Curva abstracta nosingular a través del cuerpo...

$V =$ 

$$k \subset \Gamma(V) \subset \mathcal{O}_p(V) \subset K(V)$$

- Toda curva suave es $\simeq$ curva abstracta. (abierto)
- Toda curva abstracta es $\simeq$ curva proy. suave.

igo

29/Abril/21

Fin del álgebra

We now come to the definition of an abstract nonsingular curve. Let K be a function field of dimension 1 over k (i.e., a finitely generated extension field of transcendence degree 1). Let C_K be the set of all discrete valuation rings of K/k. We will sometimes call the elements of C_K *points*, and write $P \in C_K$, where P stands for the valuation ring R_P. Note that the set C_K is infinite, because it contains all the local rings of any nonsingular curve with function field K; those local rings are all distinct (6.4), and there are infinitely many of them (Ex. 4.8). We make C_K into a topological space by taking the closed sets to be the finite subsets and the whole space. If $U \subseteq C_K$ is an open subset of C_K, we define the ring of *regular functions* on U to be $\mathcal{O}(U) = \bigcap_{P \in U} R_P$. An element $f \in \mathcal{O}(U)$ defines a function from U to k by taking $f(P)$ to be the residue of f modulo the maximal ideal of R_P. (Note by (6.6) that for any $R \in C_K$, the residue field of R is k.) If two elements $f, g \in \mathcal{O}(U)$ define the same function, then $f - g \in \mathfrak{m}_P$ for infinitely many $P \in C_K$, so by (6.5) and its proof, $f = g$. Thus we can identify the elements of $\mathcal{O}(U)$ with functions from U to k. Note also by (6.5) that any $f \in K$ is a regular function on some open set U. Thus the function field of C_K, defined as in §3, is just K.

Definition. An *abstract nonsingular curve* is an open subset $U \subseteq C_K$, where K is a function field of dimension 1 over k, with the induced topology, and the induced notion of regular functions on its open subsets.

§2.8 Operando con ideales.

Def.- $I, J \subset R$ ideales.

$I \cdot J = \langle a \cdot b : a \in I, b \in J \rangle$

$I_1, \dots, I_n \Rightarrow \prod_{i=1}^{n} I_i = \langle a_1 \cdots a_n : a_i \in I_i \rangle$

Obs.- Si $I = \langle a_1, \dots, a_r \rangle \Rightarrow I^n = \langle a_1^{i_1} \cdots a_n^{i_n} : a_i \in I, \sum i_j = n \rangle$

$\left[\left(\sum r_{i_1} a_i\right) \cdots \left(\sum r_{i_n} a_i\right) \dots \right]$

Def.- $\underset{\text{ideal}}{I} \subset \underset{\text{anillo}}{R} \subset \underset{\text{anillo}}{S} \Rightarrow IS$ es el ideal en S generado por I $\left[I^n S = (IS)^n \right]$.

Def.- I, J ideales en $R \Rightarrow I + J = \{ a + b \,/\, b \in J, a \in I \}$ es ideal.

I, J comaximales si $I + J = R$.

Lema: (1) $IJ \subset I \cap J$ $\forall I, J$ ideales

(2) Si I, J comaximales $\Rightarrow$ $IJ = I \cap J$.

$$(0) \subset I \cdot J \subset I \cap J \begin{array}{l} \subset I \\ \subset J \end{array} \subset I + J \subset R$$

§2.9 Ideales con un número finito de puntos

$$\left[\begin{array}{l} \underset{\substack{\parallel \mathbb{A}^2_k \\ V(f,g)}}{V(I)} = \{P_1, \dots, P_N\} \iff \dim_k\left(k[X_1, \dots, X_n]/I\right) \text{ es finito} \\ \qquad\qquad\qquad\qquad\qquad\qquad N \leq \end{array} \right]$$

coprimos

« local $\longrightarrow$ global »

Prop: $I \subset k[x_1,\dots,x_n]$ ideal, $k=\overline{k}$. Suponer $V(I)=\{P_1,\dots,P_N\}$.

$\Rightarrow$ $\exists$ isomorfismo natural

$$\varphi: k[x_1,\dots,x_n]/I \xrightarrow{\sim} \mathcal{O}_{P_1}(\mathbb{A}^n_k)/I\mathcal{O}_{P_1}(\mathbb{A}^n_k) \times \dots \times \mathcal{O}_{P_N}(\mathbb{A}^n_k)/I\mathcal{O}_{P_N}(\mathbb{A}^n_k)$$

$$\left[\mathcal{O}_{P_i}(\mathbb{A}^n_k) = k[x_1,\dots,x_n]_{(x_1-a_1,\dots,x_n-a_n)}\right]$$

Dem: • La idea (Fulton) (1) morfismo ± claro ✓✓✓

(2) 1-1 y sobre, encontrar una "base" en $R=k[x_1,\dots,x_n]/I$ que represente a $e_1,\dots,e_n$.

$e_1 = 1\times 0\times\dots\times 0$, $e_n = 0\times\dots\times 0\times 1$

✓ e_i :)

- Sea $\mathcal{O}_{P_i}(\mathbb{A}^n_k) =: \mathcal{O}_i$
- Sea I_i = ideal max de $P_i \supset I$
- $R := k[x_1, \dots, x_n]/I$ y $R_i := \mathcal{O}_i / I\mathcal{O}_i$.

Tenemos $R \xrightarrow{\varphi_i} R_i$ $\left[\begin{array}{l} k[x_1,\dots,x_n] \hookrightarrow \mathcal{O}_i \to \mathcal{O}_i/I\mathcal{O}_i \\ I \mapsto 0, \quad k[x_1,\dots,x_n]/I \xrightarrow{\varphi_i} \mathcal{O}_i/I\mathcal{O}_i \end{array} \right]$

$$\begin{tikzcd} & R_1 \times \dots \times R_N \arrow[d, "\pi_i"] \\ R \arrow[ur, "\exists!"] \arrow[r, "\varphi_i"'] & R_i \end{tikzcd}$$

$\therefore \quad \varphi : R \longrightarrow \prod_{i=1}^N R_i$.

- Nullstellensatz: $\mathrm{Rad}(I) = \bigcap_{i=1}^N I_i$.

así $\left(\bigcap_{i=1}^N I_i\right)^d \underset{\text{Tarea 1}}{\subset} I \quad \forall d > 0$.

$\| $ Tarea 2

$(I_1 \cdots I_N)^d = \bigcap_{i=1}^N I_i^d$

I, J ideales
I f.g.
$I \subset \mathrm{Rad}(J)$
$\Rightarrow \exists N$
$I^N \subset J$

- Sean $F_i \in k[x_1, \dots, x_n]$ tal que $F_i(P_j) = 0 \ \forall i \neq j$
  $F_i(P_i) = 1$
- Sean $E_i = 1 - (1 - F_i^d)^d$, $i = 1, \dots, N$.

Tarea 3: $\overline{E_i} =: e_i \in R$, $e_i^2 = e_i$, $e_i e_j = 0 \quad i \neq j$
y $\sum_{i=1}^{N} e_i = 1$.

(*) Si $G \in k[x_1, \dots, x_n]$, $G(P_i) \neq 0$
$\Rightarrow \exists t \in R$ tal que $t \cdot g = e_i$, $g = \overline{G} \in R$.

Dem: Asumir $G(P_i) = 1$. Sea $H = 1 - G$.
$\Rightarrow (1-H)(E_i + HE_i + \dots + H^{d-1}E_i) = E_i - H^d E_i$.
Como $H \in I_i \Rightarrow H^d E_i \in I$.
$\Rightarrow \underbrace{(1-h)}_{g} \underbrace{(e_i + he_i + \dots + h^{d-1}e_i)}_{t} = e_i$ en R. ∎

$(\varphi \text{ es } 1\text{-}1)$: $\quad \varphi : R \longrightarrow R_1 \times \cdots \times R_N$

$\varphi(f) = 0 \qquad R_i = \mathcal{O}_i / I\mathcal{O}_i \ni \bar{f} = 0$

$\bar{F} = f$

$\Rightarrow F \in I\mathcal{O}_i \Rightarrow \exists G_i$ tal que $\boxed{G_i F} \in I$

$G_i(P_i) \neq 0$

$\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbb{A}^n_k) \supset k[x_1, \ldots, x_n] \supset k$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \cup$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad I$

Aplicar $(*)$ al G_i $(G_i(P_i) \neq 0)$ $\exists t_i \quad t_i \cdot g_i = e_i$

$$\Rightarrow \quad f = \underbrace{\left(\sum_{i=1}^N e_i\right)}_{1} f = \sum_{i=1}^N e_i f = \sum_{i=1}^N t_i \cdot \underbrace{g_i f}_{0} = 0.$$

en R

(φ es solución) Como $E_i(P_i) = 1$, $\varphi_i(e_i)$ es unidad en R_i.

Como $\varphi_i(e_i)\varphi_i(e_j) = \varphi_i(e_i e_j) = 0$ si $i \neq j$

$\Rightarrow \varphi_i(e_j) = 0 \quad i \neq j$.

y $\varphi_i(e_i) = \varphi_i(e_1 + \dots + e_N) = \varphi_i(1) = 1$.

Sea $z = \left(\frac{a_1}{s_1}, \dots, \frac{a_N}{s_N}\right) \in \prod_{i=1}^{N} R_i$ y $s_j(P_j) \neq 0$.

$\Rightarrow \exists t_i,\ t_i s_i = e_i$.

$\Rightarrow \frac{a_i}{s_i} \underset{\checkmark}{=} a_i t_i$

$a_i = a_i t_i s_i = a_i e_i$

$a_i(1 - e_i) \in \cancel{I_i}\ \mathcal{O}_i$

$$\varphi_i\left(\sum t_j a_j e_j\right) = \varphi_i(t_i a_i) = \frac{a_i}{s_i}$$

$$\therefore \quad \varphi\left(\sum t_j a_j e_j\right) = z \qquad \blacksquare$$

Cor 1: $\dim_k\left(k[x_1,\dots,x_n]/I\right) = \sum_{i=1}^{N} \underbrace{\dim_k\left(\mathcal{O}_i/I\mathcal{O}_i\right)}_{\text{mult de } P_i}$

Cor 2: Si $V(I)=\{P\}$,

$$k[x_1,\dots,x_n]/I \simeq \mathcal{O}_P(\mathbb{A}^n_k)/I\mathcal{O}_P(\mathbb{A}^n_k)$$

$\left[\ \dim_k \mathcal{O}_P(\mathbb{A}^2_k) = \text{mult en } p \text{ de la intersección } f \text{ y } g.\ \right]$