Robert A. 10/10/14 (1)

→ (A,Θ) principally polarized ab. var. $|_{k=\bar{k}}$, char $k \neq 2$.
Θ divisor $\dim \Gamma(A, \mathcal{O}_A(\Theta)) = 1$, Θ irreducible, $(-1)^*\mathcal{O}_A(\Theta) \cong \mathcal{O}_A(\Theta)$

→ Let $\varphi : A \to |2\Theta| \cong \mathbb{P}^{2^g-1}$ be the morphism assoc. to 2Θ.

Def: Kummer variety of $A = Km(A) := A/\langle -1 \rangle$.

Thm: φ factors as
$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathbb{P}^{2^g-1} \\ & {\scriptstyle \pi}\searrow & \nearrow{\scriptstyle \phi} \\ & & Km(A) \end{array}$$
ϕ embedding.

→ Let $A = Jac(C)$, C not hyperelliptic, $p_0 \in C$ $\quad \alpha_{p_0} : C \hookrightarrow Jac(C) = Pic^0(C)$, $q \mapsto [q - p_0]$

Prop: $\Theta \cap t^*_{p-q}\Theta \subseteq t^*_{p-r}\Theta \cup t^*_{s-q}\Theta$, $\forall p \neq q, r, s \in C$
and ↑ is reducible.

If $H^0(J, 2\Theta) = \langle \Theta_1, \dots, \Theta_{2^g} \rangle$, the previous prop. implies that

$$c_1\,\Theta_\alpha\underbrace{\left(\tfrac{1}{2}p + \tfrac{1}{2}s - \tfrac{1}{2}r - \tfrac{1}{2}q\right)}_{x} + c_2\Theta_\alpha\underbrace{\left(\tfrac{1}{2}p + \tfrac{1}{2}r - \tfrac{1}{2}s - \tfrac{1}{2}q\right)}_{y} + c_3\Theta_\alpha\underbrace{\left(\tfrac{1}{2}p + \tfrac{1}{2}q - \tfrac{1}{2}r - \tfrac{1}{2}s\right)}_{z} = 0$$

for certain constants c_1, c_2, c_3 not all 0, $\forall r = 1, \dots, 2^g$.

$\Rightarrow [\Theta_1(x) : \dots : \Theta_{2^g}(x)], [\Theta_1(y) : \dots : \Theta_{2^g}(y)], [\Theta_1(z) : \dots : \Theta_{2^g}(z)]$ are collinear

Cor: $Km(J)$ has 4-dim family of trisecant lines.

Thm: (Krichever) If $Km(A)$ has one trisecant $\Rightarrow (A,\Theta)$ is isomorphic to a Jacobian.

Def: (A,Θ) is a Prym variety if $\exists$ Jac$(J,\tilde{\Theta})$ s.t. $A \hookrightarrow J$ and $\tilde{\Theta}|_A \equiv 2\Theta$.

Thm : W_1 = image of C in $\mathrm{Jac}(C)$. $c_1, c_2, c_3 \in W_1$, $x \in \mathrm{Jac}(C)$
Then,
$$2x + c_1 + c_2 + c_3 \in W_1 \iff \varphi(x+c_1), \varphi(x+c_2), \varphi(x+c_3) \text{ are colinear in } \mathbb{P}^{2^g-1}$$

(A, Θ) ppav Θ irred. symm. . $H \subseteq A$ finite closed subscheme, $\mathcal{L}$ ample line bundle on A.

Recall : A sheaf $\mathcal{F}$ on A is IT_i if $H^j(A, \mathcal{F} \otimes P) = \begin{cases} \neq 0 & j = i \\ = 0 & j \neq i \end{cases}$
$\forall\, P \in \mathrm{Pic}^0(A)$.

Recall : $\mathcal{F}$ $IT_i \Rightarrow R\phi(\mathcal{F}) := R\pi_{\hat{A},*}(\mathcal{P} \overset{L}{\otimes} \pi_A^* \mathcal{F})$, $\mathcal{P}$: Poincaré line bundle on $A \times \hat{A}$, then $R\phi(\mathcal{F}) \simeq \hat{\mathcal{F}}[i]$ and $\hat{\mathcal{F}}$ is locally free.

Recall : We have $A \simeq \hat{A}$, so we identify them.
(Mukai) $\Rightarrow$ ($\mathcal{L}$ ample $\Rightarrow \mathcal{L}$ is IT_0).
$\Rightarrow \hat{\mathcal{L}} := R\phi(\mathcal{L})$ is locally free , $\mathcal{L} \to \mathcal{L}|_H$ restriction
$\mathcal{L}|_H$ is also IT_0.
Let $\alpha(\mathcal{L}, H) : \underbrace{R\phi(\mathcal{L}) \longrightarrow R\phi(\mathcal{L}|_H)}_{\text{morphisms of sheaves}}$ correspond to restriction.

Def : $Z^i(\mathcal{L}, H) :=$ zero locus of $\Lambda^i \alpha(\mathcal{L}, H)$.

$Z^i(\mathcal{L}, H) \subseteq Z^{i+1}(\mathcal{L}, H)$, $Z^{i+1}(\mathcal{L}, H) \setminus Z^i(\mathcal{L}, H)$
consists of points where rank $\alpha(\mathcal{L}, H)$ is i.

$\rightarrow$ $\Delta \subseteq A \times A$ diagonal

$$\beta(\mathcal{L}, H) : \Gamma(A, \mathcal{L}) \otimes_K \mathcal{O}_A \longrightarrow \pi_{2*}(\pi_1^* \mathcal{L}|_{(H,0)+\Delta})$$
restriction $\left(\overset{\shortparallel}{\pi_{2*}(\pi_1^* \mathcal{L})} \right)$

$\mathcal{U}^i(\mathcal{L},H) :=$ zero locus of $\Lambda^i \beta(\mathcal{L},H)$

Lemma: Let $\phi_{\mathcal{L}} : A \to \hat{A} \simeq A$ given by $\phi_{\mathcal{L}}(x) = \phi_{\Theta}^{-1}(t_x^*\mathcal{L} \otimes \mathcal{L}^{-1})$.
Then $\phi_{\mathcal{L}}^{-1}(Z^i(\mathcal{L},H)) = U^i(\mathcal{L},H)$.

Thm: Let $\mathcal{L} = \mathcal{O}_A(2\Theta)$, $H = \{c_1, \dots, c_{n+2}\}$. Let $\{\Theta_\sigma : \sigma \in (\mathbb{Z}/2\mathbb{Z})^g\}$ a basis for $\Gamma(A, 2\Theta)$. Let $\zeta \in A$ s.t. $2\zeta = c_1 + \dots + c_{n+2}$.
Then,
$U^{n+2}(\mathcal{L},H)$ is scheme theor. defined by the equations
$\det(\Theta_{\sigma_{\lambda_i}}(z - \zeta + c_j)) = 0 \quad \forall (\sigma_{\lambda_1}, \dots, \sigma_{\lambda_{n+2}}) \in ((\mathbb{Z}/2)^g)^{n+2}$

$\rightarrow$ Kempf $\Rightarrow$ $Z^{n+2}(\mathcal{L},H)$ is a translation of $-W_n$ where W_n is where W_n is the image of $\mathrm{Sym}^n C$ in J in particular $Z^3(\mathcal{L},H)$ is a translate of $C = -W_1$.

$\mathcal{L} = \mathcal{O}_A(2\Theta)$, $H = \{c_1, \dots, c_{n+2}\} \subseteq C$.

Robert thinks: $H = \{c_1, c_2, c_3\} \subseteq C$
$\Rightarrow$ the points of $U^3(\mathcal{L},H) = 2^{-1}(Z^3(\mathcal{L},H))$ are those x st. $\varphi(x+c_1), \varphi(x+c_2), \varphi(x+c_3)$ are collinear.

Thm: (A,Θ) is a Jacobian iff $\exists$ at least one point on $Z^3(\mathcal{L}, c_1, c_2, c_3)$ that is not 2-torsion.