Mixed Hodge Structures

Pedro Montero and Roberto Villaflor

1 Introduction

1.1 Pure Hodge Structures

Definition 1. A **Hodge structure of weight** k in a $\mathbb{Q}$-vector space H is a decreasing filtration $F^pH_{\mathbb{C}}$, on the complexified vector space $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$, such that

$$F^p \oplus \overline{F^q} = H_{\mathbb{C}}, \quad \forall p+q=k+1.$$

It induces a **Hodge decomposition**

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} := F^p \cap \overline{F^q}.$$

In particular $F^p = H^{p,q} \oplus H^{p+1,q-1} \oplus \cdots \oplus H^{k,0}$ and $\overline{H^{p,q}} = H^{q,p}$. We denote

$$Gr_F^p H_{\mathbb{C}} := F^p/F^{p+1} \simeq H^{p,q}.$$

Theorem 1 (Hodge decomposition)**.** Let X be a compact Kähler manifold of dimension n, then for every $0 \leq k \leq 2n$, the k-th Betti cohomology (or singular cohomology) $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k, induced by the decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \ , \quad H^{p,q}(X) := \frac{\{\text{closed } (p,q)\text{-forms}\}}{\{\text{exact } (p,q)\text{-forms}\}} \simeq H^{p,q}_{\overline{\partial}}(X) \simeq H^q(X,\Omega_X^p).$$

This decomposition is compatible with the cup products with the polarization $\theta \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$, and so it also induces a Hodge structure of weight k on $H^k(X,\mathbb{Q})_{\text{prim}}$.

Example 1. More examples of Hodge structures in the cohomology of varieties:

1. Let $f: Y \to X$ be a surjective holomorphic map between compact complex manifolds. If Y is Kähler, then $f^*: H^k(X,\mathbb{Q}) \hookrightarrow H^k(Y,\mathbb{Q})$ induces a Hodge structure of weight k on $H^k(X,\mathbb{Q})$.

2. Let X be a compact complex manifold bimeromorphic to a compact Kähler manifold Y. If $f: Z \to X$ is the morphism obtained by resolving the indeterminacy of the bimeromorphism, then Z is compact Kähler (since it is a blow-up of Y) and so $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k. In this case and in the previous one

$$Gr_F^p H^k(X,\mathbb{C}) \simeq H^q(X,\Omega_X^p).$$

3. An **almost Kähler orbifold** is an orbifold (or V-manifold, i.e. whose singularities are quotient of the unit ball by a finite subrgroup of $\mathrm{GL}_n(\mathbb{C})$) X for which there exists a manifold Y bimeromorphic to a Kähler manifold and a **proper modification** (i.e. proper holomorphic map biholomorphic outside a nowhere dense analytic subset) $f: Y \to X$ which is surjective. If X is a compact almost Kähler orbifold, then $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k. In this case

$$Gr_F^p H^k(X,\mathbb{C}) \simeq H^q(X,\widetilde{\Omega}_X^p) = H^q(X^{ns},\Omega^p_{X^{ns}}).$$

Definition 2. A **morphism** of Hodge structures of weight k, $\alpha : (H, F) \to (H', F')$, is a $\mathbb{Q}$-linear map $\alpha : H \to H'$ respecting the filtrations, i.e. $\alpha_{\mathbb{C}}(F^p) \subseteq F'^p$.

Remark. All maps in cohomology coming from geometry are morphisms of Hodge structures (when both cohomology groups have Hodge structures), e.g. pull-back maps, push-forward maps, cup products, Gysin (or Thom) maps, etc. But usually they do not respect the Hodge filtration, for instance if X is compact Kähler with polarization $\theta \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ and $Y \subseteq X$ is a smooth hypersurface, the Gysin map

$$\alpha : H^k(Y, \mathbb{Q}) \xrightarrow{\cup\theta} H^{k+2}(X, \mathbb{Q})$$

do not respect the Hodge filtrations since they have different weights. In fact, it satisfies

$$\alpha(F^p H^k(Y, \mathbb{C})) \subseteq F^{p+1} H^{k+2}(X, \mathbb{C}) \quad \text{or equivalently} \quad \alpha(H^{p,q}(Y)) \subseteq H^{p+1,q+1}(X)$$

for this reason we say it is a **morphism of Hodge structures of type** $(1, 1)$.

1.2 Mixed Hodge Structures

Definition 3. A **mixed Hodge structure** in a $\mathbb{Q}$-vector space H is given by an increasing filtration $W_k H$ called the **weight filtration**, and a decreasing filtration $F^p H_{\mathbb{C}}$ called the **Hodge filtration**, such that the induced Hodge filtration on each

$$Gr_k^W H := W_k / W_{k-1}$$

is a Hodge structure of weight k.

Example 2. Mixed Hodge structures on varieties arise when one consider non-compact varieties. This structure encodes cohomological information coming from the compactifications, but at the same time is canonical (i.e. independent of the chosen compactification):

1. Every (pure) Hodge structure of weight k on H determines a mixed Hodge structure given by
$$W_k H := H \ , \quad W_m H := 0 \ \text{ for } m < k.$$

2. Let X be a smooth complete intersection and $Y \subseteq X$ be a smooth very ample divisor. Then $H^k(U, \mathbb{Q})$ has a mixed Hodge structure, where $U := X \setminus Y$. In fact
$$W_{k+1} H^k(U, \mathbb{Q}) = H^k(U, \mathbb{Q}) \ , \quad W_m H^k(U, \mathbb{Q}) = 0 \ \text{ for } m < k,$$
$$Gr_k^W H^k(U, \mathbb{Q}) \simeq H^k(X, \mathbb{Q})_{\text{prim}} \ , \ Gr_{k+1}^W H^k(U, \mathbb{Q}) \simeq H^{k-1}(Y, \mathbb{Q})_{\text{prim}}.$$

3. Let X be a smooth complete intersection in $\mathbb{P}^N$, $Y, Z \subseteq X$ be two smooth very ample divisors s.t. $Y \cap Z$ is also smooth (transversal). Then $U := X \setminus (Y \cup Z)$ has a mixed Hodge structure of the form
$$W_{k+2} H^k(U, \mathbb{Q}) = H^k(U, \mathbb{Q}) \ , \quad W_m H^k(U, \mathbb{Q}) = 0 \ \text{ for } m < k,$$
$$Gr_k^W H^k(U, \mathbb{Q}) \simeq H^k(X, \mathbb{Q})_{\text{prim}} \ , \ Gr_{k+1}^W H^k(U, \mathbb{Q}) \simeq H^{k-1}(Y, \mathbb{Q})_{\text{prim}} \oplus H^{k-1}(Z, \mathbb{Q})_{\text{prim}},$$
$$Gr_{k+2}^W H^k(U, \mathbb{Q}) \simeq H^{k-2}(Y \cap Z, \mathbb{Q})_{\text{prim}}.$$

4. In general for any smooth algebraic variety U, we can induce a mixed Hodge structure on it once we find a smooth compactification $U \hookrightarrow X$ with $Y = X \setminus U$ a normal crossing divisor. Moreover, for any algebraic variety X, $H^k(X, \mathbb{Q})$ has a mixed Hodge structure, also there are mixed Hodge structures in other cohomology groups such as $H^k_c(U, \mathbb{Q})$, $H^k(X, Y, \mathbb{Q})$, also in homotopy groups and other topological invariants.

Remark. As in the case of pure Hodge structures, all cohomology maps coming from geometry are morphisms of mixed Hodge structures in some sense. This is the case for instance of the residue map

$$res : H^k(X \setminus Y, \mathbb{Q}) \to H^{k-1}(Y, \mathbb{Q}),$$

which is a morphism of mixed Hodge structures of type $(-1, -1)$.

1.3 Main Theorem

Theorem 2 (Deligne)**.** Every morphism of mixed Hodge structures

$$\alpha : (H, W, F) \to (H', W', F')$$

is **strict** in the following sense

$$\mathrm{Im}(\alpha) \cap W'_m H' = \alpha(W_m H),$$

$$\mathrm{Im}(\alpha) \cap F'^P H'_{\mathbb{C}} = \alpha(F^p H_{\mathbb{C}}).$$

Corollary 1. Every morphism of mixed Hodge structures which is an isomorphism of $\mathbb{Q}$-vector spaces is an isomorphism of mixed Hodge structures.

Corollary 2. If U is a smooth algebraic variety with smooth compactifications $U \hookrightarrow X$, $U \hookrightarrow Y$ with boundary a normal crossing divisor, then both compactifications induce the same mixed Hodge structure on U.

Proof Taking Z a resolution of $\overline{\Delta_U} \subseteq X \times Y$ such that $U \simeq \Delta_U \hookrightarrow Z$ has a normal crossing divisor in the boundary, we get the isomorphisms of mixed Hodge structures

$$\begin{array}{ccc} & (H^k(U, \mathbb{Q}), W(Z), F(Z)) & \\ \overset{\mathrm{pr}_1^*}{\nearrow} & & \overset{\mathrm{pr}_2^*}{\nwarrow} \\ (H^k(U, \mathbb{Q}), W(X), F(X)) & & (H^k(U, \mathbb{Q}), W(Y), F(Y)). \end{array}$$

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The following corollary is very useful for computations.

Corollary 3. Mixed Hodge structures respect exact sequences.

Example 3. Using the above corollary we can obtain a lot of information of the mixed Hodge structure of a cohomology group once we put it inside an exact sequence.

1. Let X be a compact Kähler manifold and $Y \subseteq X$ be a smooth hypersurface. Then the Leray-Thom-Gysin sequence

$$\cdots \to H^k(X,\mathbb{Q}) \to H^k(U,\mathbb{Q}) \xrightarrow{res} H^{k-1}(Y,\mathbb{Q}) \xrightarrow{\cup\theta} H^{k+1}(X,\mathbb{Q}) \to \cdots$$

   gives us $W_m H^k(U,\mathbb{Q}) = 0$ for $m < k$, also $Gr_r^W H^k(U,\mathbb{Q}) = 0$ for $r > k+1$, and

$$0 \to Gr_{k+1}^W H^k(U,\mathbb{Q}) \xrightarrow{res} H^{k-1}(Y,\mathbb{Q}) \xrightarrow{\cup\theta} H^{k+1}(X,\mathbb{Q}) \to Gr_{k+1}^W H^{k+1}(U,\mathbb{Q}) \to 0.$$

   This sequence also exists in the context of orbifolds.

2. Similar computations can be done using Mayer-Vietoris sequences (usual one and with compact support), the long exact sequence of a pair (X,Y), etc.

1.4 Spectral Sequences

Up to now we have not explained how mixed Hodge structures are constructed. It turns out that to construct the mixed Hodge structures mentioned before on the cohomology of a variety X, what we really do is to construct a **mixed Hodge complex of sheaves**

$$\mathcal{K}^\bullet = (\mathcal{K}^\bullet, W, (\mathcal{K}^\bullet_{\mathbb{C}}, W, F), \beta)$$

where $\mathcal{K}^\bullet$ is a complex of $\mathbb{Q}$-vector spaces such that

$$\mathbb{H}^k(X,\mathcal{K}^\bullet) \simeq H^k(X,\mathbb{Q})$$

(or the respective cohomology group where the mixed Hodge structure will be defined), W is an increasing filtration on the complex $\mathcal{K}^\bullet$ which induces the weight filtration by

$$W_m H^k(X,\mathbb{Q}) \simeq W_m \mathbb{H}^k(X,\mathcal{K}^\bullet) := \mathrm{Im}(\mathbb{H}^k(X, W_{m-k}\mathcal{K}^\bullet) \to \mathbb{H}^k(X,\mathcal{K}^\bullet)),$$

$(\mathcal{K}^\bullet_{\mathbb{C}}, W, F)$ is a bifiltered complex of sheaves such that

$$\mathbb{H}^k(X,\mathcal{K}^\bullet_{\mathbb{C}}) \simeq H^k(X,\mathbb{C})$$

induces a bifiltered $\mathbb{C}$-vector space $(H^k(X,\mathbb{C}), W, F)$, and

$$\beta : (\mathcal{K}^\bullet \otimes \mathbb{C}, W) \dashrightarrow (\mathcal{K}^\bullet_{\mathbb{C}}, W)$$

is a pseudo-isomorphism inducing an isomorphism of filtered $\mathbb{C}$-vector spaces

$$(\mathbb{H}^k(X,\mathcal{K}^\bullet), W) \otimes \mathbb{C} \simeq (\mathbb{H}^k(X,\mathcal{K}^\bullet_{\mathbb{C}}), W),$$

such that the resulting filtrations on $(H^k(X,\mathbb{Q}), W, F)$ form a mixed Hodge structure.

Remark. In practice, we do not need to know $\mathcal{K}^\bullet$ in order to compute the mixed Hodge structure, it is enough to know its existence and to use $(\mathcal{K}^\bullet_{\mathbb{C}}, W, F)$ to compute it.

Example 4. For U a smooth variety, X a smooth compactification such that $Y := X \setminus U$ is a normal crossing divisor, the mixed Hodge structure on U is induced the mixed Hodge complex of sheaves

$$(\Omega_X(\log Y)^\bullet, W, F)$$

where

$$W_m\Omega_X(\log Y)^\bullet := \Omega_X^{\bullet-m} \wedge \Omega_X^m(\log Y) \quad \text{and} \quad F^p\Omega_X(\log Y)^\bullet := \Omega_X(\log Y)^{\bullet \geq p}.$$

Note that

$$Gr_m^W \Omega_X(\log Y)^\bullet \simeq \Omega_{Y(m)}^{\bullet-m} \quad \text{and} \quad Gr_F^p \Omega_X(\log Y)^\bullet = \Omega_X(\log Y)^p[-p],$$

where $Y(m)$ is the disjoint union of all subvarieties of Y given locally as the intersection of m local components of Y. The same holds for orbifolds replacing Ω by $\widetilde{\Omega}$ and the normal crossing divisor by a V-normal crossing divisor.

In order to compute the mixed Hodge structure we need to know what is the relation between the grading induced by the weight and Hodge filtrations on $H^k(X, \mathbb{C})$ and the respective gradings at the level of complexes of sheaves (which are usually simple to describe). This information is encoded in their associated **spectral sequences** whose behavior is described as follows.

Theorem 3. For a mixed Hodge structure induced on $H^k(X, \mathbb{Q})$ by a mixed Hodge complex of sheaves $\mathcal{K}^\bullet$, the spectral sequence associated to the weight filtration is given by

$$E_1^{-m,m+k} = \mathbb{H}^k(X, Gr_m^W \mathcal{K}^\bullet)$$

and degenerates at E_2. This means that

$$Gr_{m+k}^W H^k(X, \mathbb{Q}) \simeq E_2^{-m,m+k} = H(E_1^{-m-1,m+k} \xrightarrow{d_1} E_1^{-m,m+k} \xrightarrow{d_1} E_1^{-m+1,m+k}).$$

And the spectral sequence associated to the Hodge filtration degenerates at E_1 and is given by

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr_F^p \mathcal{K}_{\mathbb{C}}^\bullet) \simeq Gr_F^p H^k(X, \mathbb{C}).$$

Example 5. In our previous example the spectral sequence of the weight filtration corresponds to

$$E_1^{-m,m+k} = \mathbb{H}^k(X, \Omega_{Y(m)}^{\bullet-m}) \simeq H^{k-m}(Y(m), \mathbb{Q}),$$

the map $d_1 : H^{k-m}(Y(m), \mathbb{Q}) \to H^{k-m+2}(Y(m-1), \mathbb{Q})$ is induced by the Gysin morphisms in each component. And the spectral sequence of the Hodge filtration corresponds to

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \Omega_X(\log Y)^p[-p]) = H^q(X, \Omega_X(\log Y)^p).$$