Sukhendu M. Derived Categories V

Goal: Give further details about the structure and construction of $\mathcal{D}(A)$

→ **Recall**: $\mathcal{A}$ = abelian category (= Mod-R, Coh X, QCoh X)
and $Kom(\mathcal{A})$ = complexes of objects in $\mathcal{A}$.
We defined the $K(\mathcal{A})$: i.e. up to homotopy.
homotopy between $f, g \in Hom_{Kom(A)}(A^\cdot, B^\cdot)$ if $\exists$ collection of maps $h^i : A^i \to B^{i-1}$
s.t. $f - g = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.

Exercise: Show that $K(\mathcal{A})$ is a well-defined additive category using the foll. prop:

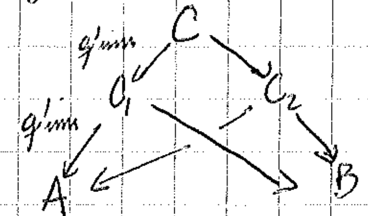
Prop'l (i) Homotopy is equiv relation
(ii) Homotopically trivial morphisms in $Kom(A)$ form an ideal in $Mor(Kom A)$)
i.e. if $f \sim 0 \in Hom_{Kom(A)}(A, B)$ and $g \in Hom_{Kom}(B, C)$ $h \in Hom_{Kom}(D, A)$

$\Rightarrow g \circ f \circ h \sim 0$ in $Hom_{Kom}(D, C)$

(iii) If $f \sim g \Rightarrow H^i(f) = H^i(g)$ $\forall i$.
(iv) If $f : A \to B$, $g : B \to A$ are morphisms with $g \circ f \sim 1_A$ $f \circ g \sim 1_B$
Then f, g are q'isms and $H^i(g) = H^i(f)^{-1}$.

→ **Recall**: $\mathcal{D}(A)$: $ob(\mathcal{D}(A)) = ob(K(A))$ and $Hom_{\mathcal{D}(A)}(A, B) = \left\{ A \xleftarrow{f} C \xrightarrow{g} B \;\middle|\; f \text{ qism},\ f, g \in Mor \right\} / \sim$

$\sim$: Two $A \xleftarrow{q'ism} C_1 \to B$, $A \xleftarrow{q'ism} C_2 \to B$ are equivalent if $\exists$ $C_1 \xleftarrow{q'ism} C \to C_2$ such that the following diag commutes in $K(A)$:



Remark (1) as $C \to A$ is q'ism
$\Rightarrow C \to C_2 \to A$ is q-ism.
(2) $K(A)$ is introduced as an intermediate step because 2 roofs cannot be composed to get a roof diagram in $Kom(A)$

Not: If $f: A \to B$ is a q-isom, we denote it as $A \overset{f}{\multimap} B$ (2)

Composition let $\begin{array}{ccc} & C_1 & \\ \swarrow\!\!\circ & & \searrow \\ A & & B \end{array}$, $\begin{array}{ccc} & C_2 & \\ \swarrow\!\!\circ & & \searrow \\ B & & C \end{array}$ $\in Mor(\mathcal{D}(\mathcal{A}))$

(i) We want $\begin{array}{ccc} & C_0 & \\ \swarrow\!\!\circ & & \searrow \\ C_1 & & C_2 \end{array}$ & (ii) we need to show it is unique.

→ Need the mapping cone for this.

Recall: If $A^\cdot \xrightarrow{f} B^\cdot \in Mor(Kom(\mathcal{A}))$

$\Rightarrow$ $C(f) \in Kom(\mathcal{A})$ is defined as $C(f)^i = A^{i+1} \oplus B^i$ $d^i_{C(f)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$

$\exists$ a Nat'l inclusion $\tau_f: B \to C(f)$

" projection $\pi_f: C(f) \to A[1]$.

Exercise: (1) $B \to C(f) \to A[1]$ is a short exact seq. of complexes

(2) $A \to B \to C(f)$, $h \sim 0$

Remark: (1) $\Rightarrow$ l.e.s. in coh $\cdots \to H^i(B) \to H^i(C(f)) \to H^{i+1}(A) \to \cdots$

→ Note that any commutative diagr

$$\begin{array}{ccccccc} A_1 & \xrightarrow{f_1} & B_1 & \longrightarrow & C(f_1) & \longrightarrow & A_1[1] \\ \downarrow & & \downarrow & & \exists\, \vdots & & \downarrow \\ A_2 & \xrightarrow{f_2} & B_2 & \longrightarrow & C(f_2) & \longrightarrow & A_2[1] \end{array}$$

(Recall TR 3)

Prop 1- Let $f: A \to B \in Mor(Kom(\mathcal{A}))$ and $A \xrightarrow{f} B \xrightarrow{\tau} C(f) \xrightarrow{\pi} A[1]$ be the Δ above. Then, $\exists g: A[1] \to C(\tau) \in Mor(Kom\mathcal{A})$ which is an isom in $K(\mathcal{A})$ s.t. comm. diagr in $K(\mathcal{A})$:

$$\begin{array}{ccccccccc} C(\tau)[-1] & \longrightarrow & B & \longrightarrow & C(f) & \xrightarrow{\tau_\tau} & C(\tau) & \xrightarrow{\pi_\tau} & B[1] \\ \uparrow{\scriptstyle g[-1]} & & \| & & \| & ② & \uparrow{\scriptstyle g} & ① & \| \\ A & \xrightarrow{f} & B & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A[1] & \xrightarrow{-f} & B[1] \end{array}$$

(Recall TR2)

sketch of pf: Define $g: A[1] \longrightarrow C(\tau)$ as

$$A[1]^i = A^{i+1} \xrightarrow[(-f^{i+1}, 1, 0)]{} C(\tau)^i = B^{i+1} \oplus C(f)^i = B^{i+1} \oplus A^{i+1} \oplus B^i$$

Exercise: Check that $g \in Hom_{Kom(A)}(A[1], C(\tau))$

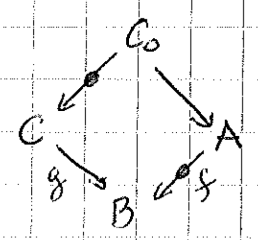
Define $h: C(\tau) \longrightarrow A[1]$ as $h = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Exercise: Check that $h = g^{-1}$ in $K(A)$.

The commutativity in ① is clear in $Kom(A)$. But ② does not commute in $Kom(A)$, only in $K(A)$.

Exercise: Check that ② commutes in $K(A)$ using $h \circ \tau_\tau = \pi$ and $g^{-1} = h$ in $K(A)$.

Prop: Suppose $f: A^\cdot \to B^\cdot$ is a q'ism and $g: C^\cdot \to B^\cdot$ is an arbitrary morphism in $Kom(A)$. Then $\exists$ a comm. diag in $Kom(A)$:



proof:

$$\begin{array}{ccccccccc} C(\tau\circ g)[-1] & \longrightarrow & C & \xrightarrow{\tau\circ g} & C(f) & \longrightarrow & C(\tau\circ g) \\ \downarrow & & \downarrow g & & \downarrow & & \exists\downarrow \\ C(\tau)[-1] \simeq A & \xrightarrow{f} & B & \xrightarrow{\tau} & C(f) & \xrightarrow{\pi} & A[1] \simeq C(\tau) \end{array}$$

Take $C_0 := C(\tau\circ g)[-1]$ and then ① is the required diagr. Only need to show that h is a q'ism take l.e.s. in cohom.:

$$H^i(C(f)) = 0 \quad \forall i$$

since $H^i(f)$ is an isomorphism $\forall i$

$\Rightarrow H^i(h)$ is an isom $\forall i$ ie h is a q'ism ∎

Cor: Composition in $\mathcal{D}(\mathcal{A})$, as proposed above, is well-defined.

Pf: Use Co in the previous.

* **Exercise**: Check that the equiv. class $A \xleftarrow{} C_0 \to C$ is unique.

Exercise: Check that $\mathcal{D}(\mathcal{A})$ is an additive category.

Exercise: Check that $\mathcal{A} \xrightarrow{i} K(\mathcal{A})$ & $\mathcal{A} \xrightarrow{j} \mathcal{D}(\mathcal{A})$

$A \mapsto \cdots 0 \to A \to 0 \cdots$ $\quad$ $A \mapsto \cdots 0 \to A \to 0 \cdots$

are fully-faithful, i.e., $Hom_{\mathcal{A}}(A,B) \simeq Hom_{\mathcal{D}(\mathcal{A})}(j(A), j(B))$ etc.

Exercise: Let $0 \to A \xrightarrow{f} B \to C \to 0$ be a s.e.s. in $\mathcal{A}$.

$\Rightarrow \exists$ isomorphisms

$$\begin{array}{ccccc} j(A) & \xrightarrow{j(f)} & j(B) & \longrightarrow & j(C) \\ \| & & \| & & \downarrow \wr \\ j(A) & \xrightarrow{j(f)} & j(B) & \xrightarrow{\tau} & C(\widetilde{j(f)}) \end{array}$$

Exercise: (i) Let $A \in ob(\mathcal{D}(\mathcal{A}))$. Then, $A \cong 0$ iff $\mathcal{H}^i(A) = 0 \ \forall i$

(ii) $f \in Hom_{\mathcal{D}(\mathcal{A})}(A,B)$. Show that $f = 0 \Leftrightarrow \exists$ q-isom $g: C \to A$ s.t. $f \circ g \sim 0$.

Remark: It may happen that $f: A \to B$ satisfies $\mathcal{H}^i(f) = 0 \ \forall i$ but $f \neq 0$ in $Mor(\mathcal{D}(\mathcal{A}))$.

Ex: Let $A \hookrightarrow B \twoheadrightarrow C$ s.e.s. in $\mathcal{A}$ $\Rightarrow A \to B \to C \to A[1]$ the corresp. Δ with $C \simeq C(f)$. Note the $\mathcal{H}^0(h)$: $\mathcal{H}^0(C) \to \mathcal{H}^0(A[1])$, where $\mathcal{H}^0(C) = C$ and $\mathcal{H}^0(A[1]) = 0$

& $\mathcal{H}^{-1}(h)$: $\mathcal{H}^{-1}(C) \to \mathcal{H}^{-1}(A[1]) = A$, where $\mathcal{H}^{-1}(C) = 0$ $\Rightarrow \mathcal{H}^i(h) = 0 \ \forall i$.

But in general $h \neq 0$ in $\mathcal{D}(\mathcal{A})$.

Reason: $h \in Hom_{\mathcal{D}(\mathcal{A})}(C, A[1]) = Ext^1(C, A)$ (we will show)

is the extension class of the seq. (*), $\neq 0$ in gen'l.

<u>Defn</u> a Δ : $A_1 \to A_2 \to A_3 \to A_1[1]$ in $K(\mathcal{A})$, resp. $\mathcal{D}(\mathcal{A})$, is called <u>distinguished</u> if it is isom. to a Δ of the form $A \xrightarrow{f} B \xrightarrow{\iota} C(f) \xrightarrow{\pi} A[1]$ in $K(\mathcal{A})$, resp. $\mathcal{D}(\mathcal{A})$.

<u>Prop</u> With the distinguished Δ as in definition above and the natural shift of complexes, $A \mapsto A[1]$, $K(\mathcal{A})$, resp. $\mathcal{D}(\mathcal{A})$ become Δ'ed categories.

<u>Pf</u> See Gelfand–Manin (~~Lehrgen. org~~).

★ <u>Exercise</u>: Let $\mathcal{A} = Vect_{finite}(k)$; f. dim vector spaces over k.
Show that $\mathcal{D}(\mathcal{A}) = \prod_{i\in\mathbb{Z}} \mathcal{A}$, ie, $\forall A \in \mathcal{D}(\mathcal{A})$, there is an isom $A \simeq \bigoplus_{i\in\mathbb{Z}} H^i(A)[-i]$ (complex with zero differentials).