→ Preliminaries : (D, T_D) graded category (equivalence), $F: D \to D'$ functor between graded categories is called graded if there exists a natural isomorphism $t_F : F \circ T_D \xrightarrow{\sim} T_{D'} \circ F$.

→ A natural transformation μ between graded functors is called graded if

$$\begin{tikzcd} F \circ T \arrow[r, "t_F"] \arrow[d, "\mu^T"'] \arrow[dr, phantom, "\circ"] & T \circ F \arrow[d, "T\mu"] \\ G \circ T \arrow[r, "t_G"] & T \circ G \end{tikzcd}$$

→ $F: D \to D'$ functor is graded and exact if $X \to Y \to Z \to TX$
$\Rightarrow$ $FX \to FY \to FZ \to FTX = T(FX)$ is exact.

Def. (Serre Functor) $\mathcal{D}$ k-linear category (k a field) with finite dimensional Hom's. A covariant additive functor $S: \mathcal{D} \to \mathcal{D}$ is called a Serre functor if

- ·) S is an equivalence.
- ·) There are bi-functorial isoms $\varphi_{A,B} : \mathrm{Hom}_{\mathcal{D}}(A,B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(A, SB)^* \ \forall A,B \in \mathcal{D}$.

Properties :

1) Any autoequivalence $\Phi: \mathcal{D} \to \mathcal{D}$ commutes with a Serre functor
$$\Phi \circ S \xrightarrow{\sim} S \circ \Phi$$
2) If $\mathcal{D}$ is Δed $\Rightarrow$ S is exact.
3) Any two Serre functors are connected by a canonical functorial isomorphism with commutes with $\varphi_{A,B}$.

Proof(3) : S, S' two Serre functor in $\mathcal{D}$, $A \in ob(\mathcal{D})$.

$$\mathrm{Hom}(A,A) \simeq \mathrm{Hom}(A, S(A))^* \simeq \mathrm{Hom}(S(A), S'(A))$$
$$1_A^{\varphi} \rightsquigarrow SA \xrightarrow{\sim} S'A$$

<u>Reconstruction</u> : X = smooth ~~alg.~~ proj. variety$/_k$, $n = \dim X$, $\mathcal{D} = \mathcal{D}^b_{coh}(X)$
ω_X = canonical sheaf
$\Rightarrow$ The Serre functor is $(\cdot)\otimes\omega_X[n]$ due to the Serre-Grothendieck duality $Ext^i(F,G) = Ext^{n-i}(G, F\otimes\omega_X)^*$.

$\rightarrow$ For a closed point $x \in X$, let $k(x)$ be the residue field.
Notation : $Hom^i(P,Q) = Ext^i(P,Q) = Hom(P, Q[i])$.

<u>Defn</u> A <u>point object</u> in $\mathcal{D}$ in codim s is an obj. P of $\mathcal{D}$ that satisfies
i) $SP \simeq P[s]$
ii) $Hom^{<0}(P,P) = 0$
iii) $Hom^0(P,P) = K(P)$ is a field extension of k.

<u>Prop</u>: (Ident. of closed points) Say ω_X is ample or antiample. Then $P \in D^b_{coh}(X)$ is a point object (codim n is the only possible) iff $P \cong \mathcal{O}_x[r]$, $r\in\mathbb{Z}$, isomorphic to the skyscraper sheaf of a closed point $x \in X$.

proof, $x \in X$ closed pt $\Rightarrow$ $k(x)$ skyscraper sheaf $\Rightarrow$ $k(x)$ is a pt obj in $D^b_{coh}(X)$ of codim n.
i) $k(x)\otimes\omega_X[n] \simeq k(x)[n]$ ii) $Hom^{<0}(k(x),k(x)) = 0$ iii) $Hom^0(k(x),k(x)) = k(x)$.

Let P be a point object in D of codim s. Let $\mathcal{H}^i$ be cohomology sheaves of P.

(i) $s = n$ (easy) and $\mathcal{H}^i\otimes\omega_X = \mathcal{H}^i$ by definition
Since ω_X is ample or antiample, $\mathcal{H}^i$ are finite lenght sheaves $\operatorname{supp}\mathcal{H}^i = \{$isolated closed pts$\}$.
$\Rightarrow$ Support P = finite points (bounded der. Cat.)

Ex: $P \in D^b(\mathrm{coh}(X))$, with $\mathcal{H}^i$ sup is 0 dim $\forall i$
with at least two different pts $\Rightarrow$ P decompose
as $P_1 \oplus P_2$ $\Rightarrow$ $\mathrm{Hom}(P,P)$ cannot be a field.

Consider the spectral sequence : $E_2^{p,q} = \bigoplus_{k-j=q} \mathrm{Ext}^p(\mathcal{H}^j, \mathcal{H}^k) \Rightarrow \mathrm{Hom}^{p+q}(P,P)$

$\mathcal{F}, \mathcal{G}$ finite length supported on the same single pts
$\Rightarrow$ $\exists$ nontrivial from one to the other it sends the generators of the first to the socle of the second.
(M R-mod the socle of M is the sum of the minimal nonzero submod of M).

Consider $\mathrm{Hom}^0(\mathcal{H}^j, \mathcal{H}^k) \neq 0$ with minimal $k-j$ this space survives at E_∞
$\Rightarrow$ $k-j=0$. $\Rightarrow$ all but one cohomology sheaves are trivial ∎