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**GEOMETRÍA ALGEBRAICA**
**MPG3972**

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In writing this book, I have attempted to present what is essential for a basic course in algebraic geometry. I wanted to make accessible to the nonspecialist an area of mathematics whose results up to now have been widely scattered, and linked only by unpublished "folklore." While I have reorganized the material and rewritten proofs, the book is mostly a synthesis of what I have learned from my teachers, my colleagues, and my students. They have helped in ways too numerous to recount. I owe especial thanks to Oscar Zariski, J.-P. Serre, David Mumford, and Arthur Ogus for their support and encouragement.

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## §1. Variedades afines.

Generalidades algebraicas: Un *anillo* es un anillo conmutativo $(A,+,\cdot)$ con $1\neq 0$. Un anillo sin divisores del cero es un *dominio*. Una unidad es un elemento invertible, y si todo $a\in A\setminus\{0\}$ es unidad, entonces $A$ se llama *cuerpo*.

Ej: $\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$, $\mathbb{F}_{p^s}$

$k$ cuerpo, $k[x_1,\dots,x_n]\subset k(x_1,\dots,x_n)$

clausuras algebraicas, etc.

Aquí detenemos el preludio algebraico, un recuento rápido se da al inicio de "Algebraic Curves" de W. Fulton, y desarrollo un poco más a lo largo del libro. (Para mucho más ver Atiyah y Macdonald "Introduction to commutative algebra".)

- Sea $k$ cuerpo. (típicamente $k=\overline{k}$)

- $\mathbb{A}^n_k = k^n$ es el <u>espacio afín</u> con la <u>topología Zariski</u>: cerrado = <u>conjunto algebraico</u>, ie, existe conjunto $S \subset k[x_1, \dots, x_n]$ tal que

$$Z = Z(f, f \in S) := \{ p \in \mathbb{A}^n_k : f(p) = 0 \ \forall f \in S \}.$$

- Si $I = \langle S \rangle \Rightarrow Z = Z(f, f \in I)$.
Como todo ideal de $k[x_1, \dots, x_n]$ es finitamente generado $\Rightarrow I = (f_1, \dots, f_m)$ y
(base Hilbert)

$$Z = Z(f_1, \dots, f_m).$$

$Z = Z(f)$ se llama <u>hipersuperficie</u>. Así todo $Z$ es intersección finita de hipersuperficies.

<u>Ej</u> $Z_1 = Z(x)$ en $\mathbb{A}^1_k \Rightarrow Z_1 = \{0\}$

$Z_2 = Z(x^n) = Z_1$.

- El <u>anillo de coordenadas</u> de $Z = Z(I)$ se define como

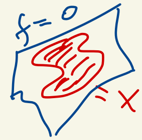
$$\Gamma(Z) := k[x_1, \dots, x_n] / I(Z)$$

y así depende de $I$. (Si $k=\overline{k}$ e $I$ primo $\Rightarrow \Gamma(Z)$ se identifica con las "funciones polinomiales")

En geometría algebraica clásica (semestre pasado), buscábamos "el mejor" de los $I$ que definen a $Z$:

Dado $X$ conjunto en $\mathbb{A}^n_k$,

$$I(X) = \{ f \in k[x_1,\dots,x_n] : f(p)=0 \ \forall p \in X \}.$$




Si $Z = Z(I) \Rightarrow Z = Z(I(Z))$, definen lo mismo y tenemos $I \subseteq I(Z)$. Luego redefinir $\Gamma(Z)$

- Todo conjunto algebraico $Z$ ($\cancel{k=\bar{k}}$) se puede expresar de forma única

$k=\bar{k}$ $\quad f = f_1^{\alpha_1}\cdots f_m^{\alpha_m}$

$Z(f) = Z(f_1)\cup\dots\cup Z(f_m)$

$$Z = Z_1 \cup Z_2 \cup \dots \cup Z_n$$

donde los $Z_i$ son irreducibles (no vacíos) con $Z_i \not\subseteq Z_j$.

Una variedad afín es un conj. alg. irreducible.

Ejl $Z(x) = Z(x^n)$ es variedad.

Para una variedad $Z$ el anillo de coordenadas se eligió $k[x_1,\dots,x_n]/I(Z)$ y así:

$$Z \text{ es variedad} \iff \Gamma(Z) \text{ es dominio}$$

Ejl.- Intentar ejemplo mejorado.

$Z = Z(y^2 - x^3) \subseteq \mathbb{A}^2_k$


$y^2 = x^3$ es variedad

char $k=2$

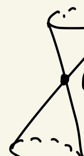
$(x+y+z)^2 = 0$

$Z=$

$Z = Z(x^2+y^2+z^2) \subseteq \mathbb{A}^3_k$

$k = \mathbb{R} \Rightarrow \{(0,0,0)\} = Z$

$k = \mathbb{C}$




Sea $\underbrace{(a,b,c)}_{=P} \in Z \Rightarrow (ta,tb,tc) \in Z \Rightarrow \overleftrightarrow{OP}$ rectas $\subseteq Z$ cono

Terminaremos con Nullstellensatz y dimensión ...

- El teorema de los ceros de Hilbert (Nullstellensatz) nos dice: Si $k=\overline{k}$, entonces

$$Z(I) \neq \emptyset \iff I \subsetneq k[x_1,\dots,x_n].$$

Y con el argumento de 1 página de Rabinowitsch se demuestra fácil: $I(Z(I)) = \sqrt{I}$.

Tenemos las siguientes correspondencias 1-1 entre

geometría $\qquad$ Álgebra

$$X \longmapsto I(X)$$
$$Z(I) \longleftarrow\!\shortmid\ I$$

$$\left\{\begin{array}{c}\text{puntos de}\\ \mathbb{A}^n_k\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{ideales maximales}\\ \text{de } k[x_1,\dots,x_n]\end{array}\right\}$$

$$(a_1,\dots,a_n) \longleftrightarrow (x_1-a_1,\dots,x_n-a_n)$$

$$\left\{\begin{array}{c}\text{variedades de}\\ \mathbb{A}^n_k\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{ideales primos}\\ \text{de } k[x_1,\dots,x_n]\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{Conjuntos algebraicos}\\ \text{de } \mathbb{A}^n_k\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{ideales radicales}\\ \text{de } k[x_1,\dots,x_n]\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{hipersuperficies irreducibles}\\ \text{de } \mathbb{A}^n_k\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{polinomios irred.}\\ \text{salvo mult. por } k^*\end{array}\right\}$$

y $\quad Z(I)$ finito $\iff k[x_1,\dots,x_n]/I$ de dimensión finita.

- Para un espacio topológico $Z$, la <u>dimensión</u> de $Z$ ( $\dim(Z)$ ) es el supremo sobre todos los $n$ para los cuales es posible encontrar una cadena

$$\emptyset \subsetneq X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$$

de conjuntos cerrados irreducibles de $Z$. Así tenemos una equivalencia con la dimensión de Krull de $\Gamma(Z)$ (ver Hart. p.6). [supremo de las alturas de los ideales primos]

- [Hart, Theorem 1.8A] $k$ cuerpo, $B$ dominio f.g. $k$-álgebra. Entonces $\dim B$ = grado trascendencia del cuerpo de fracciones de $B$ sobre $k$

- [Prop 1.13]

Una variedad $X \subset \mathbb{A}^n_k$ tiene dimensión $n-1$ $\iff$ $X = Z(f)$, $f$ polinomio no constante irreducible