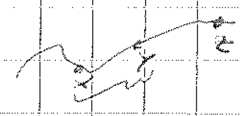
Derived categories IV   Sukhendu Mehrotra   12/9/2014

Eg. E = elliptic curve $J^0(E)$ = Jacobian $\simeq E$

$$\begin{array}{c} P = \text{Poincaré bundle} \to E\times E \\ E \xleftarrow{p} \qquad \xrightarrow{q} E \end{array} \qquad \begin{array}{l} \Phi_P : D(E) \to D(E) \\ \quad T \mapsto Rq_*(p^*(T)\otimes P) \end{array}$$

Remark: Any fine moduli space of sheaves on a smooth proj var X comes naturally with an integral transform: P = universal object of M = moduli $\Rightarrow$ $\Phi_P : D(X)\to D(M)$ $T \mapsto Rq_*(p^*(T)\otimes P)$

We noted $\Phi_P(\mathcal{O}_x) = \mathcal{O}(P_0 - x)$

• Next, suppose $\xi \subseteq E$ is length 2 subscheme    $\xi = \mathrm{Spec}\, \mathcal{O}_z/m_z^2$

$\Rightarrow$ $\Phi_P(\mathcal{O}_\xi) = \mathcal{O}(P_0 - x) \oplus \mathcal{O}(P_0 - y)$ or $\mathcal{O}(P_0 - z)\otimes F$ where F is the unique nontrivial extension $0 \to \mathcal{O}_E \to F \to \mathcal{O}_E \to 0$ if $\xi = \mathrm{Spec}\,\mathcal{O}_z/m_z^2$

In $\mathrm{Ext}^1(\mathcal{O}_E, \mathcal{O}_E) = H^1(\mathcal{O}_E) = \mathbb{C}$.

This defines a map $\mathrm{Hilb}^2(E) = \mathrm{Sym}^2 E \longrightarrow M_{0,2}(E)$ = Moduli space of degree 0 rk 2 "semistable" vector bdles on E.

This gives an isomorphism (better on Category level!)

• Birational Geometry.

Conj (Bondal-Orlov) X_1, X_2 = sm. birat. proj CY varieties of dim n (ie $\omega_X \simeq \mathcal{O}_X$)

$$\Rightarrow \quad D^b(X_1) \simeq D^b(X_2)$$

Thm. (Bridgeland) Conj holds in dim 3.

<u>Idea</u>: Kawamata, Kollar: If X_1, X_2 are above, then they are related by a sequence of flops. Reduce to the case

$$\begin{tikzcd} Y \arrow[dr, "f"'] & & W \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

Y, W smooth proj; X ← mild singularity; f, g = "small contr" finitely many rat'l curves

First consider the case of the "Atiyah flop"

$$\begin{tikzcd} & \hat{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^1 \cong C \subseteq Y \arrow[dr, "f"'] & & W \supset C' \cong \mathbb{P}^1 \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

$\{xy - zw = 0\} \subseteq \mathbb{C}^4$

$\phi = Rq_* \circ Lp^* : D(Y) \to D(W)$ is an equivalence (B-O)

Transforming pts on C' via ϕ^{-1} to Y, one obtains objects $E \in D^b(Y)$ st

$$\left.\begin{array}{ll} \text{(i)} & \mathcal{H}^i(E) = 0 \text{ if } i \neq 0, -1 \\ \text{(ii)} & R^1 f_*(\mathcal{H}^0(E)) = 0 = R^0 f_*(\mathcal{H}^{-1}(E)) \\ \text{(iii)} & \mathrm{Hom}_Y(\mathcal{H}^0(E), F) = 0 \text{ if } F \text{ is any sheaf st } Rf_*F = 0. \end{array}\right\} (*)$$

Bridgeland: objects in $D(Y)$ satisfying $(*)$ form an abelian category: $\mathrm{Per}(Y/X)$

Furthermore, if $E = \phi^{-1}(\mathcal{O}_x)$, some $x \in W$, then

$$\left.\begin{array}{l} \bullet\ ch(E) = ch(pt) \text{ on } Y \\ \bullet\ \exists \text{ an exact:} \\ \qquad G \to \mathcal{O}_Y \to E \to G[1], \text{ where } G \in \mathrm{Per}(Y/X). \end{array}\right\} (+)$$

Note taking cohomology: $\underbrace{\mathcal{H}^{-1}(\mathcal{O}_Y)}_{0} \to \mathcal{H}^{-1}(E) \to \mathcal{H}^0(G) \to \mathcal{H}^0(\mathcal{O}_Y) \to \mathcal{H}^0(E) \to \underbrace{\cancel{\mathcal{H}^1(G)}}_{0}$

$\Rightarrow$ G is a sheaf.

Using this, Bridgeland constructs a moduli space M of objects E satisfying $(*)$ & $(+)$. This shows the universal object P on $Y \times M$ gives a derived equivalence $\Phi_P : D^b(Y) \to D^b(M)$. He identifies M with W.

Homological Algebra

§1. Triangulated Categories.

Def. D = additive cat

- Finite direct sums and prod. exist
- $\forall A,B \in D$, $Hom(A,B)$ = abelian groups
- composition of morphisms is bilinear

The structure of a Δ category on D is given by an additive equivalence $[1] : D \to D$ and a set of distinguished exact Δ's:

$$A \to B \to C \to A[1]$$

st. the following axioms hold.

(1) $A \xrightarrow{1} A \to 0 \to A[1]$ is distinguished $\forall A \in D$

(2) Any Δ isom to a dist. Δ is dist.

(3) Any morphism $A \xrightarrow{f} B$ can be completed to a dist. Δ

$$A \xrightarrow{f} B \xrightarrow{\exists g} C \xrightarrow{\exists h} A[1]$$

TR 1

$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is dist. iff

$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is dist.

TR 2

Given a commuting diag.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\exists h"] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

where rows are dist. Δs, $\exists$ arrow h as above

TR 3

(octahedral axiom) $\Rightarrow$

$$\begin{tikzcd}
0 \arrow[d] & & & \\
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \\
A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\exists"] & A[1] \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
& B[1] & &
\end{tikzcd}$$

2 rows exact
1 column exact
$\Downarrow$
2 column exact

TR4

Ex: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ dist Δ. Show $g \circ f = 0$.

Prop: $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is dist.
Then, $A_0 \in D$, the following sequence of groups are exact
$Hom(A_0, A) \xrightarrow{f_*} Hom(A_0, B) \xrightarrow{g_*} Hom(A_0, C)$
$Hom(C, A_0) \longrightarrow Hom(B, A_0) \longrightarrow Hom(A, A_0)$

proof: Suppose $p \in \operatorname{Ker} g_*$

$$\begin{tikzcd} A_0 \arrow[r, "1"] \arrow[d, dashed, "\exists q \text{ by TR3}"'] & A_0 \arrow[r] \arrow[d, "p"] & 0 \arrow[r] \arrow[d] & A_0[1] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "h"] & A[1] \end{tikzcd}$$

note $f_*(q) = p$ .... ∎

Def: Let D, D' be Δ'ed. Then a additive functor $F: D \to D'$ is EXACT if
(i) $F \circ [1]_D \xrightarrow{\sim} [1]_{D'} \circ F$
(ii) F takes Δ's to Δ's

Fact: If $F: D \to D'$ is exact then both and right adjoint functors of F are exact. (if they exists)

Recall: The left-adj G of F (if exists) is a functor
$G: D' \to D$ and satisfies
$Hom_{D'}(A, FB) \cong Hom_D(GA, B)$
natural in both variables.