Mixed Hodge Structures

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1 Introduction

This short lecture series given at PUC Chile is mainly based on [PS08, Laz16, Gri18].

1.1 Pure Hodge structures

Definition 1. A **Hodge structure of weight** k in a $\mathbb{Q}$-vector space H is a decreasing filtration $F^pH_{\mathbb{C}}$, on the complexified vector space $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$, such that

$$F^p \oplus \overline{F^q} = H_{\mathbb{C}}, \quad \forall p+q=k+1.$$

It induces a **Hodge decomposition**

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}, \qquad H^{p,q} := F^p \cap \overline{F^q}.$$

In particular $F^p = H^{p,q} \oplus H^{p+1,q-1} \oplus \cdots \oplus H^{k,0}$ and $\overline{H^{p,q}} = H^{q,p}$. We denote

$$Gr_F^p H_{\mathbb{C}} := F^p/F^{p+1} \simeq H^{p,q}.$$

Theorem 1 (Hodge decomposition)**.** Let X be a compact Kähler manifold of dimension n, then for every $0 \leq k \leq 2n$, the k-th Betti cohomology (or singular cohomology) $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k, induced by the decomposition

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)\ , \qquad H^{p,q}(X) := \frac{\{\text{closed } (p,q)\text{-forms}\}}{\{\text{exact } (p,q)\text{-forms}\}} \simeq H^{p,q}_{\overline{\partial}}(X) \simeq H^q(X,\Omega_X^p).$$

This decomposition is compatible with the cup products with the polarization $\theta \in H^{1,1}(X) \cap H^2(X,\mathbb{Z})$, and so it also induces a Hodge structure of weight k on $H^k(X,\mathbb{Q})_{\text{prim}}$.

Example 1. More examples of Hodge structures in the cohomology of varieties:

1. Let $f: Y \to X$ be a surjective holomorphic map between compact complex manifolds. If Y is Kähler, then $f^*: H^k(X,\mathbb{Q}) \hookrightarrow H^k(Y,\mathbb{Q})$ induces a Hodge structure of weight k on $H^k(X,\mathbb{Q})$.

2. Let X be a compact complex manifold bimeromorphic to a compact Kähler manifold Y. If $f: Z \to X$ is the morphism obtained by resolving the indeterminacy of the bimeromorphism, then Z is compact Kähler (since it is a blow-up of Y) and so $H^k(X,\mathbb{Q})$ has a Hodge structure of weight k. In this case and in the previous one

$$Gr_F^p H^k(X,\mathbb{C}) \simeq H^q(X,\Omega_X^p).$$

3. An **almost Kähler orbifold** is an orbifold (or V-manifold, i.e. whose singularities are quotient of the unit ball by a finite subrgroup of $\mathrm{GL}_n(\mathbb{C})$) X for which there exists a manifold Y bimeromorphic to a Kähler manifold and a **proper modification** (i.e. proper

holomorphic map biholomorphic outside a nowhere dense analytic subset) $f : Y \to X$ which is surjective. If X is a compact almost Kähler orbifold, then $H^k(X, \mathbb{Q})$ has a Hodge structure of weight k. In this case

$$Gr_F^p H^k(X, \mathbb{C}) \simeq H^q(X, \widetilde{\Omega}_X^p) = H^q(X^{ns}, \Omega_{X^{ns}}^p).$$

Definition 2. A **morphism** of Hodge structures of weight k, $\alpha : (H, F) \to (H', F')$, is a $\mathbb{Q}$-linear map $\alpha : H \to H'$ respecting the filtrations, i.e. $\alpha_{\mathbb{C}}(F^p) \subseteq F'^p$.

Remark. All maps in cohomology coming from geometry are morphisms of Hodge structures (when both cohomology groups have Hodge structures), e.g. pull-back maps, push-forward maps, cup products, Gysin (or Thom) maps, etc. But usually they do not respect the Hodge filtration, for instance if X is compact Kähler with polarization $\theta \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ and $Y \subseteq X$ is a smooth hypersurface, the Gysin map

$$\alpha : H^k(Y, \mathbb{Q}) \xrightarrow{\cup\theta} H^{k+2}(X, \mathbb{Q})$$

do not respect the Hodge filtrations since they have different weights. In fact, it satisfies

$$\alpha(F^p H^k(Y, \mathbb{C})) \subseteq F^{p+1} H^{k+2}(X, \mathbb{C}) \quad \text{or equivalently} \quad \alpha(H^{p,q}(Y)) \subseteq H^{p+1,q+1}(X)$$

for this reason we say it is a **morphism of Hodge structures of type** $(1, 1)$.

1.2 Mixed Hodge structures

Definition 3. A **mixed Hodge structure** in a $\mathbb{Q}$-vector space H is given by an increasing filtration $W_k H$ called the **weight filtration**, and a decreasing filtration $F^p H_{\mathbb{C}}$ called the **Hodge filtration**, such that the induced Hodge filtration on each

$$Gr_k^W H := W_k / W_{k-1}$$

is a Hodge structure of weight k.

Example 2. Mixed Hodge structures on varieties arise when one consider non-compact varieties. This structure encodes cohomological information coming from the compactifications, but at the same time is canonical (i.e. independent of the chosen compactification):

1. Every (pure) Hodge structure of weight k on H determines a mixed Hodge structure given by
$$W_k H := H \ , \quad W_m H := 0 \ \text{ for } m < k.$$

2. Let X be a smooth complete intersection and $Y \subseteq X$ be a smooth very ample divisor. Then $H^k(U, \mathbb{Q})$ has a mixed Hodge structure, where $U := X \setminus Y$. In fact
$$W_{k+1} H^k(U, \mathbb{Q}) = H^k(U, \mathbb{Q}) \ , \quad W_m H^k(U, \mathbb{Q}) = 0 \ \text{ for } m < k,$$
$$Gr_k^W H^k(U, \mathbb{Q}) \simeq H^k(X, \mathbb{Q})_{\text{prim}} \ , \ Gr_{k+1}^W H^k(U, \mathbb{Q}) \simeq H^{k-1}(Y, \mathbb{Q})_{\text{prim}}.$$

3. Let X be a smooth complete intersection in $\mathbb{P}^N$, $Y, Z \subseteq X$ be two smooth very ample divisors s.t. $Y \cap Z$ is also smooth (transversal). Then $U := X \setminus (Y \cup Z)$ has a mixed Hodge structure of the form

$$W_{k+2}H^k(U,\mathbb{Q}) = H^k(U,\mathbb{Q}) \ , \quad W_m H^k(U,\mathbb{Q}) = 0 \ \text{ for } m < k,$$

$$Gr_k^W H^k(U,\mathbb{Q}) \simeq H^k(X,\mathbb{Q})_{\text{prim}} \ , \ Gr_{k+1}^W H^k(U,\mathbb{Q}) \simeq H^{k-1}(Y,\mathbb{Q})_{\text{prim}} \oplus H^{k-1}(Z,\mathbb{Q})_{\text{prim}},$$
$$Gr_{k+2}^W H^k(U,\mathbb{Q}) \simeq H^{k-2}(Y \cap Z,\mathbb{Q})_{\text{prim}}.$$

4. In general for any smooth algebraic variety U, we can induce a mixed Hodge structure on it once we find a smooth compactification $U \hookrightarrow X$ with $Y = X \setminus U$ a normal crossing divisor. Moreover, for any algebraic variety X, $H^k(X,\mathbb{Q})$ has a mixed Hodge structure, also there are mixed Hodge structures in other cohomology groups such as $H_c^k(U,\mathbb{Q})$, $H^k(X,Y,\mathbb{Q})$, also in homotopy groups and other topological invariants.

Remark. As in the case of pure Hodge structures, all cohomology maps coming from geometry are morphisms of mixed Hodge structures in some sense. This is the case for instance of the residue map

$$res : H^k(X \setminus Y,\mathbb{Q}) \to H^{k-1}(Y,\mathbb{Q}),$$

which is a morphism of mixed Hodge structures of type $(-1,-1)$.

1.3 Main theorem

Theorem 2 (Deligne)**.** Every morphism of mixed Hodge structures

$$\alpha : (H, W, F) \to (H', W', F')$$

is **strict** in the following sense

$$\operatorname{Im}(\alpha) \cap W'_m H' = \alpha(W_m H),$$

$$\operatorname{Im}(\alpha) \cap F'^{P} H'_{\mathbb{C}} = \alpha(F^p H_{\mathbb{C}}).$$

Corollary 1. Every morphism of mixed Hodge structures which is an isomorphism of $\mathbb{Q}$-vector spaces is an isomorphism of mixed Hodge structures.

Corollary 2. If U is a smooth algebraic variety with smooth compactifications $U \hookrightarrow X$, $U \hookrightarrow Y$ with boundary a normal crossing divisor, then both compactifications induce the same mixed Hodge structure on U.

Proof Taking Z a resolution of $\overline{\Delta_U} \subseteq X \times Y$ such that $U \simeq \Delta_U \hookrightarrow Z$ has a normal crossing divisor in the boundary, we get the isomorphisms of mixed Hodge structures

$$\begin{array}{ccc} & (H^k(U,\mathbb{Q}), W(Z), F(Z)) & \\ \overset{\text{pr}_1^*}{\nearrow} & & \overset{\text{pr}_2^*}{\nwarrow} \\ (H^k(U,\mathbb{Q}), W(X), F(X)) & & (H^k(U,\mathbb{Q}), W(Y), F(Y)). \end{array}$$

■

The following corollary is very useful for computations.

Corollary 3. Mixed Hodge structures respect exact sequences.

Example 3. Using the above corollary we can obtain a lot of information of the mixed Hodge structure of a cohomology group once we put it inside an exact sequence.

1. Let X be a compact Kähler manifold and $Y \subseteq X$ be a smooth hypersurface. Then the Leray-Thom-Gysin sequence

$$\cdots \to H^k(X,\mathbb{Q}) \to H^k(U,\mathbb{Q}) \xrightarrow{res} H^{k-1}(Y,\mathbb{Q}) \xrightarrow{\cup\theta} H^{k+1}(X,\mathbb{Q}) \to \cdots$$

gives us $W_m H^k(U,\mathbb{Q}) = 0$ for $m < k$, also $Gr_r^W H^k(U,\mathbb{Q}) = 0$ for $r > k+1$, and

$$0 \to Gr_{k+1}^W H^k(U,\mathbb{Q}) \xrightarrow{res} H^{k-1}(Y,\mathbb{Q}) \xrightarrow{\cup\theta} H^{k+1}(X,\mathbb{Q}) \to Gr_{k+1}^W H^{k+1}(U,\mathbb{Q}) \to 0.$$

This sequence also exists in the context of orbifolds.

2. Similar computations can be done using Mayer-Vietoris sequences (usual one and with compact support), the long exact sequence of a pair (X,Y), etc.

1.4 Spectral sequences

Up to now we have not explained how mixed Hodge structures are constructed. It turns out that to construct the mixed Hodge structures mentioned before on the cohomology of a variety X, what we really do is to construct a **mixed Hodge complex of sheaves**

$$\mathcal{K}^\bullet = (\mathcal{K}^\bullet, W, (\mathcal{K}^\bullet_{\mathbb{C}}, W, F), \beta)$$

where $\mathcal{K}^\bullet$ is a complex of $\mathbb{Q}$-vector spaces such that

$$\mathbb{H}^k(X,\mathcal{K}^\bullet) \simeq H^k(X,\mathbb{Q})$$

(or the respective cohomology group where the mixed Hodge structure will be defined), W is an increasing filtration on the complex $\mathcal{K}^\bullet$ which induces the weight filtration by

$$W_m H^k(X,\mathbb{Q}) \simeq W_m \mathbb{H}^k(X,\mathcal{K}^\bullet) := \operatorname{Im}(\mathbb{H}^k(X, W_{m-k}\mathcal{K}^\bullet) \to \mathbb{H}^k(X,\mathcal{K}^\bullet)),$$

$(\mathcal{K}^\bullet_{\mathbb{C}}, W, F)$ is a bifiltered complex of sheaves such that

$$\mathbb{H}^k(X,\mathcal{K}^\bullet_{\mathbb{C}}) \simeq H^k(X,\mathbb{C})$$

induces a bifiltered $\mathbb{C}$-vector space $(H^k(X,\mathbb{C}), W, F)$, and

$$\beta : (\mathcal{K}^\bullet \otimes \mathbb{C}, W) \dashrightarrow (\mathcal{K}^\bullet_{\mathbb{C}}, W)$$

is a pseudo-isomorphism inducing an isomorphism of filtered $\mathbb{C}$-vector spaces

$$(\mathbb{H}^k(X,\mathcal{K}^\bullet), W) \otimes \mathbb{C} \simeq (\mathbb{H}^k(X,\mathcal{K}^\bullet_{\mathbb{C}}), W),$$

such that the resulting filtrations on $(H^k(X,\mathbb{Q}), W, F)$ form a mixed Hodge structure.

Remark. In practice, we do not need to know $\mathcal{K}^\bullet$ in order to compute the mixed Hodge structure, it is enough to know its existence and to use $(\mathcal{K}^\bullet_{\mathbb{C}}, W, F)$ to compute it.

Example 4. For U a smooth variety, X a smooth compactification such that $Y := X \setminus U$ is a normal crossing divisor, the mixed Hodge structure on U is induced the mixed Hodge complex of sheaves

$$(\Omega^\bullet_X(\log Y), W, F)$$

where

$$W_m\Omega^\bullet_X(\log Y) := \Omega^{\bullet-m}_X \wedge \Omega^m_X(\log Y) \quad \text{and} \quad F^p\Omega^\bullet_X(\log Y) := \Omega^{\bullet \geq p}_X(\log Y).$$

Note that

$$Gr^W_m\Omega^\bullet_X(\log Y) \simeq \Omega^{\bullet-m}_{Y(m)} \quad \text{and} \quad Gr^p_F\Omega^\bullet_X(\log Y) = \Omega^p_X(\log Y)[-p],$$

where $Y(m)$ is the disjoint union of all subvarieties of Y given locally as the intersection of m local components of Y. The same holds for orbifolds replacing Ω by $\widetilde{\Omega}$ and the normal crossing divisor by a V-normal crossing divisor.

In order to compute the mixed Hodge structure we need to know what is the relation between the grading induced by the weight and Hodge filtrations on $H^k(X, \mathbb{C})$ and the respective gradings at the level of complexes of sheaves (which are usually simple to describe). This information is encoded in their associated **spectral sequences** whose behavior is described as follows.

Theorem 3 (Deligne)**.** For a mixed Hodge structure induced on $H^k(X, \mathbb{Q})$ by a mixed Hodge complex of sheaves $\mathcal{K}^\bullet$, the spectral sequence associated to the weight filtration is given by

$$E_1^{-m,m+k} = \mathbb{H}^k(X, Gr^W_m\mathcal{K}^\bullet)$$

and degenerates at E_2. This means that

$$Gr^W_{m+k}H^k(X, \mathbb{Q}) \simeq E_2^{-m,m+k} = H(E_1^{-m-1,m+k} \xrightarrow{d_1} E_1^{-m,m+k} \xrightarrow{d_1} E_1^{-m+1,m+k}).$$

And the spectral sequence associated to the Hodge filtration degenerates at E_1 and is given by

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr^p_F\mathcal{K}^\bullet_{\mathbb{C}}) \simeq Gr^p_F H^k(X, \mathbb{C}).$$

Example 5. In our previous example the spectral sequence of the weight filtration corresponds to

$$E_1^{-m,m+k} = \mathbb{H}^k(X, \Omega^{\bullet-m}_{Y(m)}) \simeq H^{k-m}(Y(m), \mathbb{Q}),$$

the map $d_1 : H^{k-m}(Y(m), \mathbb{Q}) \to H^{k-m+2}(Y(m-1), \mathbb{Q})$ is induced by the Gysin morphisms in each component. And the spectral sequence of the Hodge filtration corresponds to

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, \Omega^p_X(\log Y)[-p]) = H^q(X, \Omega^p_X(\log Y)).$$

2 Applications

2.1 Vanishing theorems

One of the applications of Deligne's theorem is the proof of vanishing theorems. The idea is to use the degeneration of the spectral sequence associated to the Hodge filtration

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log Y)) \Rightarrow H^{p+q}(X \setminus Y, \mathbb{C})$$

to derive analytic vanishing results from topological vanishing results. This idea is mainly due to Kollár and Esnault-Viehweg. We will just sketch some classical applications, for further reading we refer the reader to [EV92].

Consider first the following situation: Let X be a smooth projective variety of dimension n and L be an ample line bundle with a section vanishing along a normal crossing divisor $Y \subseteq X$. It follows from Deligne's theorem that

$$H^k(X \setminus Y, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p(\log Y))$$

and so (by Atiyah–Hodge theorem $H^k(X \setminus Y, \mathbb{C}) = H^k(\Gamma(\Omega^\bullet_{X \setminus Y}), d)$, or by Andreotti–Fraenkel theorem) we conclude

$$H^q(X, \Omega_X^p(\log Y)) = 0 \quad \text{ for } p+q > n.$$

In particular we obtain a weaker version of **Kodaira vanishing theorem**

$$H^q(X, \Omega_X^n \otimes L) = 0 \quad \text{ for } q > 0.$$

In order to obtain the stronger version for any ample line bundle L we can consider a power L^N such that it has a section vanishing along a smooth divisor $H \subseteq X$ and then take the N-cyclic covering

$$f : Z \to X$$

ramified along H. It is not hard to see that Z and $D := (f^*H)_{red}$ are smooth. Moreover

$$f_*\mathcal{O}_Z = \bigoplus_{i=0}^{N-1} L^{-i}.$$

Again since $Z \setminus D$ is affine we obtain the vanishing for $p+q > n$

$$0 = H^q(Z, \Omega_Z^p(\log D)) = H^q(X, f_*\Omega_Z^p(\log D)) = \bigoplus_{i=0}^{N-1} H^q(X, \Omega_X^p(\log H) \otimes L^{-i})$$

in particular we get the desired result

$$H^q(X, \Omega_X^n \otimes L^{N-i}) = 0 \quad \text{ for } q > 0.$$

This strategy to obtain vanishing results from Deligne's theorem has been exploited by Esnault-Viehweg by means of logarithmic connections. For instance it is possible to show the following logarithmic vanishing result.

Theorem 4 ([EV92] §6.2)**.** $H^q(X, \Omega^p_X(\log H) \otimes L^{-1}) = 0$ for $p + q \neq n$.

From the above result we can go further, and use the Poincaré residue sequence

$$0 \to \Omega^p_X \to \Omega^p_X(\log H) \xrightarrow{Res} \Omega^{p-1}_H \to 0$$

to obtain inductively the **Akizuki-Nakano vanishing theorem**

$$H^q(X, \Omega^p_X \otimes L) = 0 \quad \text{for } p + q > n.$$

There are several vanishing results which can be reobtained and extended to more general situations (e.g. to positive characteristic) using these methods. Another example which has been largely extended to singular varieties by Guillen–Navarro Aznar– Pascual-Gainza–Puerta using filtered De Rham complexes (see [PS08, Theorem 7.29]) is the following:

Theorem 5 (Grauert–Riemenschneider)**.** Let X be a smooth compact complex algebraic variety of dimension n, $\pi : Y \to X$ be a proper modification with Y smooth and L an ample line bundle on X. Then

(a) $H^q(Y, \Omega^n_Y \otimes \pi^* L) = 0$ for $q > 0$,

(b) $R^q \pi_* \Omega^n_Y = 0$ for $q > 0$.

Remark. Given $f : X \to Y$ and $\mathcal{F}$ a sheaf over X we can use Leray's spectral sequence to translate global vanishing theorems into **local vanishing** results of the form

$$R^q f_* \mathcal{F} = 0.$$

This kind of local vanishing results is useful in deformation theory. In fact, we encounter situations where the **obstruction to globalize a local deformation** is encoded by cohomology groups of the form

$$H^k(Y, f_* \mathcal{F}).$$

Leray's spectral sequence gives us

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

Therefore the local vanishing $R^q f_* \mathcal{F} = 0$ for all $q > 0$ reduces the local-to-global obstruction to global vanishing results on the family (which usually translates into an analytic or topological condition on X)

$$H^k(Y, f_* \mathcal{F}) = H^k(X, \mathcal{F}) = 0.$$

Theorem 6 (Global-To-Local principle)**.** Suppose that $f : X \to Y$ is a morphism between projective varieties, q a natural number and $\mathcal{F}$ a coherent sheaf on X with the property that

$$H^q(X, \mathcal{F} \otimes f^* L) = 0$$

for all ample line bundles L on Y. Then

$$R^q f_* \mathcal{F} = 0.$$

Proof Take L sufficiently ample such that $R^q f_* \mathcal{F} \otimes L$ is globally generated and $R^j f_* \mathcal{F} \otimes L$ is acyclic for all $j = 0, 1, \ldots$. Then the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j f_* \mathcal{F} \otimes L) \Rightarrow H^{i+j}(X, \mathcal{F} \otimes f^* L)$$

degenerates at E_2 and so $H^0(Y, R^q f_* \mathcal{F} \otimes L) = H^q(X, \mathcal{F} \otimes f^* L) = 0$. Hence $R^q f_* \mathcal{F} = 0$. ■

2.2 Basis theorems

As we illustrated before, it is possible to obtain vanishing theorems such as Akizuki–Nakano

$$H^q(X, \Omega_X^p \otimes L) = 0 \quad \text{for } p + q > n$$

from the vanishing of the cohomology group $H^k(X \setminus Y, \mathbb{C}) = 0$ for $k > n$. A natural question is to ask ourselves if is it possible to go the other way around, and by this we mean the following:

Question. In the case $H^k(X \setminus Y, \mathbb{C})$ is not trivial, can we have a better understanding of it if we know well the groups $H^q(X, \Omega_X^p \otimes L)$ for $p + q = k$?

In some nice cases the answer to previous question is affirmative, and it turns out to be enough to have a stronger vanishing result due to Bott.

Definition 4. We say a complex compact algebraic variety X satisfies the **Bott vanishing theorem** if for every ample line bundle L

$$H^q(X, \Omega_X^p \otimes L) = 0 \quad \text{for all } p \geq 0,\ q > 0.$$

Example 6. Satisfying the Bott vanishing is a very special property. Some know examples are the following:

1. Bott's original vanishing theorem (1957) states it for $\mathbb{P}^n$.
2. Steenbrink (1977) extended it to weighted projective spaces.
3. Danilov (1978), Batyrev–Cox (1993) proved it for complete simplicial toric varieties.
4. Totaro (2019) proved it for the quintic Del Pezzo surface, and characterized K3 surfaces with Picard number 1 satisfying Bott vanishing as those of degree 20 or ≥ 24. For higher Picard number, K3 surfaces satisfying the Bott vanishing do not contain elliptic curves of low degree nor are hyperplane sections of Fano 3-folds.
5. Torres (2020) proved it for stable GIT quotients of $(\mathbb{P}^1)^n$ by the action of PGL_2.

To link the Bott vanishing with the mixed Hodge structure of $X \setminus Y$ we need to change the usual Hodge filtration on $\Omega_X^\bullet(\log Y)$ by another filtration.

Proposition 1. Let X be a compact algebraic variety (smooth or orbifold) and $Y \subseteq X$ be an ample normal crossing divisor (or V-normal crossing respectively). There is a natural filtered quasi-isomorphism of filtered complexes

$$\Omega_X^{\geq p}(\log Y) \hookrightarrow P^p \Omega_X^\bullet(*Y)$$

and so we can compute

$$F^p H^k(X \setminus Y, \mathbb{C}) \simeq \mathbb{H}^k(X, P^p \Omega_X^\bullet(*Y)))$$

where $P^\bullet \Omega_X^\bullet(*Y)$ is the **pole order filtration** given by

$$
\begin{array}{lccccccc}
\vdots & & & & & & & \\
P^{-1}: & \mathcal{O}_X(2Y) \longrightarrow & \Omega^1_X(3Y) \longrightarrow & \Omega^2_X(4Y) \longrightarrow & \cdots \to & \Omega^{n+1}_X((n+3)Y) & \longrightarrow & 0 \\
\cup & & & & & & & \\
P^{0}: & \mathcal{O}_X(Y) \longrightarrow & \Omega^1_X(2Y) \longrightarrow & \Omega^2_X(3Y) \longrightarrow & \cdots \to & \Omega^{n+1}_X((n+2)Y) & \longrightarrow & 0 \\
\cup & & & & & & & \\
P^{1}: & 0 \longrightarrow & \Omega^1_X(Y) \longrightarrow & \Omega^2_X(2Y) \longrightarrow & \cdots \to & \Omega^{n+1}_X((n+1)Y) & \longrightarrow & 0 \\
\cup & & & & & & & \\
P^{2}: & 0 \longrightarrow & 0 \longrightarrow & \Omega^2_X(Y) \longrightarrow & \cdots \longrightarrow & \Omega^{n+1}_X(nY) & \longrightarrow & 0 \\
\cup & & & & & & & \\
\vdots & & & & & & & \\
P^{k}: & 0 \longrightarrow & \cdots \longrightarrow & 0 \longrightarrow & \Omega^k_X(Y) \longrightarrow & \cdots & \longrightarrow & 0 \\
\cup & & & & & & & \\
\vdots & & & & & & & \\
P^{n+2} = 0 & & & & & & &
\end{array}
$$

In particular if X satisfies the Bott vanishing theorem, then

$$F^p H^k(X \setminus Y, \mathbb{C}) \simeq H^k(\Gamma(X, P^p\Omega^\bullet_X(*Y)))$$

and consequently

$$H^{k-p}(X, \Omega^p_X(\log Y)) = Gr^p_F H^k(X \setminus Y, \mathbb{C}) \simeq \frac{H^0(X, \Omega^{k,closed}_X((k-p+1)Y))}{dH^0(X, \Omega^{k-1}_X((k-p)Y)) + H^0(X, \Omega^k_X((k-p)Y))}.$$

Corollary 4. In the case $Y \subseteq X$ is a smooth hypersurface (or quasi-smooth when X is an orbifold) and $H^k(X, \mathbb{Q})_{\text{prim}} = 0$, then the mixed Hodge structure of $H^k(X \setminus Y, \mathbb{C})$ is pure of weight $k+1$, i.e. $Gr^W_m H^k(X \setminus Y, \mathbb{Q}) = 0$ for $m \neq k+1$ and

$$Gr^W_{k+1} H^k(X \setminus Y, \mathbb{Q}) \xrightarrow[Res]{\sim} H^{k-1}(Y, \mathbb{Q})_{\text{prim}}.$$

In particular, when $X = \mathbb{P}^n$ and $Y = \{F = 0\}$ with $\deg F = d$, we get for $p + q = n - 1$ that

$$H^{p,q}(Y)_{\text{prim}} \simeq H^q(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(\log Y)) \simeq \frac{H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((n-p)Y))}{dH^0(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}((n-p-1)Y)) + H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((n-p-1)Y))}.$$

Identifying

$$H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((n-p)Y)) = \frac{\Omega}{F^{n-p}} \cdot \mathbb{C}[x_0, \ldots, x_n]_{d(n-p)-n-1}$$

we get **Griffiths basis theorem**

$$H^{p,q}(Y)_{\text{prim}} \simeq \left(\frac{\mathbb{C}[x_0, \ldots, x_n]}{\langle \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \rangle} \right)_{d(n-p)-n-1} = R^F_{d(n-p)-n-1}.$$

Remark. Similar basis theorems due to Steenbrink and Batyrev–Cox can be obtained for weighted hypersurfaces and quasi-smooth hypersurfaces of complete simplicial toric varieties. In those cases the Jacobian ring must be replaced by a **graded Jacobian ring** where in the weighted case, each variable has its grade given by the weight, while in the toric case the grading is given by the Class group $Cl(X_\Sigma)$ and so $R^F = S/Jac(F)$ is a quotient of the **Cox ring** $S = \mathbb{C}[z_1, \ldots, z_k]$ where $\deg(z_i) = D_i \in Cl(X_\Sigma)$.

Remark. When X has non-trivial primitive cohomology and/or the divisor Y has more components, it is possible to obtain similar basis results, but now we will obtain a basis compatible with the weight filtration also. Hence the basis will be given as a package of basis for each pure Hodge structure on the graded parts of the weight filtration (see for example [Ste77]).

3 Variations of Hodge Structures and Degenerations

3.1 Local systems and connections

Definition 5. Let X be a complex manifold. A **local system** over X is a locally constant sheaf $\mathbb{V}$ defined over X.

Example 7.

1. Let G be an abelian group (e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$). Every constant sheaf $\underline{G}$ is a local system.

2. Another natural way to produce locally constant sheaves is as follows: Let $f : X \to Y$ be a map such that all the fibers have the same homotopy type, then the sheaf $R^k f_* \underline{G}$ is a local system on Y. In fact, at every $y \in Y$ its fiber corresponds to $H^k(X_y, G)$, where $X_y = f^{-1}(y)$.

Remark. In the case $\mathbb{V}_\mathbb{Z}$ is a local system of finitely generated abelian groups on a complex manifold X, then $\mathcal{V} := \mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_X$ is a **holomorphic vector bundle** on X. For instance if $\mathbb{V}_\mathbb{Z} = R^k f_* \underline{\mathbb{Z}}$ for some $f : X \to Y$ proper and smooth, then $\mathcal{V}$ corresponds to the vector bundle on Y with fibers

$$\mathcal{V}_y = H^k(X_y, \mathbb{C}) \simeq H^k_{\mathrm{dR}}(X_y).$$

In fact $\mathcal{V} \simeq R^k f_* \Omega^\bullet_{X/Y} =: \mathscr{H}^k_{\mathrm{dR}}(X/Y)$ is the **De Rham cohomology bundle**.

Definition 6. A **holomorphic connection** on a holomorphic vector bundle $\mathcal{V}$ is a map

$$\nabla : \mathcal{V} \to \Omega^1_X \otimes \mathcal{V}$$

satisfying the **Leibniz rule** on local sections

$$\nabla(f \cdot s) = df \otimes s + f \nabla(s),$$

for every holomorphic function f on X and s a section of $\mathcal{V}$.

Remark. Given a connection on $\mathcal{V}$ we can naturally extend it to a connection on $\Omega^p_X \otimes \mathcal{V}$ by letting

$$\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^p \omega \otimes \nabla(s).$$

We say the connection ∇ if **flat** or **integrable** if $\nabla \circ \nabla = 0$. In such a case it induces a **De Rham complex**

$$\Omega^\bullet_X(\mathcal{V}) := [0 \to \mathcal{V} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{V} \xrightarrow{\nabla} \Omega^2_X \otimes \mathcal{V} \xrightarrow{\nabla} \cdots].$$

Example 8. If $\mathbb{V}$ is a local system of finite $\mathbb{C}$-vector spaces on X, then $\mathcal{V} = \mathbb{V} \otimes_\mathbb{C} \mathcal{O}_X$ admits a natural flat connection defined as follows: Consider $s_1, \ldots, s_k$ be a local set of generators of $\mathbb{V}$, then every local section of $\mathcal{V}$ is a combination $s = f_1 \cdot s_1 + \cdots + f_k \cdot s_k$ where $f_1, \ldots, f_k$ are holomorphic functions on X. We define the **Gauss–Manin connection** on $\mathcal{V}$ as

$$\nabla(s) := df_1 \otimes s_1 + \cdots + df_k \otimes s_k.$$

In particular $\nabla(\mathbb{V}) = 0$. Conversely given any flat connection ∇ on a holomorphic vector bundle $\mathcal{V}$ we can define a local system $\mathbb{V} := \ker(\nabla)$ such that ∇ is the associated Gauss–Manin connection on $\mathbb{V} \otimes_\mathbb{C} \mathcal{O}_X = \mathcal{V}$.

Definition 7. A **variation of Hodge structure of weight** k on a complex manifold X consists of the following data:

1. a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated abelian groups on X,
2. a finite decreasing filtration $\mathcal{F}^p$ on $\mathcal{V} = \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X$ by holomorphic subbundles (the **Hodge filtration**).

These data should satisfy the following conditions:

1. for each $x \in X$, the Hodge filtration $\mathcal{F}^p(x)$ of $\mathbb{V}(x) = \mathbb{V}_x \otimes_{\mathbb{Z}} \mathbb{C}$ defines a pure Hodge structure of weight k on the finitely generated abelian group $\mathbb{V}_x$,
2. the Gauss–Manin connection ∇ on $\mathcal{V}$ satisfies the **Griffiths transversality condition**
$$\nabla(\mathcal{F}^p) \subseteq \Omega^1_X \otimes \mathcal{F}^{p-1}.$$

Theorem 7 (Griffiths)**.** If $f : X \to Y$ is a smooth proper family of Kähler manifolds, then $\mathbb{V}_{\mathbb{Z}} = R^k f_* \underline{\mathbb{Z}}$ and $\mathcal{F}^p = R^k f_* \Omega^{\bullet \geq p}_{X/Y}$ constitute a variation of Hodge structure of weight k on Y. Moreover the spectral sequence associated to the Hodge filtration degenerates at E_1
$$E_1^{p,q} = R^q f_* \Omega^p_{X/Y} \Rightarrow R^{p+q} f_* \Omega^{\bullet}_{X/Y} = \mathscr{H}^{p+q}_{\mathrm{dR}}(X/Y).$$

3.2 Logarithmic connections

Definition 8. Let X be a complex manifold and $Y \subseteq X$ be a simple normal crossing divisor and $U := X \setminus Y$. Let $\mathcal{V}$ be a holomorphic vector bundle on X and let Δ be a holomorphic connection on $\mathcal{V}|_U$. We say Δ has **logarithmic poles along** Y if it extends to a morphism
$$\nabla : \mathcal{V} \to \Omega^1_X(\log Y) \otimes \mathcal{V}$$
which satisfies the Leibniz rule. If Y_k is an irreducible component of Y, the residue map
$$Res_{Y_k} : \Omega^1_X(\log Y) \to \mathcal{O}_{Y_k}$$
induces a map
$$\mathcal{V} \xrightarrow{\nabla} \Omega^1_X(\log Y) \otimes \mathcal{V} \xrightarrow{Res_{Y_k} \otimes id} \mathcal{O}_{Y_k} \otimes \mathcal{V}$$
which by Leibniz rule factors through $\mathcal{O}_X(-Y_k) \otimes \mathcal{V}$, giving us the **residue of the connection along** Y_k
$$\mathrm{res}_{Y_k}(\nabla) \in \mathrm{End}(\mathcal{O}_{Y_k} \otimes \mathcal{V}).$$
In case Y_k is compact, the characteristic polynomial of $res_{Y_k}(\Delta)$ has constant coefficients.

Theorem 8 (Riemann–Hilbert correspondence)**.** Let X be a complex manifold and Y be a normal crossing divisor, then the assignment
$$(\tilde{\mathcal{V}}, \nabla) \mapsto (\mathcal{V}, \nabla)|_{X \setminus Y}$$
gives an equivalence
$$\left\{ \begin{matrix} \text{regular meromorphic extensions to } X \\ \text{of vector bundles on } X \setminus Y \text{ equipped} \\ \text{with a flat logarithmic connection} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{vector bundles on } X \setminus Y \\ \text{equipped with a flat} \\ \text{connection} \end{matrix} \right\}$$

3.3 Variations of Hodge structures over a punctured disc

Consider the special case where X is the unit disc $\Delta \subseteq \mathbb{C}$ and $Y = 0$ is the origin. Consider $\mathcal{V}$ a holomorphic vector bundle on Δ and a connection ∇ with logarithmic poles along 0. We let $\Delta^* = \Delta \setminus \{0\}$ and let T be the **monodromy automorphism** of $\mathbb{V} := \ker(\nabla|_{\Delta^*})$ determined by a counter-clockwise loop around 0.

Proposition 2. T can be extended to an automorphism of $\mathcal{V}$ whose restriction T_0 to $\mathcal{V}(0)$ is given by

$$T_0 = \exp(-2\pi i \mathrm{res}_0(\nabla)).$$

On the other hand, for every bundle $\mathcal{V}$ on Δ^* equipped with a flat connection ∇, there exists a logarithmic connection extending it over Δ. Let us sketch the construction of this extension and explain how to describe T_0 in terms of T. Let

$$\mathbb{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$$

be the upper half plane, which is the universal covering of Δ^* via the map

$$\varepsilon : \tau \in \mathbb{H} \mapsto e^{2\pi i \tau} \in \Delta^*.$$

We define the **canonical fibre**

$$\mathbb{V}_\infty := H^0(\mathbb{H}, \varepsilon^* \mathbb{V})$$

the $\mathbb{C}$-vector space of multivalued sections of $\mathbb{V}$. Assume by the moment that the monodromy T is **unipotent** (i.e. $T - I$ is nilpotent) and let

$$N := -\frac{1}{2\pi i} \log T = \frac{1}{2\pi i} \sum_{k>0} \frac{(I-T)^k}{k}.$$

For any holomorphic section s of $\varepsilon^* \mathcal{V}$ we define a new holomorphic section $\varphi(s)$ by the rule

$$\varphi(s)(u) := [\exp(2\pi i u N)] s(u) = \sum_{k \geq 0} \frac{(2\pi i)^k}{k!} u^k N^k s(u).$$

If $s \in \mathbb{V}_\infty$ it transforms through the rule

$$s(u+1) = T s(u),$$

so $\varphi(s)$ is invariant under $u \mapsto u+1$, hence descends to a section of $\mathcal{V}|_{\Delta^*}$. So with $j : \Delta^* \hookrightarrow \Delta$ the inclusion, $\varphi(\mathbb{V}_\infty) \subseteq H^0(\Delta, j_* \mathcal{V})$ and we set

$$\tilde{\mathcal{V}} := \varphi(\mathbb{V}_\infty) \otimes_{\mathbb{C}} \mathcal{O}_\Delta \subseteq j_* \mathcal{V}.$$

We have

$$\nabla(\varphi(s)u) = 2\pi i N[\varphi(s)] \otimes du = 2\pi i N[\varphi(s)] \otimes \varepsilon^* \left(\frac{dt}{t}\right),$$

and so we obtain a logarithmic connection $\tilde{\nabla}$ on $\tilde{\mathcal{V}}$ with residue N at 0. This is called the **canonical extension** of $(\mathcal{V}, \nabla)$, and it gives us

$$\varphi : \mathbb{V}_\infty \xrightarrow{\sim} \tilde{\mathcal{V}}(0).$$

Remark. In the general case, the monodromy operator T is **quasi-unipotent** (at least for a polarized variation od Hodge structure), i.e. $T = T_s T_u$ has a Jordan decomposition with T_u unipotent and T_s semisimple. And a similar analysis applies for $N = \frac{-1}{2\pi i}\log T_u$.

Example 9. In the geometric case, $f : X \to \Delta$ a proper map, smooth over Δ^* with $E = f^{-1}(0)$ a reduced simple normal crossing divisor, the monodromy T on $\mathbb{V} = R^k(f|_{X\setminus E})_*\underline{\mathbb{C}}$ is unipotent.

Definition 9. Given a nilpotent endomorphism N of a finite dimensional vector space V, the **weight filtration of** N **centered at** k is the unique increasing filtration $W = W(N,k)$ of V with the properties

1. $N(W_i) \subseteq W_{i-2}$, $i \geq 2$,
2. the map
$$N^l : \mathrm{Gr}^W_{k+l}V \to \mathrm{Gr}^W_{k-l}V$$
is an isomorphism for all $l \geq 0$.

Theorem 9 (Schmid)**.** The Hodge bundles $\mathcal{F}^p$ of a variation of Hodge structure $\mathcal{V} = \mathbb{V} \otimes \mathcal{O}_{\Delta^*}$ of weight k, extend to holomorphic subbundles $\tilde{\mathcal{F}}^p$ of $\tilde{\mathcal{V}}$, and the triple

$$\mathbb{V}^{\mathrm{Hdg}}_\infty := (\tilde{\mathcal{V}}(0)_{\mathbb{Z}}, W_\bullet(N,k), \tilde{\mathcal{F}}^\bullet(0))$$

is a mixed Hodge structure.

Theorem 10. Let $f : X \to \Delta$ be a proper map, smooth over Δ^* with $E = f^{-1}(0)$ a reduced simple normal crossing divisor with all its irreducible components Kähler. The **canonical fibre** is

$$X_\infty := X \times_{\Delta^*} \mathbb{H}.$$

For $\mathbb{V} = R^k(f|_{X\setminus E})_*\underline{\mathbb{Z}}$, $\mathbb{V}^{\mathrm{Hdg}}_\infty$ is a mixed Hodge structure on $\tilde{\mathcal{V}}(0)_{\mathbb{Z}} = H^k(X_\infty, \mathbb{Z})$. The **monodromy weight spectral sequence**

$$E_1^{-r,q+r} = \bigoplus_k H^{q-r-2k}(E(2k+r+1), \mathbb{Q}) \Rightarrow H^q(X_\infty, \mathbb{Q})$$

degenerates at E_2. And the **Hodge spectral sequence**

$$E_1^{p,q} = H^q(E, \Omega^p_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \Rightarrow H^{p+q}(X_\infty, \mathbb{C})$$

degenerates at E_1.

Corollary 5. Under the same hypotheses, if $\epsilon > 0$ is small enough, then for all $t \in \Delta^*$ with $|t| < \epsilon$ the Hodge spectral sequence

$$E_1^{p,q} = H^q(X_t, \Omega^p_{X_t}) \Rightarrow H^{p+q}(X_t, \mathbb{C})$$

degenerates at E_1. Moreover $\dim F^p H^k(X_\infty) = \dim F^p H^k(X_t)$.

Remark. The canonical fibre X_∞ is homotopic to any fibre X_t. The total space X can be retracted to E. Hence the inclusion composed with the retraction $X_t \hookrightarrow X \to E$ can be seen as a specialization map. The map induced in cohomology

$$sp : H^k(E) \to H^k(X_t) \simeq H^k(X_\infty)$$

called also the **specialization map** is a morphism of mixed Hodge structures.

Theorem 11 (Local invariant cycle theorem)**.** Let $X \to \Delta$ be a **Kähler degeneration** centered at 0. Then we have the exact sequence

$$H^k(E, \mathbb{Q}) \xrightarrow{sp} H^k(X_\infty, \mathbb{Q}) \xrightarrow{T-I} H^k(X_\infty, \mathbb{Q}).$$

4 Hodge theory and moduli

The construction and study of moduli spaces is in general a hard task. One of the main difficulties relies in the fact that in general is not easy to construct good compactifications of a moduli space (we mean having good local and global properties, and mainly having a good control or geometric interpretation of the boundary points). In order to construct moduli spaces there are three classical methods: Geometric Invariant Theory (GIT), Hodge Theory and the one introduced by Kollár–Shepherd-Barron–Alexeev (KSBA) by means of the Minimal Model Program. We will briefly outline the second approach.

4.1 Period domain and period map

The philosophy behind the Hodge theoretic approach to moduli spaces is trying to replace each variety by its Hodge structure (which can be thought as a linearization of it). In view of the big amount of geometric data it usually encodes, one expects that the Hodge structure can somehow determine the original variety (Torelli). In this section we will introduce this replacement map.

Definition 10. Let X be a smooth complex variety. A **period** of X is an integral of the form

$$\int_\delta \omega \in \mathbb{C} \qquad \delta \in H_k(X,\mathbb{Z})\ ,\ \omega \in H^k(X,\mathbb{C}).$$

In case $X \to \Delta$ is a smooth locally trivial fibration, for $\omega \in H^k_{\mathrm{dR}}(X/\Delta)$ and $\delta_0 \in H_k(X_0,\mathbb{Z})$ we can see periods as holomorphic functions on $t \in \Delta$

$$p(t) = \int_{\delta_t} \omega(t)$$

where $\delta_t \in H_k(X_t,\mathbb{Z})$ is obtained by monodromy of δ_0 along a path between 0 and t.

Example 10. If E is an elliptic curve, $\delta_1, \delta_2 \in H_1(E,\mathbb{Z})$ is a basis of the homology and $\omega \in H^{1,0}(E) = H^0(\Omega^1_E)$ is a global holomorphic form, then

$$E \simeq \mathbb{C}\Big/\left(\mathbb{Z}\int_{\delta_1}\omega \oplus \mathbb{Z}\int_{\delta_2}\omega\right) \simeq \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau), \qquad \tau = \frac{\int_{\delta_1}\omega}{\int_{\delta_2}\omega} \in \mathbb{H}.$$

Remark. For a smooth locally trivial fibration $X \to T$, $\omega \in H^k_{\mathrm{dR}}(X/T)$ and $\delta_0 \in H_k(X_0,\mathbb{Z})$ the period function is not well-defined, since it might be multivalued unless the monodromy action is trivial on δ_0. We want to globalize the period map, for this it is necessary to eliminate the ambiguity coming from the monodromy action and from the choice of the form ω.

Definition 11. Let $(H, \langle\cdot,\cdot\rangle)$ be a (real) **polarized** Hodge structure of weight k. Its associated **period domain D** is the space of all Hodge filtrations $0 = F^{k+1} \subseteq F^k \subseteq \cdots \subseteq F^1 \subseteq F^0 = H_{\mathbb{C}}$ inducing a polarized Hodge structure of weight k on $(H, \langle\cdot,\cdot\rangle)$ preserving the same Hodge numbers. This means that the filtrations satisfy the **Hodge-Riemann bilinear relations**:

1. $F^p = (F^{k-p+1})^{\perp}$,

2. $(-1)^{k(k-1)} i^{p-q}\langle\alpha,\overline{\alpha}\rangle > 0$ for $\alpha \in H^{p,q} = F^p \cap \overline{F^q}$.

Proposition 3. The period domain $\mathbf{D}$ is an homogeneous space

$$\mathbf{D} = G/V.$$

Moreover it is a semi-algebraic open subset of the projective homogeneous variety

$$\check{\mathbf{D}} = G_{\mathbb{C}}/B$$

parametrizing all filtrations satisfying the first bilinear relation. Where $G_{\mathbb{C}} = \mathrm{Aut}(H_{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ and B is the subgroup fixing a reference filtration satisfying the first bilinear relation.

Example 11. Let E be an elliptic curve and $H = H^1_{\mathrm{dR}}(E, \mathbb{R})$ with the polarization $\langle \alpha, \beta \rangle = \int_E \alpha \wedge \beta$. Then $G_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$, $B = \{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in G_{\mathbb{C}}\}$ and

$$\check{\mathbf{D}} = G_{\mathbb{C}}/B \simeq \mathbb{P}^1.$$

In this space the second bilinear relation translates into

$$\mathbf{D} = \mathbb{H}.$$

Example 12. In general for any smooth curve C of genus g and $H = H^1_{\mathrm{dR}}(C, \mathbb{R})$ we have

$$G_{\mathbb{C}} = \mathrm{Sp}(g, \mathbb{C}) = \left\{ M \in \mathrm{GL}(2g, \mathbb{C}) : M^{\mathrm{tr}} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \right\},$$

B is the block upper triangular subgroup, $\check{\mathbf{D}} = \mathbf{Gr}(g, 2g)$ and

$$\mathbf{D} = \mathbb{H}^g = \{A \in M_{g \times g}(\mathbb{C}) : A = A^{\mathrm{tr}}\ ,\ \mathrm{Im}(A) > 0\}.$$

Definition 12. Let $X \to T$ be a smooth proper family of Kähler manifolds. Fix some $0 \in T$. For every $t \in T$ choose a path from 0 to t in T and pick $\phi_t : X_0 \to X_t$ a diffeomorphism determined by the path (using the local trivializations). Then

$$F_t^\bullet := \phi_t^* F^\bullet H^k(X_t, \mathbb{C})_{\mathrm{prim}}$$

defines a Hodge filtration on $H^k(X_0, \mathbb{R})_{\mathrm{prim}}$. If we fix a basis of $H^k(X_0, \mathbb{R})_{\mathrm{prim}}$ we can transport it to the Euclidean space $H = \mathbb{R}^n$. The **period map** is given by

$$\mathscr{P} : t \in T \mapsto F_t^\bullet \in \Gamma \backslash \mathbf{D},$$

where $\mathbf{D}$ is the period domain of H and $\Gamma = \mathrm{Im}(\pi_1(T, 0) \to \mathrm{Aut}(H))$ is the monodromy representation.

Proposition 4. Let X be a compact Kähler manifold with $\delta_j \in H_k(X, \mathbb{Q})_{\mathrm{prim}}$ a basis. Assume $k = 2m$ or $2m+1$ and let ω_i be a basis of F^m compatible with the filtration $F^k \subseteq F^{k-1} \subseteq \cdots \subseteq F^m$. Then the Hodge structure is determined by the row space of the period matrix

$$Z_{ij} = \int_{\delta_j} \omega_i.$$

Corollary 6. The period map is holomorphic.

Remark. In order to use the period map to study moduli spaces we want to know under which conditions $\mathscr{P}$ is an embedding. The differential of the period map has a nice description in cohomological terms (due to Griffiths transversality and the description of the tangent space of a Grassmannian)

$$d\mathscr{P} : \mathsf{T}_t T \to \bigoplus_{p=0}^{k} \hom(F^p/F^{p+1}, F^{p-1}/F^p) \subseteq \mathsf{T}_{\mathscr{P}(t)}\mathbf{D}.$$

This is why it is not hard to show that $d\mathscr{P}$ is injective in a variety of cases, the so called **infinitesimal Torelli theorem**. The hard part is to show global injectivity of $\mathscr{P}$, which is known as the **global Torelli problem**. This theorem is known to hold for abelian varieties (trivial), curves (classical Torelli theorem), K3 surfaces and other K3 like situations (cubic fourfolds and Hyperkähler manifolds). It is also known that the **generic Torelli** ($\mathscr{P}$ generically injective) holds for most hypersurfaces (Donagi). In general the global and generic Torelli do not always hold (e.g. del Pezzos), but various forms of Torelli are expected to hold quite generally. On the other hand, with few exceptions the period map is not dominant.

4.2 Compactification of classical period domain

In view of the rich structure of the space $\Gamma\backslash\mathbf{D}$ it is expected that this space has a nice compactification. We separate the analysis depending on the structure of $\mathbf{D}$.

Definition 13. We say that a period domain $\mathbf{D}$ is a **classical period domain** if it is a Hermitian symmetric domain. This is the case of:

(1) weight 1 Hodge structures (abelian variety type): $\mathbf{D} \simeq \mathbb{H}^g$,

(2) weight 2 Hodge structures with $h^{2,0} = 1$ (K3 type): $\mathbf{D} \simeq \mathrm{SO}(2, n)/S(O(2) \times O(n))$ is a Type IV domain.

In this context $\Gamma\backslash\mathbf{D}$ is a **locally symmetric variety**.

Definition 14. Let $\Gamma\backslash\mathbf{D}$ be a locally symmetric variety. The **Satake-Bailly-Borel compactification** is defined by

$$(\Gamma\backslash\mathbf{D})^* := \mathrm{Proj}\ \mathcal{A}(\Gamma),$$

where $\mathcal{A}(\Gamma)$ is the algebra of Γ-automorphic forms on $\mathbf{D}$ (which is finitely generated).

Theorem 12 (Borel's extension theorem)**.** Let $\Gamma\backslash\mathbf{D}$ be a locally symmetric variety, T be a smooth variety and $\overline{T}$ be a smooth simple normal crossing compactification of T. Then any locally liftable map $T \to \Gamma\backslash\mathbf{D}$ extends to a regular map $\overline{T} \to (\Gamma\backslash\mathbf{D})^*$.

Remark. Using automorphic forms one can obtain some geometric consequences. In fact, when Γ acts freely, the automorphic forms would be pluricanonical forms on $\Gamma\backslash\mathbf{D}$. Thus one can use automorphic forms to prove that some moduli spaces are of general type. The case of ppavs is well understood.

Theorem 13 (Tai, Mumford)**.** $\mathcal{A}_g$ is of general type for $g > 6$.

The main disadvantage of the SBB compactification is that it is quite small, and thus it does not reflect accurately the geometry of degenerations. Also, it tends to be quite singular. For instance

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_0,$$

and thus the boundary has codimension g. Similarly, for Type IV domains (K3 type period domains) the boundary is 1-dimensional. To rectify this issue, Mumford et al. have introduced **toroidal compactifications** $\overline{\Gamma\backslash\mathbf{D}}^{\Sigma}$, which depend on a choice of an admissible rational polyhedral decomposition Σ of a certain cone. Toroidal compactifications come equipped with natural forgetful maps

$$\overline{\Gamma\backslash\mathbf{D}}^{\Sigma} \to (\Gamma\backslash\mathbf{D})^*.$$

Remark. From the Hodge theoretic viewpoint, we know that the limit of a degeneration of Hodge structures is a mixed Hodge structure. The data encoded in the boundary of the SBB compactification corresponds to the data of the graded pieces $\mathrm{Gr}_{\bullet}^{W} H$ of the mixed Hodge structure. On the other hand, the toroidal compactification encodes the full limit mixed Hodge structure. It is unclear which of the toroidal compactifications should have a geometric meaning, but there are some known facts on the modular meaning of certain toroidal compactifications (mainly in the abelian variety case), see [Laz16] for more comments.

5 Studying the KSBA boundary via Hodge theory

5.1 Some words about KSBA compactification

For varieties of **general type** (i.e. with pluricanonical maps $\varphi_{mK_X} : X \dashrightarrow \mathbb{P}^{N_m}$ generically injective for $m \gg 0$, e.g. when K_X is ample) Kollár, Sheperd-Barron and Alexeev have shown that it is possible to construct the moduli spaces of these varieties with prescribed invariants. In the case of curves the main invariant is the **genus** $g(C) = h^0(K_C) = h^1(\mathcal{O}_C) \geq 2$. In the case of surfaces of general type the main numerical invariants are

- $p_g = h^0(K_S) = h^2(\mathcal{O}_S)$ the **geometric genus**,
- $q = h^0(\Omega^1_S) = h^1(\mathcal{O}_S)$ the **irregularity**,
- $K_S^2 = c_1(S)^2 > 0$ the **first Chern number**,
- $\chi_{top}(S) = c_2(S)$ the **second Chern number** or **topological Euler characteristic**,
- $\chi(S) = 1 - q + p_g = \frac{1}{12}(K_S^2 + \chi_{top}(S))$ the **algebraic Euler characteristic**.

Theorem 14 (Kollár–Shepherd-Barron, Alexeev)**.** For general type varieties with given numerical invariants there exits a moduli space $\mathcal{M}$ with a canonical completion $\overline{\mathcal{M}}$.

Given a family $\mathcal{X}^* \xrightarrow{\pi} \Delta^*$ of smooth varieties $X_t = \pi^{-1}(t)$ for $t \neq 0$, by the theorem there is defined a unique limit variety X_0 (after possibly a base change) that fills in the family $\mathcal{X} \to \Delta$ where the following conditions hold

- $\mathcal{X}$ has **canonical singularities** over X_0,
- $\mathcal{X}$ is of **relative general type** and **minimal**.

This means that X_0 should be reduced and

- mK_{X_0} is a line bundle for some $m > 0$,
- K_{X_0} is ample.

Such an X_0 is called **stable**, and the resulting family $\mathcal{X} \to \Delta$ is called a **$\mathbb{Q}$-Gorenstein smoothing** of X_0. Moreover, we say X_0 is a **Gorenstein** degeneration if K_{X_0} is a line bundle, i.e. if it has **index** $m = 1$. If the index $m \geq 2$ we say it is **non-Gorenstein**.

Remark. For curves, the singularities an stable curve can admit are only nodes. For surfaces the singularity types occurring on stable surfaces have been classified. Gorenstein degenerations admit only canonical singularities, also known as **Du Val** or **ADE singularities**. They are equivalent to isolated hypersurface singularities $f(x, y, z) = 0$. For example A_n is given by

$$x^{n+1} + y^2 + z^2 = 0.$$

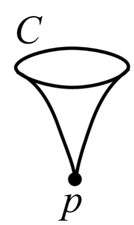
Example 13. Examples of isolated singularities appearing in non-Gorenstein degenerations of surfaces of general type are:

(a) Finite quotient singularities of the form $\frac{1}{d}(1, a)$ with $\gcd(d, a) = 1$. They are given by the quotient of $\mathbb{C}^2$ by the action of the cyclic group generated by

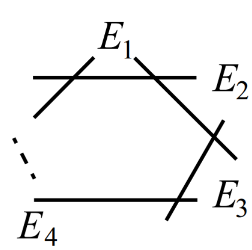
$$(x, y) \mapsto (\zeta x, \zeta^a y)$$

where $\zeta = e^{\frac{2\pi i}{d}}$. Among these are the **Wahl singularities** $\frac{1}{n^2}(1, na - 1)$.

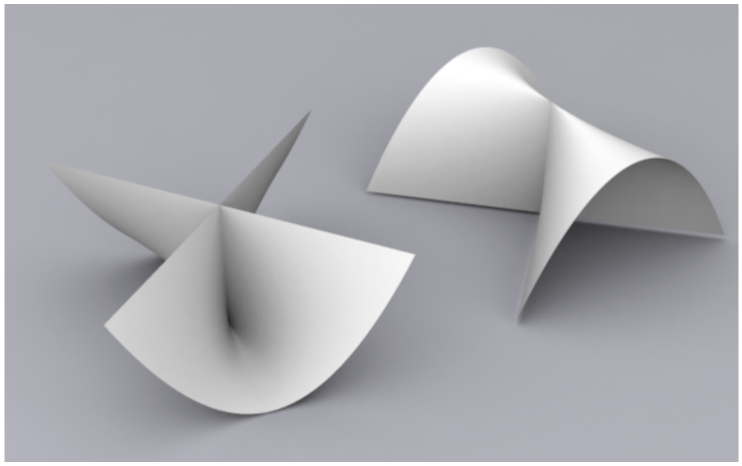
(b) **Simple elliptic singularities** (X, p) where X is a normal surface having an isolated singularity at p. The minimal resolution $(\tilde{X}, C) \to (X, p)$ is given by contracting an elliptic curve $C \subseteq \tilde{X}$ with $C^2 = -d < 0$. The number d is the degree of the singularity. To say that $(\tilde{X}, C)$ is minimal means that there are no (-1)-curves meeting C. The assumption that (X, p) is smoothable implies $1 \leq d \leq 9$.



(c) **Cusp singularities** (X, p) where the minimal resolution $(\tilde{X}, D) \to (X, p)$ has for D a cycle of $\mathbb{P}^1$'s E_i with $E_i^2 \leq -2$ and at least one $E_i^2 \leq -3$.



An example of non-normal singularity appearing in non-Gorenstein degenerations is X with a smooth double curve with pinch points (locally a **Whitney swallowtail** $x^2y = z^2$).

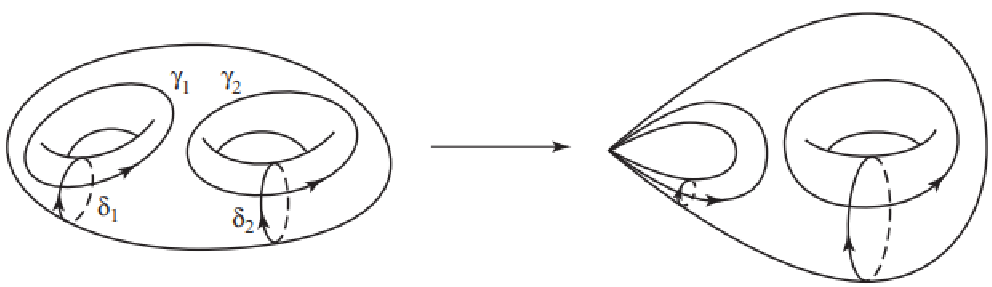


5.2 LMHS at the boundary of the period map

Example 14. To see the difference between the information provided by $\mathrm{Gr}^W_\bullet H$ and the LMHS on H consider the following example: Let S_t be the family of genus 2 curves

$$y^2 = x(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-t)$$

and let us consider the LMHS $(H, W_\bullet, F^\bullet)$ as $t \to 0$.



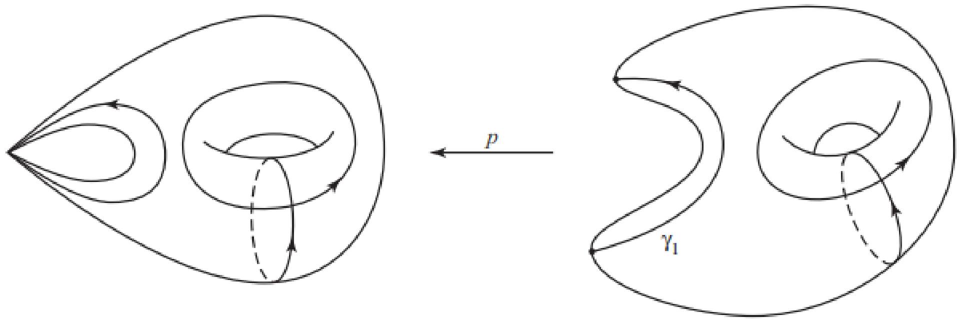
We know $H \simeq \mathbb{Q}^4$, moreover it can be shown that the MHS concentrates on weights 0, 1 and 2. It turns out that

$$W_1 = H^1(S_0, \mathbb{Q})$$

is of dimension 3, and

$$\mathrm{Gr}^W_1 = H^1(\tilde{S}_0, \mathbb{Q})$$

is of dimension 2 ($\tilde{S}_0$ is the normalization of S_0).



It can be shown that further geometric information can be obtained by using that

$$W_2/W_0 = H^1(S', \mathbb{Q})$$

where $S' = S_0^{sm}$ is open the smooth part of S_0 (which also has a mixed Hodge structure). This information, describes the two points $p, q \in \tilde{S_0}$ obtained after normalizing the singularity. Hence it gives us all the data necessary to reconstruct S_0.

Remark. In general, recall that associated to a family

$$\mathcal{X}^* \to \Delta^*$$

of degenerating smooth varieties there is a monodromy operator $T : H^n(X_t) \to H^n(X_t)$ such that

$$T = T_s T_u \text{ (Jordan decomposition)},$$

$$T_s^m = Id \ , \quad T_u = e^N \ \text{ with } \ N^{n+1} = 0.$$

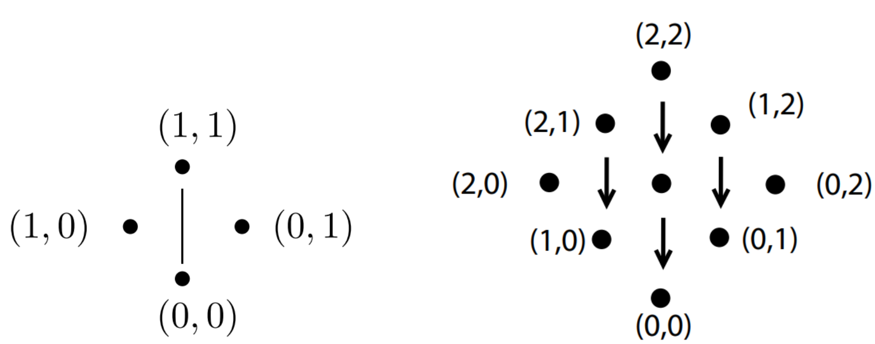
The **limit mixed Hodge structure** (LMHS) is $(H, W_\bullet(N), F^\bullet_{lim})$ with $N : W_k(N) \to W_{k-2}(N)$ such that

$$N^k : W_{n+k}(N) \xrightarrow{\sim} W_{n-k}(N).$$

The compactification of the period map associated to such a degeneration is obtained by adding at the limit the **associated graded LMHS**

$$\mathrm{Gr(LMHS)} \simeq \bigoplus_{\ell=0}^{2n} \mathrm{Gr}_\ell^{W(N)} H.$$

In general for curves and surfaces the graded LMHS look like



as explained in the previous example, the graded LMHS has less information that the whole LMHS. This is analogous to the difference between the Satake–Bailly–Borel compactification and the toroidal ones in the classical period domains.

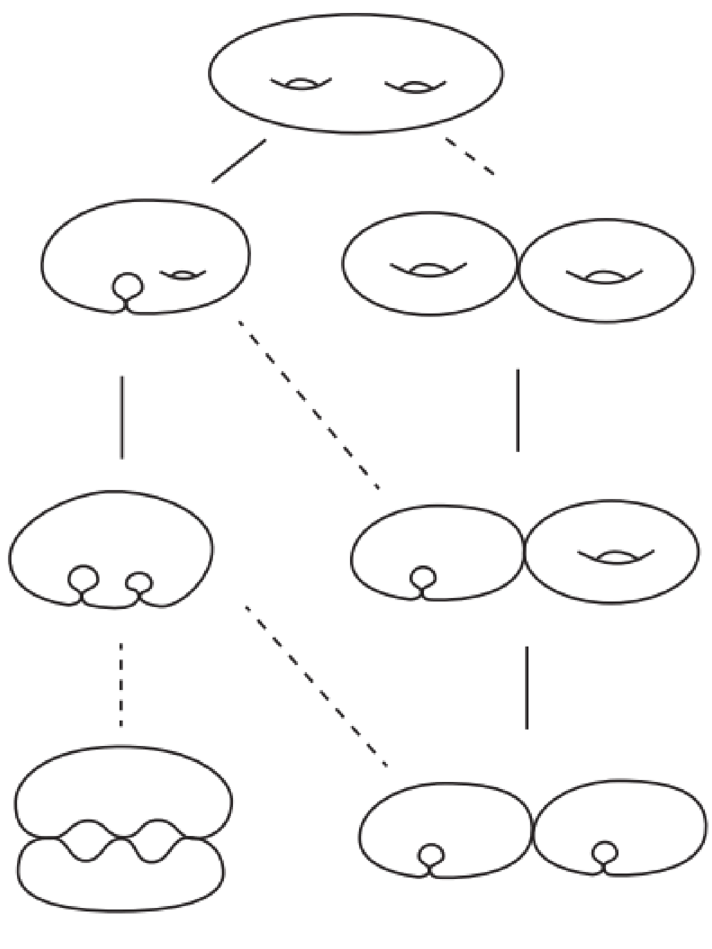
Remark. Using Lie theory the set of equivalence classes of LMHS has been classified (by M. Kerr and C. Robles, under preparation) and they form a stratified object. Informally one may say that we know how Hodge structures degenerate, the strategy is then to use this information to help understand how algebraic varieties degenerate. In general, just the numerical information of the graded LMHS detects the stratification in the KSBA boundary corresponding to Gorenstein degenerations, but not for non-Gorenstein. This is due to the following result.

Theorem 15. For a non-Gorenstein KSBA degeneration $X_t \to X$, the monodromy is finite, hence $N = 0$ and the LMHS is a pure Hodge structure on X. In other words the variation of Hodge structures extends to the singular fibre.

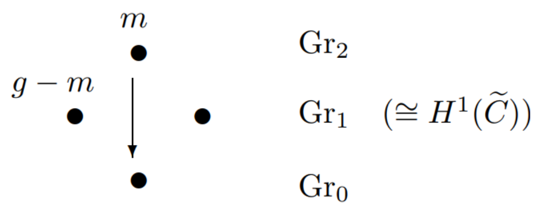
Remark. In spite of the previous fact, it is still possible to study the non-Gorenstein stratification by using refinements of the data given by the graded LMHS. For instance we can look at the irreducible decomposition of the graded LMHS and also we can take into account the spectrum of the monodromy T_s.

5.3 The Hodge picture for genus 2 curves

The stratification of the moduli of genus two curves is given by the following picture

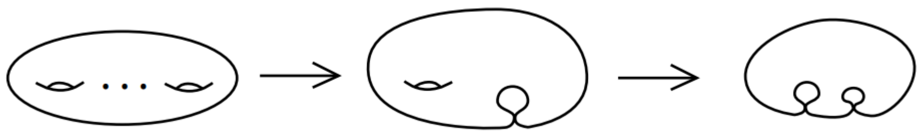


where the solid lines represent Gorenstein degenerations, while dashed lines represent non-Gorenstein ones. The graded LMHS appearing in boundary of the period map for $\overline{\mathcal{M}_g}$ is always of the form



where $m = \text{rank } N$. This gives the following stratification of $\overline{\mathcal{M}_g}^{\text{Gor}}$

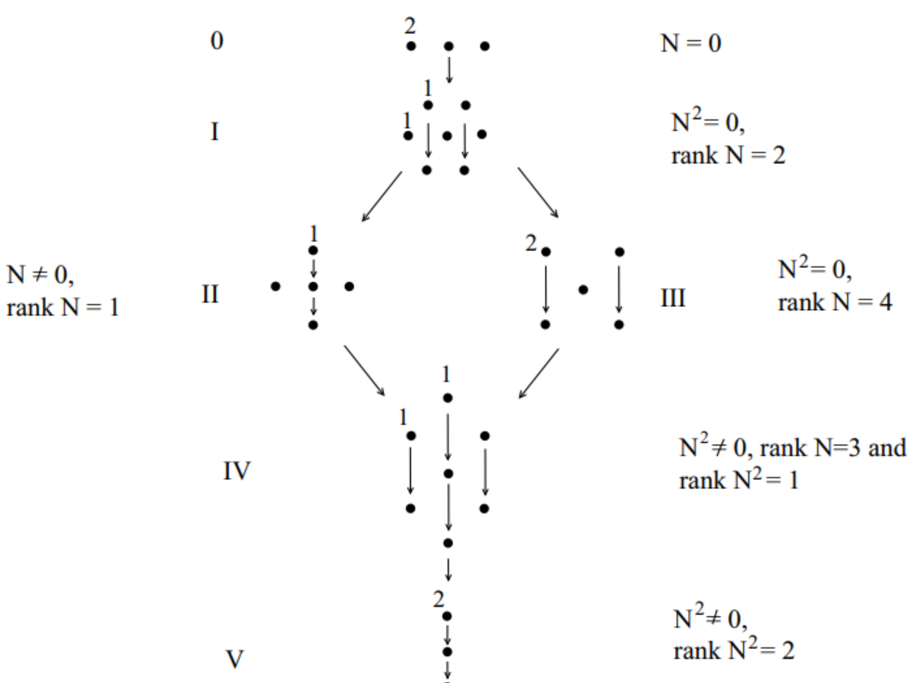
$$I_0 - I_1 - \cdots - I_g$$



The non-Gorenstein strata is detected Hodge theoretically in the genus 2 case by noting that the middle Hodge structure of Gr(LMHS) decomposes as a non-trivial direct sum of Hodge substructures.

5.4 Stratification of I-surfaces

The most complete classifications known are for **I-surfaces** and **H-surfaces** (surfaces of general type with $p_g = 2$, $q = 0$ and $K^2 = 1, 2$ respectively) where the stratification of the LMHS describes the stratification of the boundary of $\overline{\mathcal{M}_I}$ and $\overline{\mathcal{M}_H}$ (see [GGLR, Gri18]). We will sketch the Hodge theoretic picture of the strata of I-surfaces. The Gorenstein stratification is the following:



All the Hodge theoretic strata pictured above is realized at the strata of $\overline{\mathcal{M}_I}^{\mathrm{Gor}}$. Moreover, the above picture can be refined for non-Gorenstein degenerations. In the following table we describe this stratification only for simple elliptic singularities (types I_k and III_k). They have $N^2 = 0$ and the degrees d_i of the elliptic singularities are determined by the spectrum of T_s.

stratum	dimension	minimal resolution $\widetilde{X}$	$\sum_{i=1}^{k}(9-d_i)$	k	codim in $\overline{\mathcal{M}}_I$
I_0	28	canonical singularities	0	0	0
I_2	20	blow up of a K3-surface	7	1	8
I_1	19	minimal elliptic surface with $\chi(\widetilde{X})=2$	8	1	9
$\mathrm{III}_{2,2}$	12	rational surface	14	2	16
$\mathrm{III}_{1,2}$	11	rational surface	15	2	17
$\mathrm{III}_{1,1,R}$	10	rational surface	16	2	18
$\mathrm{III}_{1,1,E}$	10	blow up of an Enriques surface	16	2	18
$\mathrm{III}_{1,1,2}$	2	ruled surface with $\chi(\widetilde{X})=0$	23	3	26
$\mathrm{III}_{1,1,1}$	1	ruled surface with $\chi(\widetilde{X})=0$	24	3	27

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