Derived Categories 13/Agosto/19
(Jenia Tevelev)

X smooth proj. varieties $\rightsquigarrow$ $D^b(X)$ bounded derived category.

<u>Q</u>: $X \subset \mathbb{P}^5$ general cubic 4-fold: is it rational ?

« Kuznetsov's proposal » : Semi-orthogonal decompositions in birational geometry.

<u>Rationality</u>: $X \underset{bir}{\simeq} Y$ birational if $k(X) \simeq k(Y) \Leftrightarrow \exists\, U \subset X,\ V \subset Y$ open sets such that $U \simeq V$.

<u>Def/</u> X is called rational if $X \simeq_{bir} \mathbb{P}^n$ ie $k(X) \simeq k(t_1, \dots, t_n)$.
X is called stably rational if $X \times \mathbb{P}^m$ rational for some m.
X is called unirational if $\exists$ dominant rational map $\mathbb{P}^N \dashrightarrow X$
X is called rationally connected if $\forall p, q \in X$ general $\exists\ \mathbb{P}^1 \xrightarrow{f} X$ s.t. $p, q \in f(\mathbb{P}^1)$.

<u>Ej/</u> dim 1 : smooth alg curve C is rational $\Leftrightarrow C = \mathbb{P}^1$ and all are equivalent:
$\Big[$ If $\underset{\text{nonconstant}}{f : \mathbb{P}^1 \to C} \Rightarrow f^* : \underset{\mathbb{C}^g}{H^0(C, K_C)} \hookrightarrow \underset{0}{H^0(\mathbb{P}^1, K_{\mathbb{P}^1})} \Rightarrow g = 0 \Big]$

In fact $h^0(K_X) = \{\text{global holom. } n\text{-forms}\}$, $n = \dim X$, is a birational invariant.

Also plurigenera $h^0(rK_X), \dots$
<u>Exercise</u>: X rationally connected $\Rightarrow$ all plurigenera vanish.

In dim $= 2$, see Castelnuovo Criteria.

Artin-Mumford invariant: Thm $H^3(X, \mathbb{Z})$ not birat. invariant but $\mathrm{Tors}(H^3(X, \mathbb{Z}))$ is a stable birat. invariant, in fact,

$$X\times\mathbb{P}^r \sim_{bir} Y\times\mathbb{P}^s \Rightarrow \mathrm{Tors}(H^3(X,\mathbb{Z})) = \mathrm{Tors}(H^3(Y,\mathbb{Z})).$$

Proof/r $H^3(X\times\mathbb{P}^r,\mathbb{Z}) \simeq H^3(X,\mathbb{Z}) \oplus \underbrace{H^1(X,\mathbb{Z})}_{\text{Has no torsion by Universal coef. thm.}}$

$$\Rightarrow \mathrm{Tors}(H^3(X,\mathbb{Z})) \simeq \mathrm{Tors}(H^3(X\times\mathbb{P}^r,\mathbb{Z}))$$

Left to prove: $X\sim_{bir} Y \Rightarrow \mathrm{Tors}(H^3(X,\mathbb{Z})) \simeq \mathrm{Tors}(H^3(Y,\mathbb{Z}))$

Wlodarczyk's Weak Factorization

Thm: $X\sim_{bir} Y \iff \exists$ sequence

90's

$$\begin{tikzcd}[column sep=small] & Z_1 \arrow[dl] \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & & \cdots & & Z_n \arrow[dl] \arrow[dr] & \\ X_0 = X & & X_1 & & X_2 & & \cdots & & X_n = Y \end{tikzcd}$$

where every morphism $Z_i\to X_i$ or $Z_i\to X_{i-1}$ is a blow-up at a smooth subvarieties:

$$\begin{tikzcd} \mathrm{Exc} = D \arrow[r, phantom, "\subset"] \arrow[d] & Bl_W(X) \arrow[d] \\ W \arrow[r, phantom, "\underset{\text{subv.}}{\subset}"] & X \end{tikzcd} \qquad X\setminus W \simeq Bl_W X\setminus D.$$

Therefore it suffices to show that $\mathrm{Tors}\, H^3(X,\mathbb{Z}) = \mathrm{Tors}\, H^3(Bl_W X,\mathbb{Z})$

$$H^3(Bl_W X,\mathbb{Z}) \simeq H^3(X,\mathbb{Z}) \oplus \underbrace{H^1(W,\mathbb{Z})}_{\text{No torsion}}.$$

Artin-Mumford example: $S\subset\mathbb{P}^3$ quartic surface $p\in S$ node (A_1 sing.)
« K3 surface »

$$X \xrightarrow{2:1} \mathbb{P}^3 \text{ ramified over } S.$$

Near p, X has local equation $w^2 = f(x,y,z)$ where $f(x,y,z)$ local equation of a quartic, Then X also has a node at p.

Projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ ~~proj~~ from p, then it is double cover $S \overset{2:1}{\dashrightarrow} \mathbb{P}^2$

$\Rightarrow Bl_p S \xrightarrow{2:1} \mathbb{P}^2$ ramified along a sextic curve. Then Artin-Mumford impose the condition $C = E_1\cup E_2$

$\Rightarrow$ S has 10 nodes total.

Artin-Mumford: $H^3(X,\mathbb{Z})$ has torsion (if $C = E_1 \cup E_2$) and so it is not rational.

Simple geometric argument shows that X has a rational double cover $\Rightarrow$ X is unirational (true any C)

Claire Voisin: X is not rational for a general C.

<u>Semi-orthogonal decompositions</u>: Linear algebra version

- V vector space ($\dim V < \infty$) with a bilinear form $(\cdot,\cdot)$
  $(u,v) = u^t B v$, B = Gram matrix
  <u>Do not</u> assume that B is symmetric or skew-symmetric (Calabi-Yau condition on geom. context), but assume it is not degenerate (ie, B invertible $\Leftrightarrow \forall u \in V, u \neq 0, \exists v\ (u,v) \neq 0$).

- $\mathcal{U} \subset V$ is called admissible if $(\cdot,\cdot)/_{\mathcal{U}}$ is not degenerate

Equivalently, $\mathcal{U}^\perp = \{ v \in V : (u,v) = 0\ \forall u \in \mathcal{U} \}$
$^\perp\mathcal{U} = \{ v \in V : (v,u) = 0\ \forall u \in \mathcal{U} \}$

$\mathcal{U}$ is admissible $\Leftrightarrow V = \mathcal{U} \oplus {}^\perp\mathcal{U} \Leftrightarrow V = \mathcal{U}^\perp \oplus \mathcal{U}$ (Two step semi-orthogonal decom.)

Equivalently, $\mathcal{U} \subset V$ is admissible $\Leftrightarrow i : \mathcal{U} \hookrightarrow V$ (linear map) has a left and right adjoint maps $i^*, i^! : V \longrightarrow \mathcal{U}$

$\forall u \in \mathcal{U}, v \in V$ $\quad (i(u), v) = (u, i^!(v)) \quad (v, i(u)) = (i^*(v), u)$

Indeed, $i^!(v)$ is a projection of v onto $\mathcal{U}$ along $^\perp\mathcal{U}$
$V = \mathcal{U} \oplus {}^\perp\mathcal{U}$
$v \mapsto i^!(v)$
and $i^*(v)$ is a projection of v onto $\mathcal{U}$ along $\mathcal{U}^\perp$.

Def 1 A Semi-orthog. decomp (SOD) of V is a decomposition of $V = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots \oplus \mathcal{U}_r$ such that $(\mathcal{U}_i, \mathcal{U}_j) = 0$ $i > j$.

Special case: $\dim \mathcal{U}_i = 1$, $\mathcal{U}_i = \langle e_i \rangle$ we can assume $(e_i, e_i) = 1$

Then we say: Semi-orthonormal basis $\{e_1, \dots, e_n\}$ of V $(e_i, e_i) = 1$ $(e_i, e_j) = 0$ $\forall i > j$.

So Gram matrix $B = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

Example: $B = \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$, $\delta \in \mathbb{Z}_{>0}$.

$\{e_1, e_2\}$ is orthonormal. Define $e_i \in V$ $\forall i \in \mathbb{Z}$,
$e_{i-1} + e_{i+1} = \delta e_i$ $\forall i$

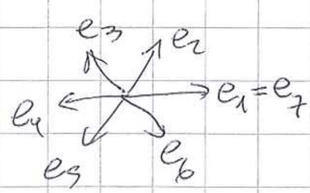
Claim: $\{e_i, e_{i+1}\}$ is also an orthonormal basis with same B.

Induction on i: Suppose $\{e_{i-1}, e_i\}$ and show $\{e_i, e_{i+1}\}$ is orthonormal
$e_{i+1} = \delta e_i - e_{i-1}$

$$(e_{i+1}, e_{i+1}) = \delta^2 \underbrace{(e_i, e_i)}_{1} - \cancel{\delta \underbrace{(e_i, e_{i-1})}_{0}} - \delta \underbrace{(e_{i-1}, e_i)}_{\delta} + (e_{i+1}, e_{i+1}) = 1$$

~~e_1, e_2~~ $\dots, e_{-1}, e_0, e_1, e_2, e_3, e_4, \dots$ mutations.

$\delta = 1$



$\delta > 1 \Rightarrow$ all e_i's are different (∞ many)

Mutations of semi-orthogonal decomp:

$$V = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots \oplus \mathcal{U}_i \oplus \mathcal{U}_{i+1} \oplus \dots \oplus \mathcal{U}_n$$

$$V = \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \dots \oplus \mathcal{U}_i \oplus \mathcal{U}_{i+1} \oplus \dots \oplus \mathcal{U}_n \qquad \text{SOD}$$

$$\xleftarrow{\sigma_i} \qquad \xrightarrow{\sigma_i^{-1}}$$

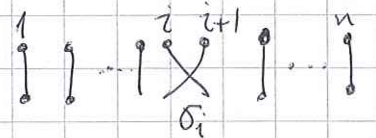
$$\mathcal{U}_1 \oplus \dots \oplus \underbrace{L(\mathcal{U}_{i+1})}_{\mathcal{U}_i} \oplus \mathcal{U}_i \oplus \dots \oplus \mathcal{U}_n \qquad \mathcal{U}_1 \oplus \dots \oplus \mathcal{U}_{i+1} \oplus \underbrace{R(\mathcal{U}_i)}_{\mathcal{U}_{i+1}} \oplus \dots \oplus \mathcal{U}_n$$

unique block
to have SOD

This gives an action of the braid group B_n on the set of SOD.

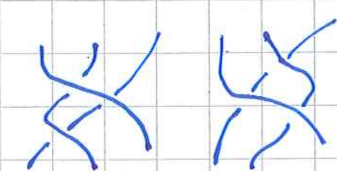
(grupo de trenzas)

Artin generators :



Artin relations : $\sigma_i \sigma_j = \sigma_j \sigma_i \quad i > j+1$

Artin braid relations : $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$



These relations are satisfied for mutations of SOD.