X smooth proj. variety $\rightsquigarrow K_0(X) \otimes \mathbb{Q} \rightsquigarrow D^b(X)$

"Mukai pairing"

$K_0(X) = \{$ group generated by $[\mathcal{E}]$ for every vector bundle $\mathcal{E}$ on $X\}$
relations $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_r \to 0$ exact
$\Rightarrow \sum (-1)^i [\mathcal{E}_i] = 0$.

Ej/ $X = \mathbb{P}^1$

Theorem (Grothendieck): Every vector bundle $\mathcal{E} \cong \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1}(a_i)$

$\Rightarrow [\mathcal{E}] = \sum_{i=1}^{s} [\mathcal{O}(a_i)]$ on $K_0(\mathbb{P}^1)$.

$K_0(\mathbb{P}^1)$ is generated by $[\mathcal{O}(a_i)]$. "Koszul seq" $0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$

Then $2[\mathcal{O}_{\mathbb{P}^1}] = [\mathcal{O}(-1)] + [\mathcal{O}(1)]$. By tensoring with $\mathcal{O}(a)$

$\Rightarrow \quad 2[\mathcal{O}(a)] = [\mathcal{O}(a-1)] + [\mathcal{O}(a+1)]$.

$\Rightarrow K_0(\mathbb{P}^1)$ is spanned by $[\mathcal{O}]$ and $[\mathcal{O}(1)]$, in fact
$K_0(\mathbb{P}^1) = \mathbb{Z}[\mathcal{O}] \oplus \mathbb{Z}[\mathcal{O}(1)]$.

To prove lin. independence.

Def/ A function $\{$vector bundles to $X\} \to \mathbb{Z}$ is called additive if $\forall$ exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \dots \to \mathcal{E}_r \to 0$ we have $\sum (-1)^i d(\mathcal{E}_i) = 0$. Such a function gives a function $d : K_0(X) \to \mathbb{Z}$, $d([\mathcal{E}]) = d(\mathcal{E})$ and is linear.

Examples:

(1) $\chi(\mathcal{E}) = \sum (-1)^i h^i(\mathcal{E})$ is an examples.
$\chi(\mathcal{O}_{\mathbb{P}^1}) = 1$, $\chi(\mathcal{O}_{\mathbb{P}^1}(1)) = 2 - 0 = 2$.

(2) $\operatorname{rank}(\mathcal{E})$, $\operatorname{rk}(\mathcal{O}) = \operatorname{rk}(\mathcal{O}(1)) = 1$.

Therefore $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)$ are independent.

<u>Remark</u>: Another set of generators: $[\mathcal{F}]$ for any coherent sheaf $\mathcal{F}$ on X and relations: $[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}]$
$\forall$ S.E.S. $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.

Indeed by <u>Hilbert's Syzygy theorem</u>, Every coherent sheaf $\mathcal{F}$ on X admits a locally free resolution:
$$0 \to \mathcal{E}_r \to \dots \to \mathcal{E}_1 \to \mathcal{F} \to 0 \quad \text{(resolution)}$$
in $K_0(X)$ $\quad [\mathcal{F}] = [\mathcal{E}_1] - [\mathcal{E}_2] + [\mathcal{E}_3] - \dots$ (finite).

<u>Mukai pairing</u> in $K_0(X)$: $\big([\mathcal{E}],[\mathcal{F}]\big) := \sum (-1)^i \dim\big(\mathrm{Ext}^i(\mathcal{E},\mathcal{F})\big)$

vector bundles $\quad$ (definition for coherent sheaves

Then, for vect. bundles, $\big([\mathcal{E}],[\mathcal{F}]\big) = \chi(\mathcal{E}^* \otimes \mathcal{F})$

It is good for K-group:

SES $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \qquad 0 \to \mathcal{E}_3^* \otimes \mathcal{F} \to \mathcal{E}_2^* \otimes \mathcal{F} \to \mathcal{E}_1^* \otimes \mathcal{F} \to 0$ is SES
$$\Rightarrow \langle \mathcal{E}_2, \mathcal{F} \rangle = \langle \mathcal{E}_1, \mathcal{F} \rangle + \langle \mathcal{E}_3, \mathcal{F} \rangle.$$

<u>Example</u>: $X = \mathbb{P}^1$ $\quad \langle \mathcal{O}, \mathcal{O} \rangle = \chi(\mathcal{O}) = 1 \qquad \langle \mathcal{O}(1), \mathcal{O} \rangle = \chi(\mathcal{O}(-1)) = 0$
$$\langle \mathcal{O}(1), \mathcal{O}(1) \rangle = \chi(\mathcal{O}) = 1 \qquad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$\langle \mathcal{O}, \mathcal{O}(1) \rangle = \chi(\mathcal{O}(1)) = 2$$

So $\mathcal{O}, \mathcal{O}(1)$ is a Semi orth basis.

<u>Mutations</u>: $2e_i = e_{i-1} + e_{i+1} \qquad \dots, e_{-2}, e_{-1}, e_0, e_1, e_2, \dots$
$\{e_i, e_{i+1}\}$ is a semi-orth. basis.
In our case $X = \mathbb{P}^1$: $\quad 2[\mathcal{O}(i)] = [\mathcal{O}(i-1)] + [\mathcal{O}(i+1)]$

Then mutations starting from $\mathcal{O}, \mathcal{O}(1)$ produce $\dots, \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \dots$
st. any two consecutive form an s. orthon. basis.

<u>Def</u>: A vector bundle $\mathcal{E}$ is called <u>exceptional</u> if $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = \begin{cases} 1 & i=0 \\ 0 & i>0. \end{cases}$
In particular $\chi(\mathcal{E},\mathcal{E}) = 1$

A sequence $\mathcal{E}_1, \dots, \mathcal{E}_r$ of exceptional vector bundles is called exceptional if $\mathrm{Ext}^p(\mathcal{E}_i, \mathcal{E}_j) = 0$ $\forall p$ if $i > j$.

$$\chi(\mathcal{E}_i, \mathcal{E}_j) = 0$$

$\Rightarrow [\mathcal{E}_1], \dots, [\mathcal{E}_r]$ is a S. Orthonormal set.

Ex: In the infinite sequence $\dots \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ every 2 consecutive line bundles $\mathcal{O}(i)$ $\mathcal{O}(i+1)$ form except. collection.

Prop: Suppose X is a Fano manifold of index r, $-K_X = rH$ where H is an ample divisor. Then $\mathcal{O}, \mathcal{O}(H), \mathcal{O}(2H), \dots, \mathcal{O}((r-1)H)$ is an except. collect. of length r.

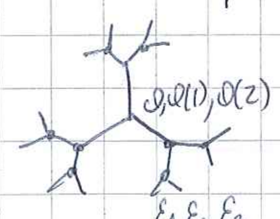
Proof: $\mathrm{Ext}^p(\mathcal{O}(iH), \mathcal{O}(jH)) = H^p(X, \mathcal{O}((j-i)H)) = H^p(X, K_X - K_X + (j-i)H)$
$i \geq j$ $\qquad = H^p(X, K_X + (r+j-i)H)$

Kodaira Vanishing $H^p(X, K_X + \text{ample}) = 0$ $\forall p > 0$.
If $r+j-i > 0 \Rightarrow H^p = 0$ $\forall p > 0$. $j - i \geq -(r-1) \Leftarrow i-j \leq r-1$
So $r+j-i > 0$.

$H^0(X, (j-i)H)$ $i \geq j$. If $i > j \Rightarrow H^0 = 0$. If $i = j$, $H^0(X, \mathcal{O}_X) = \mathbb{C}$.

Example: $X = \mathbb{P}^2$ $K_{\mathbb{P}^2} = -3H \Rightarrow \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ is exceptional collection. In fact a semi-orth. basis at a K-group.

Theorem (Rudakov) (1) Every exceptional vector bundle on $\mathbb{P}^2$ can be extended to a 3-term except. collection.
(2) All 3-term except. collections are related by mutations



$x = \mathrm{rk}(\mathcal{E}_1)$ $y = \mathrm{rk}(\mathcal{E}_2)$ $z = \mathrm{rk}(\mathcal{E}_3)$
Satisfy Markov's equation $x^2 + y^2 + z^2 = 3xyz$
Thm (Markov): All solutions are related by mutations $(x, y, z) \mapsto (x, y, 3xy - z)$
"Quadratic equation".

$(1,1,1)$ $\longrightarrow$ $(1,1,2)$

$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ $\qquad$ $\mathcal{O}, T_{\mathbb{P}^2}(-1), \mathcal{O}(1)$

Conj (Kuznetsov): X is rational $\Leftrightarrow D^b(X)$ does not contain blocks of nongeometric nature.

What is derived category!?

$Coh(X)$ = Coherent sheaves on X is "Abelian Category" $\mathcal{A} \rightsquigarrow Com^b(\mathcal{A})$ = categ. of complexes $d^2 = 0$ where morph comm diagrams $F^\circ \to G^\circ$

Then, $D^b(\mathcal{A})$ has as objects bounded complexes of elements of $\mathcal{A}$ but different morphisms

Main Feature: Suppose $f: \mathcal{F}^\circ \to \mathcal{G}^\circ$ is a morphism of complexes which is a quasi-isomorphism $H^i(f): H^i(\mathcal{F}) \to H^i(\mathcal{G})$ is an isomorphism for every i. $\left[H^i(\mathcal{F}^\circ) = \ker(d_i)/Im(d_{i-1})\right]$

Then the corresponding Morphism in $D^b(X)$ is an isomorphism.

$D^b(X) := D^b(Coh(X))$

Remark: Every coherent sheaf has a loc. free resolution (as before)

$0 \to \mathcal{E}_r \to \dots \to \mathcal{E}_1 \to \mathcal{F} \to 0$ which gives a quasi-isom

$$\begin{array}{ccccccccc} \Rightarrow & \mathcal{E}_r \to & \mathcal{E}_{r-1} \to & \dots \to & \mathcal{E}_2 \to & \mathcal{E}_1 & \to 0 \\ & & & & & \downarrow & \\ & 0 \to & 0 \to & 0 \to & 0 \to & \mathcal{F} & \to 0 \end{array}$$

$\therefore$ they are quasi-isom and so in $D^b(X)$

is isomorphic to a complex of vector bundles: $[\mathcal{E}_r \to \dots \to \mathcal{E}_1 \to 0]$

obj: $D^b(X)$ is an example of a triangulated category.

A S.O.D. of $D^b(X)$ is $\langle \mathcal{A}_1, \dots, \mathcal{A}_s \rangle$ is a collection of admissible subcategories that generate $D^b(X)$ and $Hom(\mathcal{A}_i, \mathcal{A}_j) = 0 \ \forall i > j$

Ex: $D^b(\mathbb{P}^3) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3) \rangle$, $D^b(V_3) = \langle \mathcal{A}, \mathcal{O}(-1), \mathcal{O} \rangle$ where $\mathcal{A}$ is a Calabi-Yau category but not geometric. V_3 irrational

$D^b(V_{2,2}) = \langle D^b(C_2), \mathcal{O}(-H), \mathcal{O} \rangle$ $\quad V_{2,2}$ rational.

$Q_1 \cap Q_2 \subset \mathbb{P}^5$