~~$f: X \to Y$ between smooth & proj. Can work with vector bundles~~
~~i.e. $K_f \in Q(Coh(Y))$, $K_f = \{$loc. free$\}$.~~

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$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

→ X, Y schemes F flat $\mathcal{O}$-module on $X \times Y$

$$\Phi_{X\to Y, F} : Mod(X) \to Mod(Y), \quad M \mapsto \pi_{Y*}(F \otimes \pi_X^*(M))$$

This induces $R\Phi_{X\to Y,F} : D^+(X) \to D^+(Y)$

Ex): $f: X \to Y$ $\quad \Phi_{X\to Y, \mathcal{O}_{\Gamma_f}} = f_*$ $\quad \Phi_{Y\to X, \mathcal{O}_{\Gamma_f}} = f^*$, Γ_f = graph of f

Prop: Z scheme, G object on $Y \times Z$. Then

$$R\Phi_{Y\to Z, G} \circ R\Phi_{X\to Y,F} \simeq R_{X\to Z, H} \quad H = R\pi_{X,Z,*}(\pi_{X,Y}^* F \overset{L}{\otimes} \pi_{Y,Z}^* G)$$

→ X abelian variety, $\mathcal{P}$: $Sch/k \to Sets$
$S \longmapsto \left\{ \begin{array}{l} \text{line bundles } L \text{ on } X \times_k S \text{ with} \\ \text{iso. } \sigma: L|_{\{0\}\times S} \simeq \mathcal{O}_S \end{array} \right\}$

Thm: $\mathcal{P}$ is representable by a group scheme $Pic(X)$ over k.

$Pic^0(X)$:= connected component of $Pic(X)$ that contain 0
↳ it is an abelian variety.

$S \in Sch/k$, $\mathcal{P}(S) \overset{\simeq}{\longleftrightarrow} Hom(S, Pic X)$

Ex): $i: Pic^0(X) \hookrightarrow Pic(X)$ gives a line bundle P on $X \times Pic^0(X)$
st. $P|_{\{0\}\times Pic^0(X)}$ is trivial.

P := Poincaré bundle

For $f: S \to \mathrm{Pic}^0(X)$, the line bundle associated to f is $(1\times f)^*P$

$\longrightarrow$ $\hat{X} := \mathrm{Pic}^0(X)$. $X, \hat{X}$ are isogenous but not nec. isomorphic.

Let $\phi = \Phi_{\hat{X}\to X, P} = \pi_{X,*}(P \otimes \pi_{\hat{X}}^* \,?)$

$\hat{\phi} = \Phi_{X\to\hat{X}, P}$

<u>Thm</u>: There are isomorphisms $R\phi \circ R\hat{\phi} \simeq (-1_X)^*[-g]$ and $R\hat{\phi}\circ R\phi \simeq (-1_{\hat{X}})^*[-g]$. In particular, $R\phi$ is an equivalence between $D(X)$ and $D(\hat{X})$.

<u>Proof</u>: $R\phi \circ R\hat{\phi} \simeq R\Phi_{X\to X, H}$ $\quad H = R p_{12,*}(p_{13}^* P \otimes p_{23}^* P)$

$$\begin{tikzcd} & X\times X\times\hat{X} \arrow[dl, "p_{12}"'] \arrow[dr, "p_{23}"] & \\ X\times X & & X\times\hat{X} \end{tikzcd}$$

$H \simeq m^* R p_{1,*} P$, $\quad m: X\times X \to X$ addition

(Mukai's result)

$$\begin{tikzcd} & X\times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

By Mumford: $R p_{1,*} P \simeq K(0)[-g]$ $\quad \left(\text{Mumford } H^i(X\times\hat{X}, P) = \begin{cases} K(0) & i = g \\ 0 & i \neq g \end{cases} \right)$

$\therefore$ $m^* R p_{1,*} P \simeq m^* K(0)[-g] \simeq \mathcal{O}_{\Gamma_{-1}}[-g]$

$\Rightarrow$ $R\Phi_{X\to X, H} \simeq R\Phi_{X\to X, \mathcal{O}_{\Gamma_{-1}}[-g]} \simeq (-1_X)^*[-g]$ ■

$\longrightarrow$ $\mathrm{Mod}(X) \times \mathrm{Mod}(X) \xrightarrow{\overset{R}{\star}} \mathrm{Mod}(X)$

$(M, N) \mapsto m_*(p_1^* M \otimes p_2^* N)$

(right derived functor)

$$\begin{tikzcd} & X\times X \arrow[dl, "p_1"'] \arrow[d, "m"] \arrow[dr, "p_2"] & \\ X & X & X \end{tikzcd}$$

<u>Prop</u>: $R\phi(F \overset{R}{\star} ?) \simeq R\phi(F) \overset{L}{\otimes} R\phi(?)$

$R\phi(F \overset{L}{\otimes} ?) \simeq R\phi(F) \overset{R}{\star} R\phi(?)[g]$

<u>Def:</u> • $\mathcal{F} \in Coh(X)$, X = abelian variety, $\mathcal{F}$ is a WIT-sheaf of index i if $R^j\phi(\mathcal{F}) = \begin{cases} 0 & j \neq i \\ \neq 0 & j = i \end{cases}$

($\Leftrightarrow$ $R\phi(\mathcal{F}) \simeq \mathcal{G}[-i]$ $\mathcal{G}$ sheaf)

• $\mathcal{F}$ is an IT-sheaf of index i if $H^j(X, \mathcal{F}\otimes M) = \begin{cases} 0 & j\neq i \\ \neq 0 & j = i \end{cases}$ $\forall M \in Pic^0(X)$

($IT_i \Rightarrow WIT_i$).

<u>Ex:</u> $\hat{x} \in \hat{X}$, $k(\hat{x})$ = skyscraper sheaf of $k(\hat{x})$ supported at $\hat{x}$.

$$\underset{M\in Pic^0(\hat{X})}{H^i(\hat{X}, k(\hat{x})\otimes M)} = H^i(\hat{X}, k(\hat{x})) = \begin{cases} 0 & i>0 \\ k(\hat{x}) & i=0 \end{cases} \Rightarrow k(\hat{x}) \text{ is an IT sheaf of index } 0$$

$$R\phi(k(\hat{x})) \simeq R^0\phi(k(\hat{x})) \simeq \phi(k(\hat{x})) = \pi_{X,*}(P\otimes \pi_{\hat{X}}^* k(\hat{x})) \simeq \pi_{X,*}(P\otimes \mathcal{O}_{X\times\{\hat{x}\}})$$
$$\simeq \pi_{X,*}(P|_{X\times\{\hat{x}\}}) = P_{\hat{x}} \quad (= \hat{x})$$

<u>Obs</u>: $P_{\hat{x}}$ is not IT, $P_{\hat{x}} \in Pic^0(X)$
$P_{\hat{x}} \otimes P_{-\hat{x}} \simeq \mathcal{O}_X$ $H^0(X, \mathcal{O}_X) = k$ $H^1(X, \mathcal{O}_X) \neq 0$.

<u>Cor. to Thm.</u>: If WIT_i holds for $\mathcal{F}$ and $\hat{\mathcal{F}} = R^i\phi(\mathcal{F})$. Then it does for $\hat{\mathcal{F}}$ and $i(\hat{\mathcal{F}}) = g - i(\mathcal{F})$, and $\hat{\hat{\mathcal{F}}} \simeq (-1)^*\mathcal{F}$.

<u>Ex:</u> E vector bundle on X. E is homogeneous if $t_x^* E \simeq E$ (in abelian, there are all in Pic^0)

<u>Prop:</u> $R^g\phi$ gives an equivalence between the category of homog. vector bundles on $\hat{X}$ and the cat. of Coh sheaves with finite support.

Let (X, L) be a principally pol. ab. var. (ppav) (i.e. $X \xrightarrow[\sim]{\text{rat}} \hat{X}$)

<u>Thm</u>: (1) $(R\phi)^2 \simeq (-1_X)^*[-g]$
(2) $\hat{L} \simeq L^{-1}$ $\quad \widehat{L^{-1}} \simeq (-1)^* L$
(3) $(\otimes L \circ R\phi)^3 \simeq [-g]$

I.e. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $SL(2,\mathbb{Z})$ have same relations. So $SL(2,\mathbb{Z})$ acts on $D(X)$ modulo shifts.