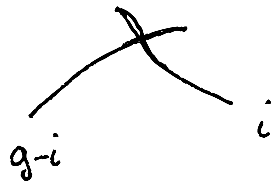
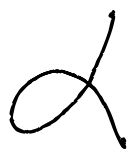
Divisores en $\overline{\mathcal{M}}_g$

Para $1 \le i \le [\frac{g}{2}]$

$\Delta_i = \overline{\Big\{}$ 

$\overline{\Big\}} = \overline{\mathrm{Im}}(\overline{\mathcal{M}}_{g-i,1} \times \overline{\mathcal{M}}_{i,1}) \to \overline{\mathcal{M}}_g$

$\dim = 3(g-i)-3+1 + 3i-3+1 = 3g-4$

$\Delta_0 = \overline{\Big\{}$  $\overline{\Big\}} \subset \overline{\mathcal{M}}_g$

$\shortparallel$

$\mathrm{Im}\,(\overline{\mathcal{M}}_{g-1,2} \longrightarrow \overline{\mathcal{M}}_g$

$[$ 

$] \longmapsto [$ 

$]$

$\partial \overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g = \Delta_0 \cup \ldots \cup \Delta_{[g/2]}$

__Def__: Un sheaf F en $\overline{\mathcal{M}}_{g,n}^{F}$ es una asignación

$$[f : \mathcal{C} \to S] \longmapsto F_f \in \mathrm{Coh}(S)$$

tal que para todo

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\ S' \arrow[r, "\beta"] & S \end{tikzcd} \qquad \beta^* \mathcal{F}_f \cong \mathcal{F}_{f'} \text{ en } S'.$$

$$\mathcal{F} \otimes \mathcal{G} := \forall f: \mathcal{C} \to S$$
$$\mathcal{F}_f \otimes \mathcal{G}_f.$$

$\mathrm{Pic}(\overline{\mathcal{M}}_{g,n}^F)$ grupo de clases de iso de l.S.

(Analogo para $CH^i(\overline{\mathcal{M}}_{g,n}^F)$)

Es suficiente asumir $\dim(S)=1$.

Def: $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g^F)$ Hodge class se define como

$$\lambda := c_1(\mathbb{E}_g) \quad \text{donde}$$

$\mathbb{E}_g(f: \mathcal{C} \to S) = f_* \omega_f$ es un vector bundle de rango g.

$$\text{fibra sobre } s \in S = H^0(C, \omega_C)$$

Thm: (1) $\mathrm{Pic}(\overline{\mathcal{M}}_g^F)_{\mathbb{Q}} \cong \mathrm{Pic}(\overline{\mathcal{M}}_g)$

y $\qquad \delta_i \longleftrightarrow [\Delta_i]$

$\qquad \delta_1 \longleftrightarrow \frac{1}{2}[\Delta_1]$.

(2) Arbarello-Cornalba, Harer

$\mathrm{Pic}(\overline{\mathcal{M}}_g^F)$ está libremente generado por $\lambda, \delta_0, \delta_1, \dots, \delta_{[g/2]}$.

Thm: $K_{\overline{M}_g} = 13\lambda - 2(\underbrace{\delta_0 + \dots + \delta_{[g/2]}}_{\delta}) \in \mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}}$ $\quad \overline{M}_g \to$ stack

y para género $g \geq 4$

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta - \delta_1$$

Rmk: $\overline{M}_g \xrightarrow{\varepsilon} \overline{\mathcal{M}}_g$ está ramificado (simple) sobre $[\Delta_1]$

$\varepsilon^*[\Delta_1] = 2\delta_1$ y Riemann-Hurwitz

$$K_{\overline{M}_g} = \varepsilon^* K_{\overline{\mathcal{M}}_g} + \delta_1 .$$

Def: ψ-classes

$$\begin{tikzcd} \mathcal{C}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi"'] \\ \overline{\mathcal{M}}_{g,n} \arrow[u, bend right, "\sigma_1 \dots"'] \arrow[u, bend right=60, "\sigma_n"'] \end{tikzcd}$$

$$\psi_i := c_1(\sigma_i^* \omega_\pi) = c_1(\sigma_i^* \Omega^1_\pi).$$

en $[$  C, p_i $]$ la fibra es $(T_{p_i}^\vee C)$.

misma historia

Asumir B curva suave y

$$B \xrightarrow{f} \overline{\mathcal{M}}_{g,n}$$

Cómo se calcula $f(B) \cdot \lambda, \psi_i$'s, δ_i's.

$$f(B) \cdot D = \deg_B (f^* D).$$

$$\begin{tikzcd} \mathcal{C}_b \arrow[r, phantom, "\subset"] \arrow[d] & \mathcal{C} \arrow[d, "\pi"] \\ b \arrow[r, phantom, "\in"] & B \end{tikzcd}$$

familia $\mathcal{C}$ es una superficie fibrada en curvas estables.

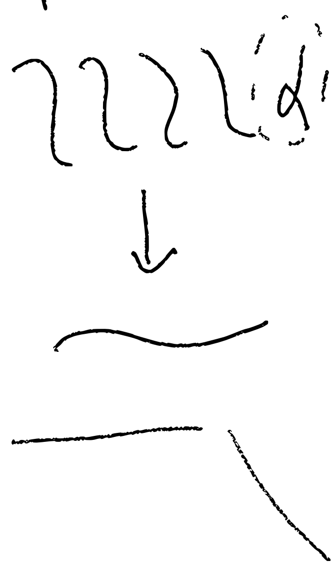
Si $\mathcal{C}$ es suave, entonces

$$f(B)\cdot\lambda = \chi(\mathcal{O}_{\mathcal{C}}) - \chi(\mathcal{O}_B)(1-g)$$

$$f(B)\cdot\delta = \chi_{top}(\mathcal{C}) - \chi_{top}(\mathcal{C})(2-2g)$$

$$f(B)\cdot\psi_i = -(\sigma_i(B)^2)$$

<u>Topología</u>:



$$\begin{array}{c}\mathcal{C}\\ \downarrow\\ B\end{array}$$

si ~~todas~~ las fibras son suaves

$$\chi_{top}(\mathcal{C}) = \chi_{top}(B)\cdot\underbrace{\chi_{top}(F)}_{(2-2g)}$$

Si $\mathcal{C}$ ~~fibras~~ nodales

$+$ # fibras nodales.

<u>Pencil de curvas de género 3</u>

$[C]\in\overline{M}_3$ genérica $\quad C \xrightarrow{|K_C|} \mathbb{P}^2 \quad \deg(C)=4$

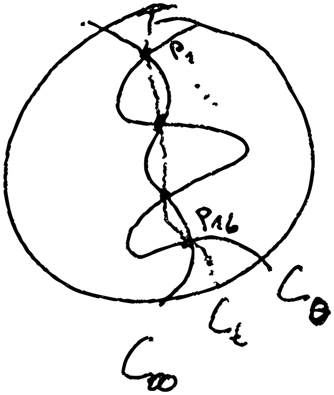
$C = V(f_4) \quad C = C_0$

$$C_0 \in |\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14} \supset \Gamma \quad \text{general}$$

$C_0 = V(f)$, $\Gamma = \mathbb{P}^1$

$t \in \mathbb{P}^1$

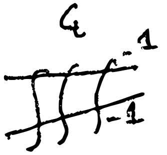
$$C_\infty = V(g) \qquad [t_0 : t_1] \quad C_t = t_0 C_0 + t_1 C_\infty = V(t_0 f + t_1 g)$$



$C_0 \cap C_\infty$ son puntos base de Γ

$$\mathbb{P}^2 \times \Gamma \supset \mathcal{S} = \{(p,t) \mid p \in C_t\}$$

$$C_0 \cdot C_\infty = 16$$



$$\begin{tikzcd} \mathcal{S} \arrow[r, "Bl_{16}"] \arrow[d, "\sigma_1, \dots, \sigma_{16}"'] & \mathbb{P}^2 \arrow[dl, dashed] \\ \Gamma & \end{tikzcd}$$

$t \in \Gamma$

induce

$$\Gamma \longrightarrow \overline{M}_{3,n} \qquad n \le 16$$

$$t \longmapsto [C_t, \sigma_1(t), \dots, \sigma_n(t)].$$

$\underline{\Gamma \cdot \delta}$: Tenemos que calcular

$$\chi_{top}(Bl_{16}\mathbb{P}^2) - \underbrace{\chi_{top}(\Gamma)}_{2}\underbrace{(2-2g)}_{-4} = 8 + \chi_{top}(Bl_{16}\mathbb{P}^2)$$

$$\chi_{top}(\mathbb{P}^2) = 3$$

$$\Downarrow$$

$$\chi_{top}(Bl_{16}\mathbb{P}^2) = 3+16 \qquad Bl_{16}\mathbb{P}^2 \xrightarrow{\varepsilon} \mathbb{P}^2$$

$$\boxed{\Gamma \cdot \delta = 19 + 8 = 27}$$

GRR $\Rightarrow$ Noether's Formula

$$\chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12} = \frac{K_S^2 + \chi_{top}(S)}{12}$$

en nuestro caso

$$K_{Bl_{16}\mathbb{P}^2} = \varepsilon^* \underbrace{K_{\mathbb{P}^2}}_{\mathcal{O}_{\mathbb{P}^2}(-3)} + E_1 + \dots + E_{16}.$$

$$c_1(Bl_{16}\mathbb{P}^2)^2 = 9 - 16 = -7$$

$$\chi(\mathcal{O}_{Bl_{16}\mathbb{P}^2}) = \frac{-7+19}{12} = 1 \rightarrow \underline{\text{invariante birracional}}$$

<u>Otra forma</u>: $\chi = h^0(\mathcal{O}_S) - \underset{h^1(K_S)}{\underset{\shortparallel}{h^1(\mathcal{O}_S)}} + \underset{h^0(K_S)}{\underset{\shortparallel}{h^2(\mathcal{O}_S)}}$

entonces $\chi(\mathcal{O}_{\mathbb{P}^1}) = 1 - g(\mathbb{P}^1) = 1.$

$$\Gamma \cdot \lambda = 1 - 1(1-g) = 3$$

$$\boxed{\Gamma \cdot \lambda = 3}$$

$$\Gamma \cdot \psi_i = -(E_i^2) = -(\sigma_i(\Gamma)^2) = 1.$$

$$K_{\overline{\mathcal{M}}_{3,n}} = 13\lambda + \psi_1 + \dots + \psi_n - 2\delta - \delta_1.$$

$$\Gamma \cdot K_{\overline{\mathcal{M}}_{3,n}} = 39 + n - 2 \cdot 27 = -15 + n.$$

<u>Corolario</u>: $\mathrm{Kod}(\overline{\mathcal{M}}_{3,13}) = -\infty$

<u>Pf</u>: $p_1, \dots, p_{13} \in \mathbb{P}^2$ generales

$$\Gamma = |\mathcal{O}_{\mathbb{P}^2}(4) \otimes I_{p_1 + \dots + p_{13}}| \simeq \mathbb{P}^1 \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$$

$\Rightarrow \Gamma$ cubre $\overline{\mathcal{M}}_{3,13}$ (es nef)

y $\Gamma \cdot K < 0$ $\square$.