# Sumas de Dedekind

*Ahora sí

## Motivación:

**The Atiyah – Singer Theorem and Elementary Number Theory**

F. Hirzebruch, D. Zagier

# Modular forma * casi

$\tau \in \mathbb{C}$ $\operatorname{im}(\tau) > 0$ $q = e^{2\pi i \tau}$

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$$

peso $\frac{1}{2}$

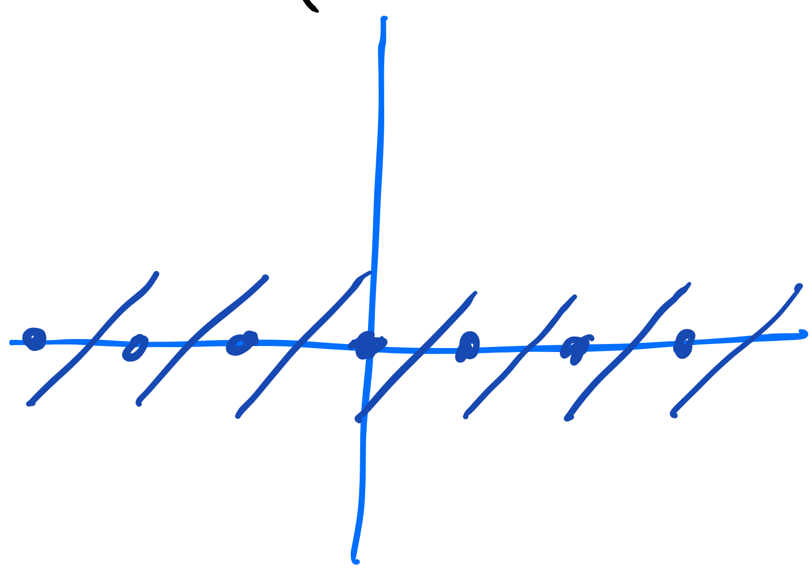
$$SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} \text{ints,} \\ \det = 1 \end{matrix} \right\}$$

## Forma modular peso $k$

* Holomorfa $\operatorname{im} > 0$
* acotada $z \to i\infty$
* $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$

# Sawtooth fct

$$((x)) = \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$



$$\left(\left(\frac{a}{b}\right)\right)$$

Como función en $a$ es periódica (largo $b$)

$$= \frac{1}{2b} \sum_{k=1}^{b-1} \left( \frac{1+\omega_b^k}{1-\omega_b^k} \right) \omega_b^{ak}$$

(keyword: Fourier)

$$= \frac{i}{2b} \sum_{k=1}^{b-1} \cot\left( \frac{\pi k}{b} \right) \omega_b^{ak}$$

# Suma de Dedekind

*Tablas

$$S(a,b) = \sum_{k=0}^{b-1} \left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right)$$

$$= \frac{1}{4b} \sum_{k=1}^{b-1} \cot\left(\frac{\pi k}{b}\right)\cot\left(\frac{\pi k a}{b}\right)$$

# Geometría discreta

$$A = \{a_1, \dots, a_d\} \subseteq \mathbb{Z}$$

$$\boxed{P_A(n) = \#\{m \in \mathbb{Z}^d \mid m_i \geq 0,\ m_1 a_1 + \dots + m_d a_d = n\}}$$

Frobenius

Coin exchange, knapsack

$d \geq 3$ ???

# Problema de la Mochila

$m_i \to$ cantidad objeto $i$

$a_i \to$ peso

$n \to$ capacidad

$p_i \to$ precio

$$\boxed{\min \sum p_i m_i}$$

Formula $p_A(n)$?

$$\left(\frac{1}{1-z^{a_1}}\right)\cdots\left(\frac{1}{1-z^{a_d}}\right)=\sum_{n=1}^{\infty} p_A(n)z^n$$

$$\boxed{p_A(n) = \text{const} \quad \left(\frac{1}{1-z^{a_1}}\right)\cdots\left(\frac{1}{1-z^{a_d}}\right)\left(\frac{1}{z^n}\right)}$$

idea Fracciones Parciales

$$\frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_n}{z^n} +$$

$$\frac{B_1}{(z-1)} + \cdots + \frac{B_d}{(z-1)^d} +$$

$$\sum_{k=1}^{a_1-1} \frac{C_{1k}}{z-w_{a_1}^k} + \cdots + \sum_{k=1}^{a_d-1} \frac{C_{dk}}{z-w_{a_d}^k}$$

* primos rel

$$P_A(n) = -B_1 + B_2 - \cdots + (-1)^d B_d$$

$$- \sum_{k=1}^{a_1-1} \frac{C_{1k}}{w_{a_1}^k} - \cdots \sum_{k=1}^{a_d-1} \frac{C_{dk}}{w_d^k}$$

# Fourier-Dedekind Sum

$$s_n(a_1, \ldots, a_m; b) := \frac{1}{b} \sum_{k=1}^{b-1} \frac{\omega_b^{kn}}{(1-\omega_b^{ka_1})(1-\omega_b^{ka_2}) \cdots (1-\omega_b^{ka_r})}$$

Formula:

$$P_A(n) = -B_1 + B_2 - \dots + (-1)^d B_d$$

$$+ S_{-n}(a_2, a_3, \dots, a_d; a_1) + \dots$$

$$+ S_{-n}(a_1, a_2, \dots, a_{d-1}; a_d)$$

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# Ejemplo

$$A = \{a, b, c\}$$

$$\frac{n^2}{2abc} + \frac{n}{2}\left(\frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}\right) +$$

$$\frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)$$

$\mathrm{poly}_A(n)$

# Facts

- $S_0(a,1;b) = -S(a,b) + \frac{b-1}{4b}$

- $P_A(0) = 1$ (Zagier recipro)

$$1 - \mathrm{Poly}_A(0) = S_0(a_2, \ldots, a_d; a_1) + \ldots + S_0(a_1, \ldots, a_{d-1}; a_d)$$

- En particular (Dedekind)

$$s(a,b)+s(b,a) = \frac{1}{12}\left(\frac{a}{b}+\frac{b}{a}+\frac{1}{ab}\right)-\frac{1}{4}$$

• Para $n=1,\dots,\sum a_i - 1$

$-\mathrm{poly}_A(-n)$ (Rademacher)

$$= s_n(a_2,a_3,\dots,a_d;a_1)+\dots$$

$$s_n(a_1,\dots,a_{d-1};a_d)$$

$$\boxed{P_A(n) = \#\{m \in \mathbb{Z}^d \mid m_i \geq 0,\ m_1 a_1 + \dots + m_d a_d = n\}}$$

$$P_A(-n) = \#\{m \in \mathbb{Z}^d \mid m_i > 0,\ m_1 a_1 + \dots + m_d a_d = n\}$$

$\rightarrow$ n pequeño da cero

Ehrhart

$P$ reticular

polinomio

$a_{d-1}$ ✓

$$L_P(z) = a_d z^d + \cdots + a_1 z + a_0$$

$$= |zP \cap \mathbb{Z}^d|$$

conexion HRR en tóricas

Serie

$$E_p(\lambda) = \sum_{n=0}^{\infty} L_p(n)\lambda^n$$

$$= \frac{\sum_{i=0}^{d} \delta_i \lambda^i}{(1-\lambda)^{d+1}}$$

$(\delta_0, \ldots, \delta_d)$ son enteros $\geq 0$

(Haase-Nill-Payne)
Crelle 08

fijo $d$

fijo $k = \max\{i \mid \delta_i \neq 0\}$

$\Rightarrow$ hay finitos vectores $\delta$ con esos datos.

Otra serie

$s \in \mathbb{C}$
$\mathrm{Re}(s) > 0$

$$F_p(s) = \sum_{n=0}^{\infty} \zeta_p(n) e^{-2\pi s n}$$

(Diaz-Robins 97 Annals)

expansión en términos de cotangentes

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arXiv "Fourier analysis on Polytopes"