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$\mathcal{A}$ = Abelian Category

→ Let $Kom^*(\mathcal{A})$ for $* = +, -, b$ to be the category of complexes A° with $A^i = 0$ $\forall\, i << 0$, $i >> 0$, $|i| >> b$, resp.

Let $K^*(\mathcal{A}) = Kom^*(\mathcal{A})/\text{hom. eq}$ and $D^*(\mathcal{A}) = K^*(\mathcal{A})[q\text{-isom}]$

Prop 1.- The natural functor $\mathcal{D}^*(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$, $* = +, -, b$ defines an equivalence of $\mathcal{D}^*(\mathcal{A})$ with the full Δ'ed subset of $\mathcal{D}(\mathcal{A})$ consisting of complexes $A \in \mathcal{D}(\mathcal{A})$ with $H^i(A^\circ) = 0$ if $i << 0$, $i >> 0$, $|i| >> 0$ resp.

→ Def 1.- An abelian cat. has enough injectives ...

Prop 3.- Let $\mathcal{A}$ be an abelian cat. with enough injectives. Then, $\forall A \in K^+(\mathcal{A})$, $\exists$ $I^\circ \in K^+(\mathcal{A})$, I^i inj $\forall i$ and a q-ism $A^\circ \to I^\circ$.

Pf: Cartan - Eilenberg.

Rmk: Note that if $\mathcal{A}$ is simply additive, we may still define $K^*(\mathcal{A})$ since hot. eq. only uses additive structure.

Prop.- Suppose $\mathcal{A}$ = abelian contains enough injectives. Then there is an equivalence $i : K^+(\mathcal{I}) \xrightarrow{\sim} \mathcal{D}^+(\mathcal{A})$ where $\mathcal{I}$ = addl subcategory of $\mathcal{A}$ of injectives.

Lemma 1: Suppose $A^\circ \to B^\circ$ is a q-ism in $Kom^+(\mathcal{A})$. For any $I^\circ \in Kom^+(\mathcal{A})$, I^i = injective $\forall i$, the induced map

$$Hom_{K^+(\mathcal{A})}(B^\circ, I^\circ) \xrightarrow{\sim} Hom_{K^+(\mathcal{A})}(A^\circ, I^\circ) \text{ is an isomorphism.}$$

Lemma 2: Let $A^\circ, I^\circ \in Kom^+(\mathcal{A})$, I^j = inj $\forall j$

Then $Hom_{K^+(\mathcal{A})}(A^\circ, I^\circ) = Hom_{D^+(\mathcal{A})}(A^\circ, I^\circ)$.

Proof. ① i is fully-faithful by lemma 2.
② i is essentially surj (ie any other object in $D^+(A)$ is isom. to an object of the form $i(I^\cdot)$) since A has enough inj. •

Proof of lemma 2: $\exists$ a natural map $\mathrm{Hom}_{K^+(A)}(A^\cdot, I^\cdot) \longrightarrow \mathrm{Hom}_{D^+(A)}(A^\cdot, I^\cdot)$

It suffices to show that given a roof

$$\begin{tikzcd} & B^\cdot \arrow[dl, "\text{qism}"'] \arrow[dr] & \\ A^\cdot \arrow[rr, dashed] & & I^\cdot \end{tikzcd}$$

$\exists$ a unique arrow $A^\cdot \longrightarrow I^\cdot$ so that the diagram commutes ie $\exists$ nat'l bijection $\mathrm{Hom}(A, I) \xrightarrow{\sim} \mathrm{Hom}(B, I)$ (True by lemma 1). •

Proof of lemma 1: Consider the exact Δ: $A^\cdot \longrightarrow B^\cdot \longrightarrow C^\cdot \longrightarrow A[1]$

$\Rightarrow \quad \cdots \to H^i(A) \xrightarrow{\sim} H^i(B) \to \underset{0}{H^i(C)} \to H^{i+1}(A) \xrightarrow{\sim} H^{i+1}(B) \longrightarrow \cdots$
ie $C^\cdot$ is acyclic.

We also have $\cdots \to \mathrm{Hom}_{K^+(A)}(C, I) \to \mathrm{Hom}_{K^+(A)}(B, I) \to \mathrm{Hom}_{K^+(A)}(A, I) \to \mathrm{Hom}_{K^+(A)}(C, I[1]) \to \cdots$

(General Fact: In a triang category, if $A \to B \to C \to A[1]$ is a dist. Δ
and I is an object $\Rightarrow$
$\cdots \longrightarrow \mathrm{Hom}(C, I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(A, I) \to \mathrm{Hom}(C, I[1]) \to \cdots$
is exact)

Enough to show $\mathrm{Hom}^{K^+(A)}(C, I^\cdot) = 0$ if $C^\cdot$ is acyclic, ie, if $g: C^\cdot \to I^\cdot \in \mathrm{Hom}_{Kom(A)}(C, I)$ then $g \sim 0$.
We will construct a homotopy h by induction on degree.

$$\begin{tikzcd}
\cdots \arrow[r] & C^{i-2} \arrow[r, "d^{i-2}"] \arrow[d, "g^{i-2}"'] & C^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "g^{i-1}"] \arrow[dl, "h^{i-1}"'] & C^{i} \arrow[r] \arrow[d, "g^{i}"] \arrow[dl, "h^{i}"'] & \cdots \\
\cdots \arrow[r] & I^{i-2} \arrow[r] & I^{i-1} \arrow[r] & I^{i} \arrow[r] & \cdots
\end{tikzcd}$$

Suppose h^j has been constructed $\forall j \leq i$.

Consider $g^i - d_I^{i-1} \circ h^i : C^i \to I^i$

$\Rightarrow$ $(g^i - d_I^{i-1} \circ h^i) \circ d^{i-1} = d_I^{i-1} \circ g^{i-1} - d_I^{i-1} \circ h^i \circ d^{i-1}$

But hyp. ind: $g^{i-1} = h^i \circ d^{i-1} + d_I^{i-2} \circ h^{i-1}$

$\therefore$ $d_I^{i-1} \circ g^{i-1} = d_I^{i-1} \circ h^i \circ d^{i-1} + 0$ $\quad \therefore$ $(g^i - d_I^{i-1} \circ h^i) \circ d^{i-1} = 0$

$\Rightarrow$ $g^i - d_I^{i-1} \circ h^i$ descends to $C^i / C^{i-1} \longrightarrow I^i$

Note ~~thus~~ we have an injection $C^i / C^{i-1} \hookrightarrow C^{i+1}$ as $C^\bullet$ is acyclic

$$\begin{tikzcd}
C^i / C^{i-1} \arrow[r, hook] \arrow[d] & C^{i+1} \arrow[dl, "h^{i+1}"] \\
I^i &
\end{tikzcd}$$

$h^{i+1} \leftarrow$ exists!

By inj of I^i, $\exists\, h^{i+1} : C^{i+1} \to I^i$. By comm, $g^i - d_I^{i-1} \circ h^i = h^{i+1} \circ d^i$

Derived functors

→ Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor, left exact. This defines

$F : K^+(\mathcal{A}) \to K^+(\mathcal{B})$

$(\dots \to A^i \to A^{i+1} \to \dots) \mapsto (\dots \to F(A^i) \to F(A^{i+1}) \to \dots)$

However if F is not exact, F ~~need~~ will not preserve acyclic objects, and will therefore not descent to a functor $\mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$.

Assuming $\mathcal{A}$ has enough inj, we define the Right-derived functor $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ which is the "best approx. to F".

Let $i^{-1} : \mathcal{D}^+(\mathcal{A}) \longrightarrow K^+(\mathcal{I})$ be the quasi inverse of i, we have the diag :

$$\begin{tikzcd} \mathcal{D}^+(\mathcal{A}) \arrow[r, shift left, "i^{-1}"] & K^+(\mathcal{I}) \arrow[l, shift left, "i"] \arrow[r, hook] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[ll, bend left, "Q_A"] & K^+(\mathcal{B}) \arrow[r, "Q_B"] & \mathcal{D}^+(\mathcal{B}) \end{tikzcd}$$

Def/- The <u>right derived functor</u> RF is defined as $Q_\beta \circ K(F) \circ i^{-1} : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$.

Prop : (i) There is a nat'l morphism of functors $Q_\beta \circ K(F) \to RF \circ Q_A$.
(ii) $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$
(iii) Suppose $\mathcal{G} : \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{D}^+(\mathcal{B})$ is an exact functor. Then any morphism of functors $Q_\beta \circ K(F) \to \mathcal{G} \circ Q_A$ functor through a unique morpism $RF \to \mathcal{G}$.

Proof/- (i) Let $A \in K^+(\mathcal{A})$ and $I = i^{-1}(A^\cdot)$. The natural transf. $1 \longrightarrow i \circ i^{-1}$ gives a functorial morphism $A \to i(I) = I^\cdot$ in $\mathcal{D}^+(\mathcal{A})$ given by a roof

$$\begin{tikzcd} & C \arrow[dl, "\text{quasi}"'] \arrow[dr] & \\ A & & I^\cdot \end{tikzcd} \quad \text{in } K^+(\mathcal{A})$$

By lemma 2, as $C \to A$ is quism and $I^\cdot$ is inj obj, $\exists!$ arrow $A \to I$ (up to homot.) s.t. that the diagram commutes. Thus we obtain a functorial morgism $KF(A) \to KF(I) = RF(A)$.

(ii) RF is exact being a comp. of exact functors,

(iii) Gelfand-Manin.

Def/✓ Let $RF : \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ be right derived functor of a left exact functor $F : \mathcal{A} \to \mathcal{B}$. Then for any $A^\cdot \in \mathcal{A}$ define $R^iF(A) = H^i(RF(A)) \in \mathcal{B}$

Exts & Homs: Let $A \in \mathcal{A}$, $F = \mathrm{Hom}(A, -): \mathcal{A} \to \mathcal{Ab}$. Then F is left exact, and if $\mathcal{A}$ has enough injectives, we can define
$RF: D^+(\mathcal{A}) \to D^+(\mathcal{Ab})$.

Define $\mathrm{Ext}^i(A, B^\cdot) = H^i(R\mathrm{Hom}(A, B^\cdot))$.

Prop. Suppose $A, B^\cdot \in \mathrm{Kom}(\mathcal{A})$, $\mathcal{A}$ with enough injectives. Then, there are nat'l isoms $\mathrm{Ext}^i_{\mathcal{A}}(A, B^\cdot) \simeq \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[i])$.

proof. Let $B^\cdot \to I^\cdot$ be injective resolution (quasi-isomorphism)
Then, $R\mathrm{Hom}(A, B^\cdot)$ is the complex.

$$\cdots \to \mathrm{Hom}(A, I^i) \xrightarrow{\delta^i} \mathrm{Hom}(A, I^{i+1}) \to \cdots$$
$$f \longmapsto d_I^i \circ f$$

$$\begin{tikzcd}
 & & 0 & & \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & I^{i-1} \arrow[r] & I^i \arrow[r] & I^{i+1} \arrow[r] & \cdots \\
 & & 0 & &
\end{tikzcd}$$

f gives that.

$\mathrm{Ext}^i(A, B^\cdot) = H^i(\mathrm{Hom}^\cdot(A, I^\cdot))$

f is a cycle in $\mathrm{Hom}(A, I^i)$
iff $f: A \to I[i]$ is a morphism complex, if f is a boundary $\delta^{i-1}(h)$ in $\mathrm{Hom}(A, I^{i-1})$ Then $f \sim 0$

$\therefore \quad H^i(\mathrm{Hom}(A, I^\cdot)) = \mathrm{Hom}_{K^+(\mathcal{A})}(A, I[i]) = \mathrm{Hom}_{D^+(\mathcal{A})}(A, B[i])$ (Lemma 1/2) ■

General statements:

① $\mathcal{A}, \mathcal{B}$ = ab. cat, suppose $F: K^+(\mathcal{A}) \to K^+(\mathcal{B})$ is exact functor. Then the right derived functor $RF: D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists whenever there is a full Δed subcat $K_F \subseteq K^+(\mathcal{A})$ adapted to F, ie

(i) if $A \in K_F$ is acyclic, then $F(A)$ is acyclic

(ii) any object in $K^+(A)$ is quasi-isom to an obj. of K_F.

Ex. Lf^* requires projectives, but there are not enough proj. in Alg. Geom. in general
$f: X \to Y$ smooth proj can work with vector bundle, ie, $K_f \subseteq Q(\mathrm{Coh}(Y))$
$K_f = \{\text{loc. free sheaves}\}$.