Pontificia Universidad Católica de Chile
Facultad de Matemáticas
Departamento de Matemática

Annihilating Cohomology
Francisco Gallardo
14/04/2024

1 Introduction

The goal of today is to kill cohomology classes in an specific setting.

The general question is very simple; if one has a cohomology class $\alpha \in H^n(S, G)$ where S is a scheme and G is a sheaf, is there a morphism $\pi\colon S' \to S$ such that $\alpha|_{S'} := \pi^*\alpha = 0$? in which case we say that α is "killed" by π. For example, it is straightforward to see that cohomology classes are locally trivial (in the respective site). But what if we put restrictions on π, like finite or proper? In this line, we have the following two theorems:

Theorem 1.1 (Bhargav Bhatt, 2011)**.** Let S be a noetherian excellent scheme, and let G be a finite flat commutative group scheme over S. Then classes in $H^n_{\text{fppf}}(S, G)$ can be killed by finite surjective maps to S for $n > 0$.

Theorem 1.2 (Bhargav Bhatt, 2011)**.** Let S be a noetherian excellent scheme, and let A be an abelian scheme over S. The classes in $H^n_{\text{fppf}}(S, A)$ can be killed by proper surjective maps to S for $n > 0$. Moreover, there exists an example of a normal affine scheme S that is essentially of finite type over $\mathbb{C}$, and an abelian scheme $A \to S$ with a class in $H^1_{\text{fppf}}(S, A)$ that cannot be killed by finite surjective maps to S.

The goal for today will be to prove Theorem 1.1. The informal idea of the proof is as follows: we take a class $\alpha \in H^n(S, G)$, we find a étale cover over which α trivializes. We then reduce to the case of Zariski covers owing to an idea of Gabber and then we finally solve the problem by hand using spectral sequences and Zariski cohomology.

2 Preliminaries

Theorem 2.1 (Raynaud)**.** Let S be any scheme and G a commutative finite, flat, locally of finite presentation group scheme over S. Then there exists an abelian S-scheme A and an S-closed immersion $G \to A$.

Proof. [BBM, Théorème 3.1.1]. □

2.1 Cohomology

Theorem 2.2 (Fppf and étale cohomology)**.** Let X be a scheme and G a smooth commutative group scheme over X. Then

$$H^i_{\text{ét}}(X, G) \cong H^i_{\text{fppf}}(X, G)$$

for all $i \geq 0$.

Proof. [G2, Théorème 11.7] □

3 Simplification of covers

In this section, we state some results about covers. The first one helps us “globalize” finite surjective covers while the second, due to Gabber, allows us to pass from étale covers to Zariski ones.

Lemma 3.1 (Extending covers)**.** Fix a noetherian scheme X. Given an open dense subscheme $U \hookrightarrow X$ and a finite surjective morphism $f\colon V \to U$, there exists a finite surjective morphism $\overline{f}\colon \overline{V} \to X$ such that $\overline{f}_U$ is isomorphic to f. Given a Zariski open cover $\mathcal{U} = \{j_i\colon U_i \to X\}$ by a finite amount of dense opens $U_i \subset X$, and finite surjective morphisms $f_i\colon V_i \to U_i$ for each i, there exists a finite surjective morphism $f\colon Z \to X$ such that f_{U_i} factors through f_i. The same claims hold if “finite surjective” is replaced “proper surjective” everywhere.

Proof. We first explain how to deal with the claims for finite morphisms. The morphism $V \to U \to X$ is quasi-finite and separated as $V \to U$ and $U \to X$ are. Since X is quasi-compact and quasi-separated we can apply Zariski's Main Theorem to $V \to X$ and find a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & U \\ {\scriptstyle j}\downarrow & & \downarrow \\ T & \xrightarrow[\ \pi\ ]{} & X \end{array}$$

where j is a quasi-compact open immersion and π is finite. Taking the schematic closure of V in T (whose underlying set is the closure of V in T since j is quasi-compact) we get the following commutative diagram:

$$\begin{array}{ccccc} & & V & \xrightarrow{\ f\ } & U \\ & \swarrow & {\scriptstyle j}\downarrow & & \downarrow \\ \overline{V} & \xrightarrow[\ i\ ]{} & T & \xrightarrow[\ \pi\ ]{} & X \end{array}$$

The morphism $\pi \circ i = \overline{f}\colon \overline{V} \to X$ is finite since i is closed immersion and π is finite.

The surjectivity is straightforward. The fact that $\overline{f}_U$ is f is more delicate and won't be proven.

For the second part, we may extend each $f_i \circ j_i$ as in the first part to obtain finite surjective morphisms $\overline{f_i}\colon \overline{V_i} \to X$ that restrict to j_i over $U_i \subseteq X$. Setting W to be the fiber product of all the $\overline{V_i}$ over X solves the problem (finite surjective is preserved by base change).

The statements replacing “finite” by “proper” follows using the exact same argument and using Nagata's compactification instead of Zariski's Main Theorem. □

Lemma 3.2 (Gabber)**.** Let $f\colon U \to X$ be a surjective étale morphism of schemes. Then there exists a finite surjective map $g\colon X' \to X$, and a Zariski open cover $\{U_i \hookrightarrow X'\}$ such that the natural map $\amalg_i U_i \to X$ factors through $U \to X$.

Proof. Gabber proved this in the affine case with the advantage that the resulting $g\colon X' \to X$ is also flat. He did this by constructing an specific (spectrum of a) ring and a cover of it. □

The next result assures that we can kill our cohomology classes étale-locally.

Proposition 3.3. Let S be the spectrum of a strictly henselian local ring, and let G be a finite flat commutative group scheme over S. Then $H^i_{\mathrm{fppf}}(S, G) = 0$ for $i > 1$.

Proof. By Theorem 2.1, there exists an abelian S-scheme A (proper, smooth with geometrically connected fibres) and an S-closed immersion $G \to A$. Let A/G denote the quotient sheaf for the fppf topology, and it happens that this is an abelian scheme over S. This is not obvious at all but we will take it for granted. Thus, we have a short exact sequence

$$0 \longrightarrow G \longrightarrow A \longrightarrow A/G \longrightarrow 0$$

of abelian sheaves on the fppf site of S. This gives rise to a long exact sequence

$$\cdots \longrightarrow H^{n-1}_{\text{fppf}}(S, A/G) \longrightarrow H^n_{\text{fppf}}(S, G) \longrightarrow H^n_{\text{fppf}}(S, A) \longrightarrow \cdots$$

Since A and A/G are smooth, we can apply Theorem 2.2 to conclude

$$H^{n-1}_{\text{fppf}}(S, A/G) = H^n_{\text{fppf}}(S, A) = 0$$

for $n \geq 2$ and hence $H^n_{\text{fppf}}(S, G) = 0$ for $n \geq 2$ as we wanted (here we use the fact that the étale cohomology of an strictly local spectrum vanishes for $n \geq 1$). □

Next, we explain how to deal with Zariski cohomology with coefficients in a finite flat group scheme.

Proposition 3.4. Let S be a normal noetherian scheme, and let $G \to S$ be a finite flat commutative group scheme. Then $H^n_{\text{Zar}}(S, G) = 0$ for $n > 0$.

Proof. We may assume that S is connected (and hence integral). As constant sheaves on irreducible topological spaces are acyclic, it will suffice to show that G restricts to a constant sheaf on the small Zariski site of S, i.e, that the restriction maps $G(S) \to G(U)$ are bijective for any non-empty open subset $U \hookrightarrow S$ (here we use the fact that S is final in the category of S-schemes). Since S is integral, it is reduced and U is dense in S. Also, as G is separated over S, the S-version of the reduced-to-separated theorem implies that $G(S) \to G(U)$ is injective.

For the surjectivity, take a section $f \colon U \to G$. The scheme theoretic image of U in G yields an integral closed subscheme $S' \hookrightarrow G$. Indeed, since U is reduced, its scheme theoretic image is the closed subscheme $\overline{f(U)} \to G$ with its reduced structure. Since U is irreducible, so is $f(U)$ and its closure. Thus, the map $S' \to S$ is a finite map that restricts to an isomorphism over U. Since a finite birational morphism from an integral scheme to an (integral) normal scheme is an isomorphism, we conclude $S' \cong S$. The map $S \cong S' \to G$ is the desired extension. □

Now we come to the proof of Theorem 1.1.

Proof. We prove this by induction. For $n = 1$, classes $\alpha \in H^1_{\text{fppf}}(S, G)$ are represented by fppf G-torsors T over S. Since finite flat is stable under base change and descends for fppf morphisms, it is local in the fppf topology. Since T and G are isomorphic locally in this topology, $T \to S$ is also finite flat. Thus, we can take our $S' \to S$ to be $T \to S$. Indeed, it is finite flat, and $T \times_S T \to T$ is the trivial T torsor (it has section). Thus $\alpha|_T = 0$.

We now fix $n > 1$ and a cohomology class $\alpha \in H^n_{\text{fppf}}(S, G)$. By Proposition 3.3 we have that α is trivial at étale stalks at all points of S. Since étale cohomology commutes with projective limits of schemes, we can show that α is trivial around an étale neighbourhood of every point, and hence there is an étale cover of S over which α trivializes. Using Lemma 3.1, after replacing S by a finite cover, we may assume that there exists a Zariski cover $\mathcal{U} = \{U_i \hookrightarrow S\}$ over which α trivializes.

The Cech to cohomology spectral sequence for this cover is

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(G)) \implies H^{p+q}_{\text{fppf}}(S, G)$$

where $\underline{H}^q(G)$ is the cohomology presheaf. What this imply, is two things:

- There is a filtration of $H^n_{\mathrm{fppf}}(S,G)$ by objects

$$0 = F^0 \subset F^1 \subset F^2 \subset \cdots \subset F^{n-1} \subset F^n = H^n_{\mathrm{fppf}}(S,G)$$

  where

$$F^i/F^{i-1} \cong E_\infty^{n-i,i}$$

  for $0 \leq i \leq n$.

- For any n,

$$E_{n+1}^{p,q} \cong \frac{\ker d_n^{p,q}}{\operatorname{im} d^{p-n,q+n-1}}.$$

  In particular, an element of $E_{n+1}^{p,q}$ "comes" from an element in $E_n^{p,q}$. Repeating this process, we get that any element of $E_\infty^{p,q}$ "comes" from one in

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(G)).$$

In particular, we have surjections $F^i \to E_\infty^{n-i,i}$ and every element on the right comes from an element $\check{H}^{n-i}(\mathcal{U}, \underline{H}^i(G))$.

Since $\alpha \in H^n(S,G)$. We want to prove that α comes from F^{n-1} or, what is equivalent, $0 = \overline{\alpha} \in E_\infty^{0,n}$. Now, this class comes from an element $\alpha' \in \check{H}^0(\mathcal{U}, \underline{H}^n(G))$. It is the case that this α' is just the image of α under the canonical map

$$H^n(S,G) \to \check{H}^0(\mathcal{U}, \underline{H}^n(G))$$

so $\alpha' = 0$ since $\alpha|_{U_i} = 0$ for any i.

We now replace our Cech-to-cohomology spectral sequence by its base change along $S' \to S$.

Then, $\alpha \in F^{n-1}$ (base change) and we want to prove that it lies on F^{n-2}. The class of $\alpha \in E_\infty^{1,n-1}$ comes from an element in $\alpha' \in \check{H}^1(\mathcal{U}, \underline{H}^{n-1}(G))$. This is the $(n-1)$-th cohomology group of the standard Cech complex

$$\prod_i H^{n-1}(U_i, G) \to \prod_{i,j} H^{n-1}(U_{ij}, G) \to \cdots$$

By induction, every term of this complex can be annihilated by a finite surjective morphisms to the corresponding schemes. By Lemma 3.2, we can a find a global finite surjective morphism $S' \to S$ such that $\alpha'|'_S = 0$.

After replacing S with S', the (again base changed) Cech spectral sequence then implies that α comes from $\check{H}^2(\mathcal{U}, \underline{H}^{n-2}(G))$ and we can repeat our process. Proceeding this way we can reduce the second index q all the way down to 0, i.e, assume that the class α lies in the image of of the map

$$\check{H}^n(\mathcal{U}, G) \to H^n_{\mathrm{fppf}}(S,G) \tag{$\bigstar$}$$

Now, we take the normalization $S' \to S$ (which is a finite morphism since S is excellent!) and do base change (noch mal). Thus we may assume that S is normal and we position ourselves in the same situation, α lies on the image of ($\bigstar$). If we prove that $\check{H}^n(\mathcal{U}, G) = 0$ we are finally done. Using again, the Cech-to-derived spectral sequence, but in the Zariski site, one can manage prove using Proposition 3.4 that in fact

$$\check{H}^n(\mathcal{U}, G) \cong H^n_{\mathrm{Zar}}(S,G)$$

and we are done. □

4 Theorem 2

It was essential for the proof of Theorem 1.1 to show that G was a Zariski constant sheaf. In the same manner we have the following lemma, which is beautiful on its own, and we state without proof:

Lemma 4.1. Let S be a regular connected excellent noetherian scheme, and let $f\colon A \to S$ be an abelian scheme. For any non-empty $U \subset S$, the restriction map $A(S) \to A(U)$ is bijective.

The regularity assumption cannot be weakened to much as we will see in the next example.

Example 4.2. Let k be a field. Let $(E, e) \subset \mathbb{P}^2_k$ be an elliptic curve, and let S be the affine cone on E with origin s. We can see S is normal since it is a hypersurface singularity of dimension 2, hence Cohen-Macaulay, and it has 0-dimensional singular locus, so regular in codimension 1.

Consider $A = S \times E$ the constant abelian scheme on E over S and set $U = S/\{s\}$. The projection $p\colon U \to E$ yields a non-constant morphism $p \times id_U \colon U \to A$. On the other hand, all sections $S \to A$ are constant. Indeed, every point of S lies on an $\mathbb{A}^1_k$ containing s. If we can prove that every map $\mathbb{A}^1_k \to E$ is constant, the claim follows.

Let $\mathbb{A}^1_k \to E$ be a non-constant map. By properness, we may extend this to a non-constant morphism $f\colon \mathbb{P}^1_k \to E$. Since $\mathbb{P}^1_k$ is proper and f is non-constant, it follows that f is finite and $K(E) \subset k(\mathbb{P}^1)$ is a finite field extension.

Assume first that $k(E) \subset k(\mathbb{P}^1_k)$ is separable. Then, by [H, IV. 2.1] we would have that the map

$$H^0(\mathbb{P}^1_k, f^*\Omega_{E/k}) \to H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1_k/k}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_k}(-2)) = 0$$

is injective. But $\Omega_{E/k}$ is a free sheaf! Thus $f^*\Omega_{E/k}$ is a free sheaf on $\mathbb{P}^1$ and, it particular, it has sections. This is a contradiction.

If the field extension is not separable we split it as $k(E) \subset k(E)^{\text{sep}} \subset k(\mathbb{P}^1)$ by a separable extension and purely inseparable one. By the equivalence of categories of curves and function fields ([H, I.§6] we have a non-singular projective curve C and a factorization of f as follows

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{f} & E \\ & {\scriptstyle F}\searrow \quad \nearrow{\scriptstyle f'} & \\ & C & \end{array}$$

By [H, IV. 2.5], $C \cong \mathbb{P}^1$ (because of purely inseparability of the corresponding field extensions). If f is non-constant, then so is f'. Since f' is separable, this cannot happen as we have already seen.

To finish, we mention that the proof of Theorem 1.2 is somehow analogous to the proof of Theorem 1.1. Instead of using normalization in the final step of the proof, de Jong's alterations [dJ] are used together with an argument by noetherian approximation.

Bibliography

[B] B. Bhatt, *Annihilating the cohomology of group schemes*, Algebra Number Theory **6** (2012), no. 7, 1561–1577, DOI 10.2140/ant.2012.6.1561. MR3007159

[BBM] P. Berthelot, L. Breen, and W. Messing, *Théorie de Dieudonné cristalline. II*, Lecture Notes in Mathematics, vol. 930, Springer-Verlag, Berlin, 1982 (French). MR0667344

[C] B. Conrad, *Deligne's notes on Nagata compactifications*, J. Ramanujan Math. Soc. **22** (2007), no. 3, 205–257. MR2356346

[G1] A. Grothendieck, *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, Institut des Hautes Études Scientifiques, Paris, 1963. Troisième édition, corrigée; Séminaire de Géométrie Algébrique, 1960/61. MR0217087

[G2] ———, *Le groupe de Brauer. III. Exemples et compléments*, Dix exposés sur la cohomologie des schémas, Adv. Stud. Pure Math., vol. 3, North-Holland, Amsterdam, 1968, pp. 88–188 (French). MR0244271

[M] J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. No. 33, Princeton University Press, Princeton, NJ, 1980. MR0559531

[dJ] A. J. de Jong, *Smoothness, semi-stability and alterations*, Inst. Hautes Études Sci. Publ. Math. **83** (1996), 51–93. MR1423020

[H] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157