

$\mathcal{A}$  = Abelian Category

→ Let  $\text{Kom}^*(\mathcal{A})$  for  $*$  = +, -, b to be the category of complexes  $A^\bullet$  with  $A^i = 0 \ \forall i \ll 0, i \gg 0, |i| \gg b$ , resp.

Let  $K^*(\mathcal{A}) = \text{Kom}^*(\mathcal{A}) / \text{hom. eq.}$  and  $D^*(\mathcal{A}) = K^*(\mathcal{A}) [\text{isom}]$

Prop 1: The natural functor  $D^*(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{A})$ ,  $*$  = +, -, b  
describes an equivalence of  $D^*(\mathcal{A})$  with the full  $\Delta$ -set subset of  $\mathcal{D}(\mathcal{A})$   
consisting of complexes  $A \in \mathcal{D}(\mathcal{A})$  with  $H^i(A^\bullet) = 0$  if  $i \ll 0, i \gg 0, |i| \gg 0$  resp.

→ Def 1: An abelian cat. has enough injectives ...

Prop 1: Let  $\mathcal{A}$  be an abelian cat. with enough injectives. Then,  $\forall A \in K^+(\mathcal{A})$ ,  
 $\exists I^\bullet \in K^+(\mathcal{A})$ ,  $I^i \text{ inj } \forall i$  and a gsm  $A^\bullet \rightarrow I^\bullet$ .

Pf: Cartan - Eilenberg.

Remark: Note that if  $\mathcal{A}$  is simply additive, we may still define  $K^*(\mathcal{A})$  since hom. eq. only uses additive structure.

Prop 1: Suppose  $\mathcal{A}$  = abelian contains enough injectives. Then there is an equivalence  $i: K^+(I) \xrightarrow{\sim} D^+(\mathcal{A})$  where  $I$  = full subcategory of  $\mathcal{A}$  of injectives.

Lemma 1: Suppose  $A^\bullet \rightarrow B^\bullet$  is a gsm in  $\text{Kom}^+(\mathcal{A})$ . For any  $I^\bullet \in \text{Kom}^+(\mathcal{A})$ ,  $I^i = \text{injective } \forall i$ , the induced map

$$\text{Hom}_{K^+(\mathcal{A})}(B^\bullet, I^\bullet) \xrightarrow{\sim} \text{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) \text{ is an isomorphism.}$$

Lemma 2: Let  $A^\bullet, I^\bullet \in \text{Kom}^+(\mathcal{A})$ ,  $I^j = \text{inj } \forall j$   
Then  $\text{Hom}_{K^+(\mathcal{A})}(A^\bullet, I^\bullet) = \text{Hom}_{D^+(\mathcal{A})}(A^\bullet, I^\bullet)$ .

Proof - ①  $i$  is fully-faithful by lemma 2.

②  $i$  is essentially surj (i.e. every other object in  $\mathcal{D}^+(A)$  is isom. to an object of the form  $i(I^\bullet)$ ) since  $k$  has enough inj.

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Proof of lemma 2:  $\exists$  a natural map  $\text{Hom}_{k^+(A)}(A, I^\bullet) \rightarrow \text{Hom}_{\mathcal{D}^+(A)}(A, I^\bullet)$

It suffices to show that given a roof

$$\begin{array}{ccc} & B^\bullet & \\ \text{qfpm} \swarrow & & \searrow \\ A^\bullet & \dashrightarrow & I^\bullet \end{array}$$

$\exists$  a unique arrow  $A^\bullet \rightarrow I^\bullet$  so that the diagram commutes in  $\mathcal{D}^+(A)$  by the fact that  $\mathcal{D}^+(A)$  has enough inj. (True by lemma 1)

Proof of lemma 1: Consider the exact  $\Delta: A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A[\bullet]$

$$\Rightarrow \dots \rightarrow H^i(A) \xrightarrow{\cong} H^i(B) \rightarrow H^i(C) \rightarrow H^{i+1}(A) \xrightarrow{\cong} H^{i+1}(B) \rightarrow \dots$$

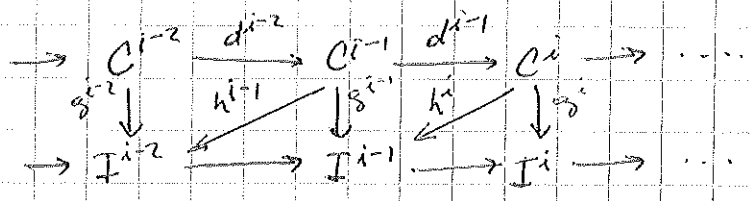
ie  $C^\bullet$  is acyclic.

we also have  $\dots \rightarrow \text{Hom}_{k^+(A)}(C, I^\bullet) \rightarrow \text{Hom}_{k^+(A)}(B, I^\bullet) \rightarrow \text{Hom}_{k^+(A)}(A, I^\bullet) \rightarrow \text{Hom}_{k^+(A)}(C, I[\bullet]) \rightarrow \dots$

(General fact: In a  $\mathcal{N}^+$  category, if  $A \rightarrow B \rightarrow C \rightarrow A[\bullet]$  is a dist.  $\Delta$  and  $I$  is an object  $\Rightarrow$   
 $\dots \rightarrow \text{Hom}(C, I) \rightarrow \text{Hom}(B, I) \rightarrow \text{Hom}(A, I) \rightarrow \text{Hom}(C, I[\bullet]) \rightarrow \dots$   
 is exact)

Enough to show  $\text{Hom}(C, I^\bullet) = 0$  if  $C^\bullet$  is acyclic, ie, if  $g: C^\bullet \rightarrow I^\bullet \in \text{Hom}_{k^+(A)}(C, I^\bullet)$  then  $g \sim 0$ .

we will construct a homotopy  $h$  by induction on degree.



Suppose  $h^j$  has been constructed  $\forall j \leq i$ .

Consider  $g^i - d_I^{i-1} \circ h^i : C^i \rightarrow I^i$

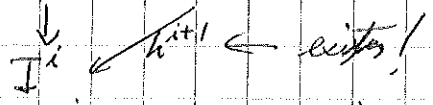
$$\Rightarrow (g^i - d_I^{i-1} \circ h^i) \circ d^{i-1} = d_I^{i-1} \circ g^{i-1} - d_I^{i-1} \circ h^i \circ d^{i-1}$$

But hyp. ind:  $g^{i-1} = h^i \circ d^{i-1} + d_I^{i-2} \circ h^{i-1}$

$$\therefore d_I^{i-1} \circ g^{i-1} = d_I^{i-1} \circ h^i \circ d^{i-1} + 0 \quad \therefore (g^i - d_I^{i-1} \circ h^i) \circ d^{i-1} = 0$$

$\Rightarrow g^i - d_I^{i-1} \circ h^i$  descends to  $C^i/C^{i-1} \rightarrow I^i$

Note that we have an injection  $C^i/C^{i-1} \hookrightarrow C^{i+1}$  as  $C^i$  is acyclic



By inj of  $I^i$ ,  $\exists h^{i+1} : C^{i+1} \rightarrow I^i$ . By com,  $g^i - d_I^{i-1} \circ h^i = h^{i+1} \circ d^i$

Derived functors

$\rightarrow$  Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an additive functor, left exact. This defines

$$F: K^+(\mathcal{A}) \rightarrow K^+(\mathcal{B})$$

$$(\rightarrow A^i \rightarrow A^{i+1} \rightarrow \dots) \mapsto (\rightarrow F(A^i) \rightarrow F(A^{i+1}) \rightarrow \dots)$$

However if  $F$  is not exact,  $F$  will not preserve acyclic objects, and will therefore not descend to a functor  $D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$ .

Assuming  $\mathcal{A}$  has enough inj, we define the Right-derived functor  $RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B})$  which is the "best approx. to  $F$ ".

Let  $i^{-1}: \mathcal{D}^+(A) \rightarrow K^+(I)$  be the quasi inverse of  $i$ , we have the diagram:

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$$\begin{array}{ccccccc} \mathcal{D}^+(A) & \xrightleftharpoons[i]{i^{-1}} & K^+(I) & \hookrightarrow & K^+(A) & \xrightarrow{K(F)} & K^+(B) \xrightarrow{Q_B} \mathcal{D}^+(B) \\ & & & & & & \uparrow Q_A \\ & & & & & & \mathcal{D}^+(A) \end{array}$$

Def - The right derived functor  $RF$  is defined as  $Q_B \circ K(F) \circ i^{-1}: \mathcal{D}^+(A) \rightarrow \mathcal{D}^+(B)$ .

Prop: (i) There is a nat'l morphism of functors  $Q_B \circ K(F) \rightarrow RF \circ Q_A$ .

(ii)  $RF: \mathcal{D}^+(A) \rightarrow \mathcal{D}^+(B)$

(iii) Suppose  $G: \mathcal{D}^+(A) \rightarrow \mathcal{D}^+(B)$  is an exact functor. Then any morphism of functors  $Q_B \circ K(F) \rightarrow G \circ Q_A$  factors through a unique morphism  $RF \rightarrow G$ .

Proof (i) Let  $A \in K^+(A)$  and  $I = i^{-1}(A)$ . The natural trans.  $1 \rightarrow i \circ i^{-1}$  gives a functorial morphism  $A \rightarrow i(I) = I$  in  $\mathcal{D}^+(A)$  given by a roof  $\begin{array}{ccc} & C & \\ \text{quasi} & \searrow & \\ A & & I \end{array}$  in  $K^+(A)$ .

By lemma 2, as  $C \rightarrow A$  is a qm and  $I$  is inj,  $\exists!$  arrow  $A \rightarrow I$  (up to homot.) so that the diagram commutes. Thus we obtain a functorial morphism  $KF(A) \rightarrow KF(I) = RF(A)$ .

(ii)  $RF$  is exact being a comp. of exact functors.

(iii) Gel'fand-Monin.  $\square$

Def - Let  $RF: \mathcal{D}^+(A) \rightarrow \mathcal{D}^+(B)$  be right derived functor of a left exact functor  $F: A \rightarrow B$ . Then for any  $A' \in A$  degree  $R^i F(A') = H^i(RF(A')) \in B$ .

Exts & Hom: Let  $A \in \mathcal{A}$ ,  $F = \text{Hom}(A, -): \mathcal{A} \rightarrow \mathcal{A}b$ . Then  $F$  is left exact, and if  $\mathcal{A}$  has enough injectives, we can define  $R^i F: \mathcal{D}^+(\mathcal{A}) \rightarrow \mathcal{D}^+(\mathcal{A}b)$ .

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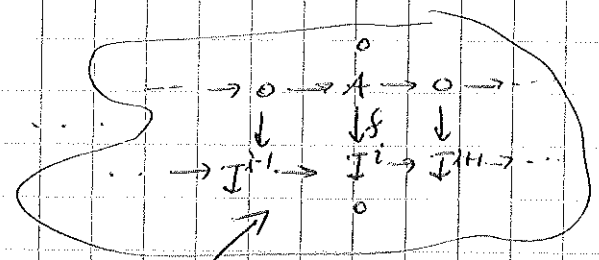
Define  $\text{Ext}^i(A, B^\bullet) = H^i(R\text{Hom}(A, B^\bullet))$ .

Prop: Suppose  $A, B^\bullet \in \mathcal{K}^+(\mathcal{A})$ ,  $\mathcal{A}$  with enough injectives. Then, there are natural isms  $\text{Ext}_{\mathcal{A}}^i(A, B) \simeq \text{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[i])$ .

proof: Let  $B^\bullet \rightarrow I^\bullet$  be injective resolution (quasi-isomorphism). Then,  $R\text{Hom}(A, B^\bullet)$  is the complex

$$\dots \rightarrow \text{Hom}(A, I^i) \xrightarrow{d_i} \text{Hom}(A, I^{i+1}) \rightarrow \dots$$

$f \mapsto d_i \circ f$



$$\text{Ext}^i(A, B^\bullet) = H^i(\text{Hom}(A, I^\bullet))$$

$f$  is a cycle in  $\text{Hom}(A, I^i)$  if  $f: A \rightarrow I^i$  is a cocycle, if  $f$  is a boundary  $d^{i-1}(u)$  in  $\text{Hom}(A, I^{i-1})$ . Then  $f \sim 0$ .

$$\therefore H^i(\text{Hom}(A, I^\bullet)) = \text{Hom}_{\mathcal{K}^+(\mathcal{A})}(A, I^i) = \text{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B^i[i]) \quad (\text{lemma 4/2})$$

General statements:

①  $\mathcal{A}, \mathcal{B} = \text{ob. cat.}$  suppose  $F: \mathcal{K}^+(\mathcal{A}) \rightarrow \mathcal{K}^+(\mathcal{B})$  is exact functor. Then the right derived functor  $R^i F: \mathcal{D}^+(\mathcal{A}) \rightarrow \mathcal{D}^+(\mathcal{B})$  exists whenever there is a full  $\mathcal{A}$ -set subset  $K_F \subseteq \mathcal{K}^+(\mathcal{A})$  adapted to  $F$ , i.e.

- (i) if  $A \in K_F$  is cyclic, then  $F(A)$  is cyclic
- (ii) any object in  $\mathcal{K}^+(\mathcal{A})$  is quasi-isomorphic to an obj. of  $K_F$ .

Ex:  $L_f^*$  requires projectives, but there are not enough proj. in  $\text{Alg. Geom.}$  in general.  $f: X \rightarrow Y$  smooth, proj. can work with vector bundles, i.e.,  $K_f \subseteq \mathcal{Q}(\text{Coh}(Y))$ .  $K_f = \{ \text{loc. free sheaves} \}$ .