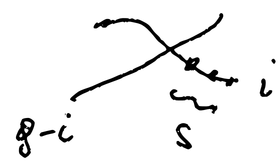
Cálculos en $\overline{\mathcal{M}}_{3,n}$

$\to$ libre

$$\operatorname{Pic}(\overline{\mathcal{M}}_{g,n})_{\mathbb{Q}} \cong \langle \lambda, \psi_1, \dots, \psi_n, \delta_{i:S} \rangle_{\mathbb{Q}}$$

boundary

$S \subset \{1, \dots, n\}$



$$\boxed{K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta}$$

$B \xrightarrow{f} \overline{\mathcal{M}}_{g,n}$ induce una fibración

$$\begin{array}{c} \mathcal{C} \\ \downarrow \uparrow \sigma_1 \dots \uparrow \sigma_n \\ B \end{array}$$

si $\mathcal{C}$ suave

$$\boxed{\begin{aligned} f(B) \cdot \lambda &= \deg(f^*\lambda) = \chi(\mathcal{O}_{\mathcal{C}}) - \chi(\mathcal{O}_B)(1-g) \\ f(B) \cdot \delta &= \chi_{top}(\mathcal{C}) - \chi_{top}(B)(2-2g) \\ f(B) \cdot \psi_i &= -(\sigma_i(B))^2 \end{aligned}}$$

$\underline{g=3}$

$\Gamma \subset |\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$

$\Gamma \cong \mathbb{P}^1$ general

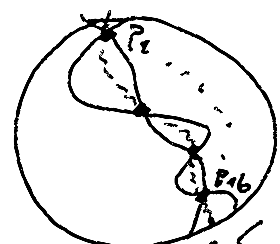
$\mathcal{D}_{univ} \subset \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(4)|$

$= \{ (p, C) \in \mathbb{P}^2 \times |\mathcal{O}_{\mathbb{P}^2}(4)| \mid p \in C \}$.

$S_\Gamma = S = \{ (p,t) \mid p \in C_t \} \subset \mathbb{P}^2 \times \Gamma$

$$\begin{array}{ccc} S & \xrightarrow[\varepsilon]{Bl_{16}\mathbb{P}^2} & \mathbb{P}^2 \\ \sigma_n \curvearrowleft \cdots \sigma_1 \curvearrowleft \big\downarrow & & \\ \Gamma & & \end{array}$$

$n \leq 16$



$C_t \quad C_0 = V(f)$

$C_\infty = V(g) \qquad [t_0 : t_1] = V(t_0 f + t_1 g)$

$\Gamma \longrightarrow \overline{\mathcal{M}}_{g,n}$

$\underline{\underline{\Gamma \cdot \delta}}$:

$$\chi_{top}(Bl_{16}\mathbb{P}^2) - \underbrace{\chi_{top}(\Gamma)}_{=2}\underbrace{(2-2g)}_{=-4} = 8 + \chi_{top}(Bl_{16}\mathbb{P}^2)$$

y $\chi_{top}(\mathbb{P}^2) = 3 \Longrightarrow \chi_{top}(Bl_{16}\mathbb{P}^2) = 3+16$

$$\boxed{\Gamma\cdot\delta = 19+8 = 27} \qquad Bl_{16}\mathbb{P}^2 \xrightarrow{\varepsilon} \mathbb{P}^2$$

GRR $\Longrightarrow$ Noether's formula

$$\chi(\mathcal{O}_S) = \frac{c_1^2+c_2}{12} = \frac{K_S^2+\chi_{top}(S)}{12}$$

en nuestro caso

$$K_{Bl_{16}\mathbb{P}^2} = \varepsilon^* \underset{\mathcal{O}_{\mathbb{P}^2}(-3)}{\underset{\parallel}{K_{\mathbb{P}^2}}} + E_1 + \dots + E_{16}$$

$$\Longrightarrow c_1(Bl_{16}\mathbb{P}^2)^2 = 9-16 = -7$$

$$\chi(\mathcal{O}_S) = \frac{-7+19}{12} = 1$$

invariante biracional

Alternativa: $\chi(\mathcal{O}_S) = \underset{1}{\underset{\parallel}{h^0(\mathcal{O}_S)}} - \underset{\underset{0.}{\underset{\parallel}{h^1(\mathcal{O}_{\mathbb{P}^2})}}}{\underset{\parallel}{h^1(\mathcal{O}_S)}} + \underset{h^0(\cancel{K_S}) \to 0}{\overset{\text{SD}}{\parallel}}{h^2(\mathcal{O}_S)}$

entonces

$$\Gamma\cdot\lambda = \underbrace{\chi(\mathcal{O}_\Gamma)}_{1} - \underbrace{\chi(\mathcal{O}_{\mathbb{P}^1})}_{1}\underbrace{(1-g)}_{-2} = 3$$

$$\boxed{\Gamma\cdot\lambda = 3}$$

$$\Gamma\cdot\psi_i = -(E_i^2) = 1$$

$$K_{\overline{\mathcal{M}}_{3,n}} = 13\lambda + \psi_1 + \dots + \psi_n - 2\delta$$

$$K\cdot\Gamma = 13\cdot 3 + n - 2\cdot 27 = -15 + n$$

<u>Corolario</u>: $\mathrm{Kod}(\overline{\mathcal{M}}_{3,13}) = -\infty$

<u>Pf</u>: $p_1, \dots, p_{13} \in \mathbb{P}^2$ generales

$$\Gamma = |\mathcal{O}_{\mathbb{P}^2}(4) \otimes I_{p_1+\dots+p_{13}}| \simeq \mathbb{P}^1 \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$$

$\Longrightarrow$ Γ cubre $\overline{\mathcal{M}}_{3,13}$ ($\Rightarrow$ uniruled) $\square$

<u>Rmk</u>: Γ nef ($\Gamma\cdot D \geq 0$ $\forall D$ eff.)

y $\Gamma\cdot K = -2 \Rightarrow K_{\overline{\mathcal{M}}_{3,13}}$ no es eff.

$$\begin{array}{ccc} D & \longrightarrow & \overline{M}_{3,14} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \overline{M}_{3,13} \end{array}$$

Rmk: D es racional!

$\Rightarrow$ D está cubierta por $\mathbb{P}^1$'s.

$\Rightarrow$ $\mathrm{Kod}(\overline{M}_{3,14}) = -\infty$.

Problema: $\mathrm{Kod}(\overline{M}_{3,15}) = -\infty$?

$$D = \mathrm{Bl}_{16}\mathbb{P}^2 \xrightarrow{\varepsilon} \mathbb{P}^2$$

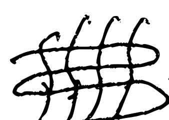
$$\tilde{l} = \varepsilon^* l \qquad l$$

$$\begin{array}{ccc} S & \longrightarrow & \mathcal{C}_{3,14} \cong \overline{M}_{3,15} \\ \downarrow & & \downarrow \\ \tilde{l} \cong \mathbb{P}^1 & \longrightarrow & \overline{M}_{3,14} \end{array}$$

Si β es la clase de una curva que cubre S y $\beta \cdot K_{\overline{M}_{3,15}} < 0$ entonces

$$\mathrm{Kod}(\overline{M}_{3,15}) = -\infty.$$

$$\begin{array}{ccc} S = \tilde{D} & \longrightarrow & D \supset \tilde{l} \\ \downarrow & & \downarrow \pi \\ \tilde{l} & \xrightarrow{4:1} & \Gamma \end{array}$$



$\longrightarrow \tilde{l}$

$\{\sigma_{15}, \ldots, \sigma_1\}$

$$\tilde{D} = \{(p,q) \in D \times \tilde{l} \mid p \in C_{\pi(q)}\}$$

$$\tilde{l} \xrightarrow{\Delta} \cup\, \sigma$$

$\tilde{\mathcal{D}} \subset \mathcal{P} \times \tilde{\ell} \xrightarrow{p_1} \tilde{\ell}$ $\quad \tilde{\mathcal{P}}$ es un divisor suave de $\tilde{\mathbb{P}}^2 \times \tilde{\ell}$

$\downarrow p_2$

$\mathcal{P}$

de clase

$$\mathcal{O}_{\mathcal{P}\times\tilde{\ell}}(\tilde{\mathcal{P}}) = p_2^* \mathcal{O}_{\mathcal{P}}(\tilde{\ell}) + p_1^* \underset{\underset{\mathcal{O}_{\ell}(4)}{\shortparallel}}{\mathcal{O}_{\Gamma}(1)}$$

$$= \mathcal{O}(\tilde{\ell}, 4pt)$$

$$\begin{cases} K_{\tilde{\mathcal{P}}} = K_{\mathcal{P}\times\tilde{\ell}}\big|_{\tilde{\mathcal{P}}} + \mathcal{O}_{\mathcal{P}\times\tilde{\ell}}(\tilde{\ell}, 4)\big|_{\tilde{\mathcal{P}}} \\ = p_1^*\left(K_{\tilde{\ell}} + \cancel{h^*}\mathcal{O}(1)\right) + p_2^*\left(K_{\mathcal{P}} + \mathcal{O}_{\mathcal{P}}(C)\right) \end{cases}$$

$$\longrightarrow \tilde{\ell} \cdot \psi_{14} = -\left(\sigma(\tilde{\ell})^2\right) = 5$$

$$\begin{array}{ccc} \tilde{\mathcal{P}} & \longrightarrow & \overline{\mathcal{M}}_{3,15} \\ \downarrow & & \downarrow \\ \tilde{\ell} & \longrightarrow & \overline{\mathcal{M}}_{3,14} \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \overline{\mathcal{M}}_{3,13} \end{array}$$

y $\tilde{\ell} \cdot (\lambda, \psi_1, \ldots, \psi_{13}, \delta) = 4\, \Gamma \cdot (\text{same})$

$$\tilde{\ell} \cdot K_{\overline{\mathcal{M}}_{3,14}} = 4\, \Gamma \cdot K_{\overline{\mathcal{M}}_{3,13}} + 5$$

$$= 4 \cdot -2 + 5 = -3 \,!!$$

Que hacemos ahora?

$$\begin{array}{ccc} \tilde{D} & \longrightarrow & \overline{\mathcal{M}}_{3,15} \\ \downarrow & & \downarrow \\ \tilde{\ell} & \longrightarrow & \overline{\mathcal{M}}_{3,14} \end{array}$$

$\tilde{D} \xrightarrow[\varphi]{4:1} Bl_{16}\mathbb{P}^2$

Basta con encontrar una curva $\alpha \in \mathrm{Pic}(\tilde{D})$

$\cup$

$\mathbb{Z}\tilde{\ell} \oplus \mathbb{Z}E_1 \oplus \dots \oplus \mathbb{Z}E_{16}$

tal que

$\dim |\mathcal{O}_{\tilde{D}}(\alpha)| \geq 1$

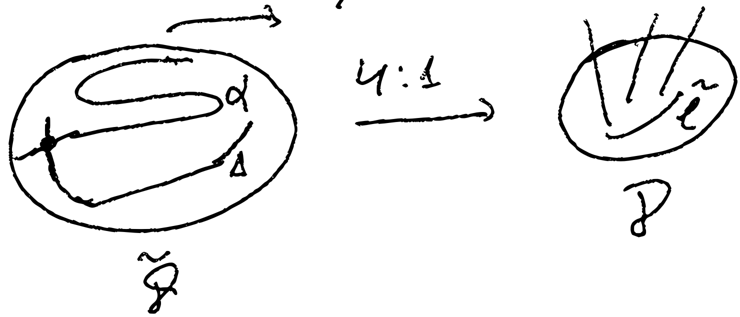
tal que $\alpha \cdot K_{\overline{\mathcal{M}}_{3,15}} \leq 0$.

Empezar con $\tilde{\ell}$

$\alpha = \varphi^* \tilde{\ell}$ ... $3:1$ a $\tilde{\ell}$

$+$

Δ



$\tilde{D}$ $\xrightarrow{4:1}$ D

Slope of $\overline{\mathcal{M}}_g$

$$D = a\lambda - b_0\delta_0 - \dots - b_{[g/2]}\delta_{[g/2]}$$

recordar que

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta - \delta_1 .$$

Si $\exists D$ es efectivo y

$$\frac{a}{b_0} < \frac{13}{2} \;, \quad \frac{a}{b_1} < \frac{13}{3} \;, \quad \frac{a}{b_i} < \frac{13}{2} \quad 2 \leq i \leq [g/2]$$

entonces $\overline{\mathcal{M}}_g$ es de tipo general

$$K_{\overline{\mathcal{M}}_g} = \varepsilon\lambda + \eta D + \text{positive boundary.} \qquad \eta > 0$$

<u>Def</u>: $\delta = \delta_0 + \dots + \delta_{[g/2]}$, $D \in \mathrm{Eff}(\overline{\mathcal{M}}_g) \subset \mathrm{Pic}(\overline{\mathcal{M}}_g)$

$$s(D) = \inf \left\{ \frac{a}{b} \;\middle|\; a, b > 0 \text{ y } a\lambda - b\delta - D = \sum_{c_i \geq 0} c_i\delta_i \right\}$$

$s(D) = \infty$ unless $D = a\lambda - \sum b_j \delta_i$

$a, b_i \geq 0 \quad \forall i$

Si $s(D) < \infty$, $\qquad D \equiv a\lambda - \sum b_i \delta_i$

$$s(D) = \frac{a}{\min b_j}$$

$$s(\overline{\mathcal{M}_g}) = \inf \{ s(D) \mid D \in \mathrm{Eff}(\overline{\mathcal{M}_g}) \}.$$

<u>Prop</u>: Si $s(\overline{\mathcal{M}_g}) < \frac{13}{2}$, entonces $\overline{\mathcal{M}_g}$ es de tipo general. Si $s(\overline{\mathcal{M}_g}) > \frac{13}{2}$, entonces $\mathrm{Kod}(\overline{\mathcal{M}_g}) = -\infty$.

<u>Q</u>: $s(\overline{\mathcal{M}_g}) = ?$

Se sabe hasta $g = 11$.

y $\qquad s(\overline{\mathcal{M}_g}) \geq O\left(\frac{1}{g}\right)$ $\qquad$ (Pandharipande)

Fix g and assume $\exists r, d$ s.t.

$$\rho(g, r, d) = g - (r+1)(g - d + r) = -1$$

$$BN_d^r = \{ [C] \in \mathcal{M}_g \mid C \text{ admits a } g_d^r \}$$

es un divisor en $\overline{M}_g$ de clase

$$BN^r_d = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum i(g-i)\delta_i\right)$$

$$s(BN^r_d) = 6 + \frac{12}{g+1}$$

$$6 + \frac{12}{g+1} < \frac{13}{2} \iff (6g+6+12)\cdot 2 < 13g+13$$

$$\iff \boxed{g \geq 24}$$

<u>Lemma</u>: $(S, H) \in \mathcal{F}_g$

$|\mathcal{O}_S(H)| \cong \mathbb{P}^g$, $C \in |\mathcal{O}_S(H)|$ genus g canonical

$\cup$
$\mathbb{P}^1$ gral
$\cong$
Γ

$\Gamma \cdot \lambda = g+1$

$\Gamma \cdot \delta_0 = 6g+18$

$$\frac{\Gamma \cdot \delta_0}{\Gamma \cdot \lambda} = 6 + \frac{12}{g+1}$$

$\Rightarrow \nexists$ D eff. tal que

$$s(D) < 6 + \frac{12}{g+1} \Rightarrow$$

D contiene $\left\{ [C] \in \mathcal{M}_g \;\middle|\; \begin{array}{l} \exists\, (S,H) \in \mathcal{F}_g \\ C \in |H| \end{array} \right\}$

$\cap$

K3 locus. $\square$

<u>Q</u>: $\lim\limits_{g \to \infty} s(\overline{\mathcal{M}}_g) = ? \in [0,6]$.

<u>Lower bounds</u>: If $R \subset \overline{\mathcal{M}}_g$ is nef then $s(\overline{\mathcal{M}}_g) \geq \dfrac{R \cdot \delta}{R \cdot \lambda}$.

$$\left[\begin{array}{l} R \cdot D \geq 0 \Rightarrow a(R \cdot \lambda) - b(R \cdot \delta) \geq 0. \\ \qquad\qquad\qquad \Rightarrow \dfrac{a}{b} \geq \dfrac{R \cdot \delta}{R \cdot \lambda} \end{array} \right]$$

<u>Upper bounds via curves</u> ?

? ? ?