Derived Category III — 5-9-14 — Subhendu Mehrotra — (1)

- Last time : $D^b(\mathbb{P}^1) = \langle \mathcal{O}(-n), \mathcal{O}(-n+1), \dots, \mathcal{O}(-1), \mathcal{O} \rangle$.

obs: $K(\mathrm{Coh}\,X) \simeq K(D^b(\mathrm{Coh}\,X))$ $\quad D^b(\mathrm{Coh}\,X) \rightsquigarrow K_0(X)$

// Grothendieck groups

$\langle \text{isom classes} \rangle / 0 \to E \to F \to G \to 0$

Eg: $X = \mathbb{P}^n$; $K_0(\mathbb{P}^n) \simeq \mathbb{Z}^{n+1}$

→ Basic results : X = smooth proj. of dim $n/_k$ E, F vector bundles on X

(Serre duality) $\quad \mathrm{Ext}^i(E, F) \underset{\text{Nat}}{\simeq} \mathrm{Ext}^{n-i}(F, E \otimes \omega_X)^\vee$ ← canonical line bundle.

→ This works in $D^b(X)$: Set $S_X : D^b(X) \longrightarrow D^b(X)$ (Serre functor)

$E, F \in D^b(X)$ $\qquad E \longmapsto E \overset{L}{\otimes} \omega_X[n]$

X = smooth & proj

(we can think on E, F as vector bundles)

i: $\mathrm{Hom}_{D^b(X)}(E, F) \underset{\text{Natural}}{\simeq} \mathrm{Hom}(F, S_X E)^\vee \quad (+)$

Grothendieck generalized this to very general schemes and to the relative setting $f : X \to Y$, f = proper map :

$\exists$ right adjoint to Rf_*, ie $\exists\, f^! : D(Y) \to D(X)$

s.t. $R\mathrm{Hom}(Rf_* E, F) \xrightarrow{\sim} R\mathrm{Hom}(E, f^! F) \quad (*)$

Ex: If X = smooth proj , $Y = pt$; then $f^! F = f^* F \otimes \omega_X[\dim X]$

use $(*)$ to prove $(+)$

→ Let $\mathcal{D}$ be a triangulated category with $\mathrm{Hom}_{\mathcal{D}}(E, F)$ being a vector space $/_k$ $\forall E, F \in \mathcal{D}$, and compositions be bilinear (ie a k-linear Δ ed) category. Then a functor $S_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is called a SERRE functor if it satisfies $(+)$.

± 1991

Thm (Bondal–Kapranov) If C = k-linear Δ'ed Cat, then any Serre functor, if it exists, is unique up to isomorphism. (2)

It is exact, and if $F: C \to D$ is exact equiv. of Δ'ed Cats. Then, $S_D \circ F \simeq F \circ S_C$ canonically.

II Thm (Bondal–Orlov)

X = smooth proj var., ω_X is ample or antiample (ie ω_X^{-1} is ample)

If Y = smooth proj. var, and $D^b(X) \xrightarrow{\sim} D^b(Y)$ (exact equiv)

$\Rightarrow X \simeq Y$.

(True for surfaces of general type)

Idea of proof:

Step 1: $\mathcal{E} \in D^b(X)$ s.t. (i) $\begin{cases} \cdot\ \mathcal{E} \simeq S_X \mathcal{E}[-\dim X] \quad \cdot\ \mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) = 0 \ \forall i < 0 \\ \cdot\ \mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}. \end{cases}$

If $\omega_X^{\pm}$ is ample $\Rightarrow$ any $\mathcal{E}$ satisfying (i) is a shift of the structure sheaf of a pt, ie, $\mathcal{E} \simeq \mathcal{O}_x[i]$ $x \in X$.

Step 2: Y = any var, smooth, quasi-proj. Suppose $\mathcal{L} \in D^b(Y)$ s.t. $\forall y \in Y \ \exists s \in \mathbb{Z}$ st

(ii) $\begin{cases} \cdot\ \mathrm{Hom}(\mathcal{L}, \mathcal{O}_y[s]) = k \\ \cdot\ \mathrm{Hom}(\mathcal{L}, \mathcal{O}_y[i]) = 0 \quad \forall i \neq s \end{cases}$ $\Rightarrow \mathcal{L}$ is line bundle, poss. shifted, ie $\mathcal{L} \simeq \underbrace{M}_{\text{l.b.}}[j]$

$\Rightarrow$ line bundles on Y map to line bundles on X.

$\Rightarrow$ we will have a map $Y \to X$ cont. bijection (because $\omega_Y^{\pm}$ ample)

Step 3: Finally $R(X) = \bigoplus \mathrm{Hom}(\mathcal{L}_X, S_X^i \mathcal{L}_X[-i\dim X])$

$= \bigoplus \mathrm{Hom}(\mathcal{L}_Y, S_Y^i \mathcal{L}_Y[-i\dim Y])$

$= R(Y)$

using $\phi(\mathcal{L}_X) = \mathcal{L}_Y$.

$X = \mathrm{Proj}(R(X)) = \mathrm{Proj}(R(Y)) = Y$ ∎

Defn: X, Y varieties, $\mathcal{E} \in D^b(X \times Y)$. the integral transform with kernel $\mathcal{E}$ is the functor: $\Phi_{\mathcal{E}} : D^b(X) \longrightarrow D^b(Y)$

$$T \mapsto Rq_*(L^*(T) \otimes^L \mathcal{E})$$

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

If $\Phi_{\mathcal{E}}$ is equiv, we say that $\Phi_{\mathcal{E}}$ is a <u>Fourier-Mukai</u> transform.

III <u>Thm</u>: (Orlov) If X, Y smooth proj. varieties, any equivalence $\phi : D^b(X) \longrightarrow D^b(Y)$ is a Fourier-Mukai transform.

IV <u>Thm</u>: (Bondal-Orlov, Bridgeland) X, Y smooth proj., $\Phi : D^b(X) \longrightarrow D^b(Y)$ an integral transform with kernel $\mathcal{P} \in D^b(X \times Y)$

For $x \in X$, let $\mathcal{P}_x = \Phi(\mathcal{O}_x)$. The Φ is <u>fully faithfull</u> $\iff$
(preserves Hom)

iff $\mathrm{Hom}(\mathcal{P}_x, \mathcal{P}_y[i]) = \begin{cases} 0 & \text{if } x \neq y \text{ or } i \notin [0, \dim X] \\ \mathbb{C} & \text{if } x = y \text{ and } i = 0 \end{cases}$.

Check if necess.

Further Φ is an <u>equivalence</u> iff $\mathcal{P}_x \otimes \omega_Y \cong \mathcal{P}_x \quad \forall x \in X$.

<u>Remark</u>:

$$\begin{tikzcd} Y \cong Y_x \arrow[r, hook, "i'"] \arrow[d, "j"] & X \times Y \arrow[r, "q"] \arrow[d, "p"] & Y \\ x \arrow[r, hook, "i"'] & X & \end{tikzcd}$$

$\Phi(i_* \mathcal{O}_x) = Rq_*(Lp^* i_* \mathcal{O}_x \otimes^L \mathcal{P})$

Proj formula

$\cancel{Rq_* Lp^* i_*(Li^* \mathcal{P})}$

$$Rq_*(i'_* \underbrace{(Lj)^* \mathcal{O}_x}_{\mathcal{O}_{Y_x}} \otimes^L \mathcal{P}) \underset{\text{Proj}}{=} Rq_* i'_*(Li'^* \mathcal{P}) = Li'^* \mathcal{P}.$$

<u>Example</u>: E = Elliptic curve$/\mathbb{C}$ P_0 = origin of E $\mathrm{Jac}(E) \cong E$

$E \times E \leftarrow \mathcal{P}$ Poincaré bdle

$$\mathcal{P} = p^* \mathcal{O}_E(-P_0) \otimes q^* \mathcal{O}_E(-P_0) \otimes \mathcal{O}_{E \times E}(\Delta)$$

For $x \in E$, $\Phi(\mathcal{O}_x) = \mathcal{O}_E(x - P_0) = \mathcal{P}_x$ $\quad \mathrm{Hom}_{D(E)}(\mathcal{P}_x, \mathcal{P}_y[i]) = \begin{cases} 0 & \text{if } x \neq y \ \forall i \\ 1 & \text{if } x = y \text{ and } i = 0 \end{cases}$

& $\omega_E \cong \mathcal{O}_E \Rightarrow \Phi$ equiv.