1^* F \otimes^L \mathcal{O}_{\Delta}) \rightarrow$   
 is exact.

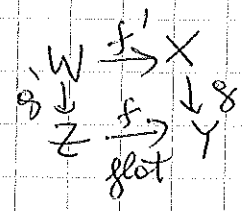
Facts: The derived functors satisfy rules similar to the usual underived functors:

Ex  
 $X =$  smooth proj.  
 $D^b(\text{Coh}(X))$

$Rf_* (Lf^* F \otimes^L G) \simeq F \otimes Rf_* G$

• Flat base change:

$Lf^* \circ Rg_* \simeq Rg'_* \circ Lf'^*$



$$R_{P_2*}(LP_1^* \mathcal{F} \otimes_{\mathbb{C}} \mathcal{O}_{\Delta}) = R_{P_2*}(LP_1^* \mathcal{F} \otimes_{\mathbb{C}} R\Delta_* \mathcal{O}_{P_1})$$

$$\begin{aligned} & \parallel \text{Proj formula} \\ & R_{P_2*} R\Delta_* (L\Delta^* LP_1^* \mathcal{F}) \\ & \parallel \text{fact} \\ & R(P_2 \circ \Delta)_* (L(P_1 \circ \Delta)^* \mathcal{F}) \\ & \parallel \parallel \\ & \mathcal{F} \end{aligned}$$

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{P_1} & \mathbb{P}^1 \\ P_2 \downarrow & & \downarrow \mathcal{S} \\ \mathbb{P}^1 & \xrightarrow{f} & \text{Spec } \mathbb{C} \end{array}$$

$$R_{P_2*} \mathcal{F} = R\Gamma(\mathcal{F}) \text{ (cohomology of } \mathcal{F} \text{)}$$

$R\Gamma(\mathcal{F})$  is a  $\mathbb{C}$ -complex of vector spaces  $H^i(R\Gamma(\mathcal{F})) = H^i(\mathbb{P}^1, \mathcal{F})$

$$\underbrace{R\Gamma(\mathcal{F}(-1))}_{\text{complex of vector spaces}} \otimes_{\mathbb{C}} \underbrace{\mathcal{O}_{\mathbb{P}^1}(-1)}_{\text{bundle of } \mathcal{O}(-1)}$$

$$\therefore R\Gamma(\mathcal{F}(-1)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} R\Gamma(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \text{exact triangle}$$

$$\Rightarrow \mathcal{F} = \text{Cone}(\Delta)$$

$$\Rightarrow D^b(\mathbb{P}^1) = \langle \mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1} \rangle$$

Fact:  $A = \text{Vect } \mathbb{C}$ ,  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \Rightarrow \text{splits}$

$$\dots \rightarrow A^{n+1} \rightarrow A^{n+2} \rightarrow \dots \Rightarrow A^0 \cong H^n(A) \mathbb{C}[n]$$

obs/r  $\{ \mathcal{O}(n), \dots, \mathcal{O} \}$  is semi-orthogonal