Seminar : Mutations in Derived Categories

§1 Mutations & Helices

X = del Pezzo surface ie $-K_X$ ample.

(1) Every exceptional collection can be extended to a full exceptional collection $\langle \mathcal{E}_1, .., \mathcal{E}_r \rangle$ in $\mathcal{D}^b(X)$.
(2) Every object in $\langle \mathcal{E}_1, .., \mathcal{E}_r \rangle$ is a sheaf (up to $[s]$)
(3) In fact, $\mathcal{E}_i$ are either excep. vector bundles or $\mathcal{O}_{\mathbb{P}^1}(d)$ $\mathbb{P}^1 \hookrightarrow X$ (-1)-curve.
(4) Braid group acts transitively on full excep. collections.
(5) in $D^b(\mathbb{P}^2)$ exceptional collections $\langle \mathcal{E}, \mathcal{F}, \mathcal{G} \rangle$ of vector (Rudakov) bundles, up to tensoring with line bundle L, are determined by their ranks (e, f, g) which satisfy $e^2 + f^2 + g^2 = 3efg$ "Markov equation".

Theorem : X smooth proj. rational surface. Then $\exists$ a full except. collection of line bundles.

Pf :- Either $X \cong \mathbb{P}^2$ or $\mathbb{F}_n$ or $X = Bl_p(S)$ with exc. divisor E.
$D^b(\mathbb{P}^2) = \langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O} \rangle$. More generally, by Orlov's proj. bundle formula, $X \xrightarrow{\pi} Y$ $\mathbb{P}^r$- bundle
SOD $D^b(X) = \langle \pi^*\mathcal{D}^b(Y)(-r), \dots, \pi^*\mathcal{D}^b(Y)(-1), \pi^*\mathcal{D}^b(Y) \rangle$
so $\mathbb{F}_n \xrightarrow{\pi} \mathbb{P}^1$ projective bundle with fiber F, N = neg. section
$\mathcal{D}^b(\mathbb{F}_n) = \langle \mathcal{O}(-N-F), \mathcal{O}(-N), \mathcal{O}(-F), \mathcal{O} \rangle$
If $X = Bl_p(S) \Rightarrow$ by Orlov's blow-up $X \xrightarrow{\pi} S$ blow-up and induction $\mathcal{D}^b(S) = \langle L_1, .., L_k \rangle$

$$\mathcal{D}^b(X) = \langle \underset{\substack{\| \\ \mathbb{P}^1}}{\mathcal{O}_E(-1)}, \pi^* L_1, \dots, \pi^* L_k \rangle.$$

torsion sheaf

<u>Lemma</u>: $\langle E_1, \dots, E_r \rangle$ exc collection $\Rightarrow \langle E_2, \dots, E_r, E_1(-K) \rangle$ exc. coll.

<u>Proof</u>: $\mathrm{Hom}(E_1(-K), E_i[n]) = \mathrm{Hom}(E_1, E_i(K)[n])$
$\underset{\text{Serre}}{=} \mathrm{Hom}(E_i, E_1[m])^* = 0 \ \forall m, i$.

$E_1, \dots, E_r$ Full coll $\Rightarrow E_2, \dots, E_r, E_1(-K)$ full collection.

$E_1(-K)$ = mutation of E_1 all the way to the right. ∎

Therefore, $\mathcal{D}^b(X) = \langle \pi^* L_1, \dots, \pi^* L_k, \mathcal{O}_E(-1) \otimes \mathcal{O}_X(-K) \rangle$
$= \langle \pi^* L_1, \dots, \pi^* L_k, \underbrace{\mathcal{O}_E}_{\text{torsion}} \rangle$ but $\pi^*(L_i)|_E \overset{\shortparallel}{} \mathcal{O}_E$

Then $\langle M_1, \dots, M_{k-1}, \mathcal{O}, \mathcal{O}_E \rangle$. Use $0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$
by $\pi^* L_k^{-1}$
see that $\langle \mathcal{O}(-E), \mathcal{O} \rangle$ is exceptional pair of line bundles, ie $H^i(X, \mathcal{O}(-E)) = 0 \ \forall i \geq 0$.

Claim: $\mathcal{D}^b(X) = \langle M_1, \dots, M_{k-1}, \mathcal{O}(-E), \cancel{\mathcal{O}_E} \rangle$
proof: $\mathrm{Ext}^k(\mathcal{O}(-E), M_i) = 0 \ \forall i, k$, $\mathrm{Ext}^k(\mathcal{O}, M_i) = \mathrm{Ext}^i(\mathcal{O}_E, M_i) = 0$.
$\Leftrightarrow \langle \mathcal{O}, \mathcal{O}_E \rangle = \langle \mathcal{O}(-E), \mathcal{O} \rangle$ in $\mathcal{D}^b(X)$ ∎

★ <u>Conjecture</u> (Orlov): $\mathcal{D}^b(X)$ has a full except collection $\Rightarrow X$ is rational.
(X any dimension)

<u>Theorem</u> (Perling) Every <u>numerical</u> exceptional coll. of maximal possible length on a numerical rational surface ($q = p_g = 0$) can be mutated on exceptional collection of objects of rank 1.

Riemann-Roch arithmetic:

$$D^b(X) \longrightarrow K_0(X) = K(\mathrm{Coh}(X)) = K(\text{vect bundles } X)$$

$$E \mapsto [E]$$

$$\underset{\Delta\,\text{exact}}{E \to F \to G} \mapsto [F] = [E] - [G]$$

$$[F[1]] = -[F]$$

$\Rightarrow$ Euler form: $\chi(E,F) = \sum_i (-1)^i \dim \underset{\mathrm{Hom}(E,F[i])}{\overset{\shortparallel}{\mathrm{Ext}^i(E,F)}}$

$$j : K_0(X) \to \mathbb{Z} \oplus NS(X) \oplus \mathbb{Z}$$

$$E \mapsto \mathrm{rk} \quad c_1 \quad \chi$$

is surjective [Take $\mathcal{O}_X, \mathcal{O}_D, \mathcal{O}_p$]

ker j = kernel of the Euler pairing because of Riemann-Roch.

rank$(E) = e$

$$\chi(E,F) = -ef\underbrace{\chi(\mathcal{O}_X)}_{=1} - c_1(E)\cdot c_1(F) + fK_X\cdot c_1(E) + e\chi(F) + f\chi(E)$$

$$\chi(E,F) = \chi(F,E) + K_X\cdot(fc_1(E) - ec_1(F))$$

Say E is numerically exceptional $\Leftrightarrow$ $\chi(E,E) = 1$. (Definition)

$$1 = -e^2 - c_1(E)^2 + 2e\chi(E) + eK_X\cdot c_1(E)$$

rank $E = e = 0 \Leftrightarrow c_1(E)^2 = -1$ "numerical exceptional (-1)-curve"

$e \neq 0 \Rightarrow \chi(E) = \ldots$

suppose that $\mathcal{E}, \mathcal{F}$ are numerically exceptional $e, f \neq 0$

$$\Rightarrow \chi(\mathcal{E},\mathcal{F}) = \underbrace{\frac{1}{2}(\chi(\mathcal{E},\mathcal{F}) - \chi(\mathcal{F},\mathcal{E}))}_{= -\frac{1}{2}K_X\cdot c_1(\mathcal{E},\mathcal{F})} + \underbrace{\frac{1}{2}(\chi(\mathcal{E},\mathcal{F}) + \chi(\mathcal{F},\mathcal{E}))}_{\frac{1}{2}\frac{c_1(\mathcal{E},\mathcal{F})^2 + e^2 + f^2}{ef}}$$

$$c_1(\mathcal{E},\mathcal{F}) = ec_1(\mathcal{F}) - fc_1(\mathcal{E})$$

Take $K_0(\mathbb{P}^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$

$\langle \mathcal{E}, \mathcal{F}, \mathcal{G} \rangle$ numerically exceptional $\chi(\mathcal{F}, \mathcal{E}) = \chi(\mathcal{G}, \mathcal{E}) = \chi(\mathcal{G}, \mathcal{F}) = 0$

$$B = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} a &= \chi(\mathcal{E}, \mathcal{F}) \\ b &= \chi(\mathcal{E}, \mathcal{G}) \\ c &= \chi(\mathcal{F}, \mathcal{G}) \end{aligned}$$

Numerical Serre duality.

$$\chi(u,v) = u^T B v = v^t B^t u = v^t B (B^{-1} B^t) u = \chi(v, \underbrace{B^{-1} B^t u}_{S(u)})$$

Let's compute $\mathrm{Tr}(B^{-1}B^t)$. First $B^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$

Then trace $= 3 - a^2 - b^2 - c^2 + abc$ same for any a, b, c.

$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ gives $\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ trace $= 3$

$\Rightarrow a^2 + b^2 + c^2 = abc$ "triple Markov" $\quad (3a, 3b, 3c) \Updownarrow (a, b, c)$ solution

$(a^2 + b^2 + c^2 = 3abc$ usual Markov $)$

Then $\chi(E,F), \chi(E,G), \chi(F,G)$ is solution of triple Markov.

$$\chi(\mathcal{E},\mathcal{F}) = \chi(\mathcal{E},\mathcal{F}) - \chi(\mathcal{F},\mathcal{E}) = -K_X \cdot c_1(\mathcal{E},\mathcal{F})$$

In $\mathbb{P}^2$, $-K_X = 3H$, so $c_1 \in \mathbb{Z}$

$\Rightarrow \chi(E, F) = 3c_1(\mathcal{E}, F)$

$\Rightarrow c_1(\mathcal{E},\mathcal{F}), c_1(\mathcal{E},\mathcal{G}), c_1(\mathcal{F}, \mathcal{G})$ are solution of the usual Markov.

<u>Lemma</u>: $c_1(\mathcal{E},\mathcal{F}) = g$. And so $c_1(\mathcal{F}, \mathcal{G}) = rk(\mathcal{E}(-K)) = e$

$\langle \mathcal{E}, \mathcal{F}, \mathcal{G} \rangle$, $\langle \mathcal{F}, \mathcal{G}, \mathcal{E}(-K) \rangle$

and so $\mathcal{E}, \mathcal{G}, \underbrace{R_{\mathcal{G}}(\mathcal{F})}_{\text{mutation}}$ $\quad c_1(\mathcal{E},\mathcal{G}) = rk(R_{\mathcal{G}}(\mathcal{F}))$

$\Rightarrow$ ranks of $\mathcal{E}, \mathcal{G}, R_{\mathcal{G}}(\mathcal{F})$ satisfy Markov $\Rightarrow$ ranks all satisfy Markov.

Lemma: $c_1(\mathcal{E},\mathcal{F}) = g$

proof: $\chi(\mathcal{G},\mathcal{F}) = \chi(\mathcal{G},\mathcal{E}) = \chi(\mathcal{F},\mathcal{E}) = 0 \qquad e,f,g \neq 0$

$$\Rightarrow \frac{3}{2} c_1(\mathcal{G},\mathcal{F}) + \frac{1}{2gf}\left(c_1(\mathcal{G},\mathcal{F})^2 + g^2 + f^2\right) = 0 \qquad /\cdot -2e^2fg$$

$$\frac{3}{2} c_1(\mathcal{G},\mathcal{E}) + \dots = 0 \qquad /\cdot 2f^2eg$$

$$\frac{3}{2} c_1(\mathcal{F},\mathcal{E}) + \dots = 0 \qquad /\cdot -2g^2ef$$

(and c_1's satisfies Markov)

odd: $c_1(\mathcal{F},\mathcal{E}) \cdot c_1(\mathcal{G},\mathcal{F}) = eg$

Rotate $c_1(\mathcal{F},\mathcal{G}) \cdot c_1(\mathcal{G}, \mathcal{E}(-K)) = fe$ rotation!

$$c_1(\mathcal{G}, \mathcal{E}(-K)) \cdot \underbrace{c_1(\mathcal{E}(-K), \mathcal{F}(-K))}_{c_1(\mathcal{E},\mathcal{F})} = fg$$

$\frac{(1)\times(3)}{(2)}$: $c_1(\mathcal{E},\mathcal{F})^2 = g^2$

$\Rightarrow c_1(\mathcal{E},\mathcal{F}) = g$ ∎

Thm (Markov) All solutions to $a^2+b^2+c^2 = 3abc$ can be obtained from $(1,1,1)$ by $(a,b,c) \longrightarrow (3bc-a, b, c)$.

proof [Hint: choose $\underline{a}$ maximal $3bc - a < a$]

Claim: Mutations at solutions correspond to mutations in the numerically exceptional collection,

proof: $\chi(E,E) = 1 \qquad K_0(X) \xrightarrow{L_E} E^\perp \quad M \mapsto \chi(E,M)E - M$

$\qquad\qquad\qquad\qquad \xrightarrow{R_E} {}^\perp E \quad M \mapsto \chi(M,E)E - M$

$(A,B,C) \rightarrow (B, R_B(A), C)$

except triple $\qquad R_B(A) = \chi(A,B)B - A$

$\mathrm{rank}\,(\chi(A,B)B - A) = \chi(A,B)\,b - a$ remember: $\chi(A,B) = 3\,G(\alpha,\beta)$

$\Rightarrow \quad = 3bc - a \qquad\qquad = 3c$

■

Corollary: All numerically exceptional triples in $K_0(\mathbb{P}^2)$ can be obtained from triples of rank 1.

Lemma on del Pezzo surface, Exceptional $E \in D^b$ is determined by $[E] \in K_0$. $\quad e \neq 0$

Pf:- Cohomology estimates, Mukai Lemma. $\mathcal{E}$ is an exceptional vector bundle.

Then show that E is stable wrt $-K$.

$[E] = [E']$ except. $\Rightarrow$ both stable, say $E \neq E'$

$\mathrm{Hom}(E,E') = 0 = \mathrm{Ext}^2(E,E')$

$\chi(E,E') \leq 0$

$\quad \| \, RR \qquad \rightarrow\leftarrow$

$\chi(E,E) = 1$

■

[restrict E to a smooth member of $|-K|$ C, $E|_C$ is simple vector bundle $\Rightarrow E|_C$ is stable by Atiyah]