Jenia : "Quotient Stacks and Their derived categories"

$D^b(\mathbb{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \rangle \qquad \mathbb{P}^n = \mathbb{A}^{n+1} \setminus 0 / \mathbb{G}_m$

<u>Thm</u> (Castravet-T) (Conjectured by Manin)

$D^b(\overline{M}_{0,n})$ admits an exceptional collection permuted by S_n.

<u>Thm</u> : $D^b(M_n)$ " " " " " " "

where $M_n = (\mathbb{P}^1)^n /\!/ PGL_2$ $\quad$ n is odd.

GIT by $\mathcal{O}(1,\ldots,1)$

$\overline{M}_{0,n} \longrightarrow \overline{M}_n \quad , \quad \overline{M}_{0,5} = \overline{M}_5 = Bl_4(\mathbb{P}^2)$

Orlov's collection : $\langle \mathcal{O}_{E_i}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$ not S_5 invariant.

<u>Defn</u> F° is generated by $\{G_\alpha^\circ\}_{\alpha \in I}$ if $\exists\, T_1, \ldots, T_s$

$T_i = \{T_i' \to T_i'' \to T_i''' \to \ldots\}$ triangles

$F^\circ \in T_s$ and every T_i contains at least 2 objects from T_j $j<i$ or $\{G_\alpha^*\}$. (up to shifts)

<u>Examples</u>: $\mathcal{F}^\circ = \{0 \ldots \to \mathcal{F}^i \to \mathcal{F}^{i+1} \to \ldots 0\}$ complex bounded both sides

naive truncation $\Rightarrow$ $\mathcal{F}^\circ$ is generated by $\mathcal{F}^i$'s (and shifts!)

Given an exact sequence of sheaves "resolution":

$0 \qquad \cdots \to G_{s-1} \to G_s \to \mathcal{F} \to 0$

$\Rightarrow$ in $D^b(X)$ we have $\mathcal{F} \underset{qis}{\sim} [\cdots \to G_{s-1} \to G_s]$

$\Rightarrow$ $\mathcal{F}$ is generated by G_i's.

Obs: If $\{G^\bullet_\alpha\}$ generates $D^b(X) \Rightarrow \langle G^\bullet_\alpha \rangle^\perp = 0$

[If $H^\bullet \in \langle G^\bullet_\alpha \rangle^\perp$, $\mathrm{Hom}(G^\bullet_\alpha, H^\bullet[s]) = 0 \ \forall s$
$\Rightarrow$ everything generated by $G^\bullet_\alpha$'s $\in {}^\perp H^\bullet$
$\Rightarrow H^\bullet$ is not generated by $G^\bullet_\alpha$'s ~~because~~ because $\mathrm{Hom}(H^\bullet, H^\bullet) \neq 0$.

Bad example: $\{k(x)\}_{x \in X} \Rightarrow {}^\perp\{k(x)\} = 0$ (Nakayama)

But $\{k(x)\}$ do not generate $D^b(X)$ if $\dim X > 0$ since only complexes supported at finitely many points can be generated.

Fact: If $\{G^\bullet_\alpha\}$ is an exceptional collection then
$\{G^\bullet_\alpha\}$ generates $D^b(X) \Leftrightarrow \{G^\bullet_\alpha\}^\perp = 0$.

Basic idea: Suppose $D^b(X)$ is generated by G_α's and $\mathcal{U} \subset X$ open subset $\Rightarrow D^b(\mathcal{U})$ is generated by $G_\alpha|_{\mathcal{U}}$.

[proof: suffices to check any $\mathcal{F} \in \mathrm{Coh}(\mathcal{U})$ is generated. Ex (Hartshorne)
There exists an $\tilde{\mathcal{F}} \in \mathrm{Coh}(X)$ s.t. $\mathcal{F} = \tilde{\mathcal{F}}|_{\mathcal{U}} \Rightarrow$ generate $\tilde{\mathcal{F}}$ by G_α's $\Rightarrow \mathcal{F}$ is generated by $G_\alpha|_{\mathcal{U}}$.
Hint: $\mathcal{U} \overset{i}{\hookrightarrow} X$, $i_*\mathcal{F} = \tilde{\mathcal{F}}$ is quasi-coherent and then we can find a coherent sheaf same restriction]

We would like $\mathbb{P}^n \underset{\text{open}}{\hookrightarrow} \mathcal{X}$ (but $\mathbb{P}^n$ is proj $\Rightarrow \mathbb{P}^n = \mathcal{X}$ for schemes).
but will work if $\mathcal{X}$ is an algebraic stack.

$$\mathbb{P}^n = \mathbb{A}^{n+1} \setminus \{0\} / \mathbb{G}_m \hookrightarrow \underbrace{\mathbb{A}^{n+1} / \mathbb{G}_m}_{\text{quot. stack}}$$

<u>Remark</u> $G \curvearrowright X$ (alg. group, alg var.) $\Rightarrow \exists$ object called quotient stack $[X/G]$

$$\mathrm{Coh}[X/G] = \mathrm{Coh}_G(X)$$ G equivariant coh sheaves on X (abelian category)

$$D^b[X/G] = D^b(\;) = D^b_G(X)$$ bounded complexes of G-equiv coherent sheaves

What is an equiv sheaf?

(1) coherent sheaf $\mathcal{F}$

(2) $\forall g \in G$ we choose an isomorphism $\theta_g : g^*\mathcal{F} \simeq \mathcal{F}$
In particular it should exist!

(3) θ_g should be compatible with the action.

$$\begin{tikzcd} (gh)^*\mathcal{F} = h^* g^*\mathcal{F} \arrow[r, "h^*\theta_g"] \arrow[dr, "\theta_{gh}"', "\sim"] & h^*\mathcal{F} \arrow[d, "\sim\theta_h"] \\ & \mathcal{F} \end{tikzcd} \qquad \forall g, h$$

This is fine if G is finite. In general, θ_g has to be "regular in g".

$G \times X \xrightarrow{a} X$ we want $\theta : a^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$ + analogous diagram.

<u>Special Case</u>: If $\mathcal{F}$ is a line (or vector) bundle then equivariant structure is equivalent to lifting $G \curvearrowright X$ to the action $G \curvearrowright L$ (action on L over pts is linear)

<u>Special subcase</u>: $X = pt$, $F =$ vector space V, $G \curvearrowright pt$ trivially
equivariant structure: $G \curvearrowright V$ linear action

$$\mathrm{Coh}[pt/G] = \{\text{finite-dim alg represent. of } G\}$$

Another special case : $X = \mathrm{Spec}\, R$. $G \curvearrowright X \Leftrightarrow G \curvearrowright R$

~~Ex~~ $\mathrm{Coh}(X) \simeq \{$ f.g. R-mod $\}$ equivariant structure on $\tilde{M}$

$\tilde{M} \leftarrow\!\shortmid M$

is the action of G on M s.t. $g(rm) = g(r)g(m)$ $r \in R, m \in M$ (algebraic)

$D^b(\mathbb{A}^{n+1})$ is generated by $\mathcal{O}$.

since $\mathrm{Coh}(\mathbb{A}^{n+1}) \Leftrightarrow$ f.g. modules over $\overset{=R}{k[x_0, \dots, x_n]} \ni M$

Hilbert syzygy thm : f. resol $0 \to R^{n+2} \to \dots R^{b_2} \to R^{b_1} \to M \to 0$

$\Rightarrow M$ is gen. by R

$D^b(\mathbb{A}^{n+1}/\mathbb{G}_m) \Rightarrow \mathrm{Coh}_{\mathbb{G}_m}(\mathbb{A}^{n+1}) = \{$ f.g mod over R with $\mathbb{G}_m$-action $\}$

But $\mathbb{G}_m \curvearrowright M \Leftrightarrow M = \oplus M_n$

$M_n = \{ m : z \cdot m = z^n m \ , z \in \mathbb{G}_m \}$

Then $\{$ f g graded R-modules $\}$ $R_d M_k \subset M_{d+k}$.

and morphisms are graded homomorphisms.

Hilbert's syzygy thm : $0 \to \dots \to \bigoplus_{i=1}^{s_2} R(-n_i) \to \bigoplus_{i=1}^{s_1} R(-k_i) \to M \to 0$

in $(n+1)$ steps

homog gen.

homog generators degrees $k_1, \dots, k_s$,

$\Rightarrow \mathcal{O}(k) := \tilde{M}(k)$

<u>Lemma</u> : $D^b_{\mathbb{G}_m}(\mathbb{A}^n)$ has a full infinite excep collection $\langle \dots, \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots \rangle$

Hilbert syzygys $\Rightarrow$ generates Then check

$\mathrm{Hom}_{\mathbb{G}_m}(\mathcal{O}(k), \mathcal{O}(m)) = \mathrm{Hom}_{\text{graded } R\text{-mod}}(R(k), R(m))$ but $1 \in R(k) \mapsto$ degree $-k$

so $k > m \Rightarrow \mathrm{Hom} = 0$ $k = m$ $\mathrm{Hom} = \mathbb{C}$ $k < m$ $\mathrm{Hom} = R_{m-k}$.

$\Rightarrow D^b_{\mathbb{G}_m}(\mathbb{A}^{n+1}\setminus\{0\})$ is generated by restr. of $\mathcal{O}(k)|_{\mathbb{A}^{n+1}-\{0\}}$ $\forall k$

$\mathcal{U} \underset{\text{open}}{\subset} X$ every G-equiv coherent sheaf on $\mathcal{U}$ is a restriction of a G-equiv coherent sheaf on X.

<u>Lemma</u>: $G \circlearrowright X$ $\quad \pi : X \to \underset{\text{scheme}}{X/G} = Y$ good quotient

$$\Rightarrow \quad D^b_{\mathbb{G}_m}(X) = D^b(X/G)$$
$$\pi^* \mathcal{F} \to \mathcal{F}$$
$$\mathcal{F} \to (\pi_* \mathcal{F})^G$$

$D^b(\mathbb{P}^n)$ is generated by $\mathcal{O}(k)|_{\mathbb{A}^{n+1}\setminus\{0\}}$ Claim: $\mathbb{A}^{n+1}\setminus 0 \xrightarrow{\pi} \mathbb{P}^n$

$\pi^* \mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}(k)|_{\mathbb{A}^{n+1}\setminus\{0\}}$.

$k=-1$ trivialization $(\mathbb{A}^{n+1}\setminus 0)\times \mathbb{C} \to \pi^*\mathcal{O}(-1)$

$$(v, \lambda) \mapsto (v, \lambda v)$$

How to show that $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ generate $D^b(\mathbb{P}^n)$? We need gen. of $\mathcal{O}(k)$

Use Koszul complex of unstable locus:

$$0 \to R(-n-1) \to \dots \to R^{\binom{n+1}{2}}(-2) \to R^{n+1}(-1) \to R \to R/(x_0, x_1, \dots, x_n) = \mathbb{C} \to 0$$

sheafifying

$$0 \to \mathcal{O}(-n-1) \to \dots \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to \mathcal{O}_0 \to 0$$

Then restrict $\mathbb{A}^{n+1}\setminus\{0\}$: $0 \to \mathcal{O}(-n-1) \to \dots \to \mathcal{O}(-1)^{n+1} \to \mathcal{O} \to 0$ $\quad \otimes \mathcal{O}(k)$

$\Rightarrow$ induction: every $\mathcal{O}(k)$ can be obtained from $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n)$ ∎

Sketch for $D^b((\mathbb{P}^1)^n /\!\!/_{GIT} PGL_2)$. First $D^b(\mathbb{P}^1)^n = \langle \mathcal{O}(\pm E) \rangle_{2^n}$ $E = (0010\dots01)$

$E \subset \{1, \dots, n\}$

$D^b_{PGL_2}((\mathbb{P}^1)^n) = \langle \mathcal{O}(E) \otimes V_\ell \rangle$ $\quad V_\ell = k[x,y]_\ell$ infinite exceptional collection.

$D^b((\mathbb{P}^1)^n_{ss} / PGL_2)$ If n odd $PGL_2 \circlearrowright (\mathbb{P}^1)_{ss}$ free

democratic linear

$\simeq D^b_{PGL_2}((\mathbb{P}^1)^n_{ss})$ $\quad$ <u>Thm</u>: Full excep coll for $D^b(GIT)$

$\mathcal{F}_{\ell, E}$ $\quad \ell + \min(|E|, n - |E|) \le r - 1$

$n = 2r + 1$