
Intro GA

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prelim

algebraicos

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Chapter 1

Affine Algebraic Sets

1.1 Algebraic Preliminaries

This section consists of a summary of some notation and facts from commutative algebra. Anyone familiar with the italicized terms and the statements made here about them should have sufficient background to read the rest of the notes.

When we speak of a *ring* we shall always mean a commutative ring with a multiplicative identity. A *ring homomorphism* from one ring to another must take the multiplicative identity of the first ring to that of the second. A *domain* or integral domain, is a ring (with at least two elements) in which the cancellation law holds. A *field* is a domain in which every nonzero element is a unit, i.e., has a multiplicative inverse.

\mathbb{Z} will denote the domain of integers, while \mathbb{Q} , \mathbb{R} , and \mathbb{C} will denote the fields of rational, real, complex numbers, respectively.

Any domain R has a quotient field K , which is a field containing R as a *subring*, and any elements in K may be written (not necessarily uniquely) as a ratio of two elements of R . Any one-to-one ring homomorphism from R to a field L extends uniquely to a ring homomorphism from K to L . Any ring homomorphism from a field to a nonzero ring is one-to-one.

For any ring R , $R[X]$ denotes the ring of polynomials with coefficients in R . The *degree* of a nonzero polynomial $\sum a_i X^i$ is the largest integer d such that $a_d \neq 0$; the polynomial is *monic* if $a_d = 1$.

The ring of polynomials in n variables over R is written $R[X_1, \dots, X_n]$. We often write $R[X, Y]$ or $R[X, Y, Z]$ when $n = 2$ or 3 . The monomials in $R[X_1, \dots, X_n]$ are the polynomials $X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$, i_j nonnegative integers; the degree of the monomial is $i_1 + \dots + i_n$. Every $F \in R[X_1, \dots, X_n]$ has a unique expression $F = \sum a_{(i)} X^{(i)}$, where the $X^{(i)}$ are the monomials, $a_{(i)} \in R$. We call F *homogeneous*, or a *form*, of degree d , if all coefficients $a_{(i)}$ are zero except for monomials of degree d . Any polynomial F has a unique expression $F = F_0 + F_1 + \dots + F_d$, where F_i is a form of degree i ; if $F_d \neq 0$, d is the *degree* of F , written $\deg(F)$. The terms F_0, F_1, F_2, \dots are called the *constant, linear, quadratic, ... terms* of F ; F is *constant* if $F = F_0$. ~~The zero polynomial is allowed~~

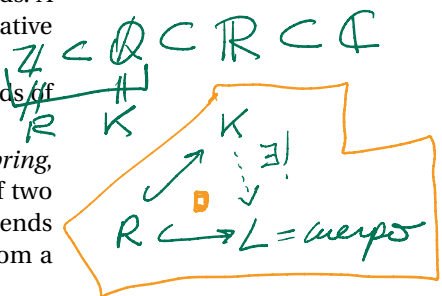
Anillo R con $+, \cdot$
y $0 \neq 1$ y conmutativo.
 $[x \cdot y = y \cdot x]$

dominio

corpo

$$K = \left\{ \frac{a}{b} : a, b \in R, b \neq 0 \right\}$$

$$\frac{a}{b} = \frac{a'}{b'} \Leftrightarrow ab' = a'b$$



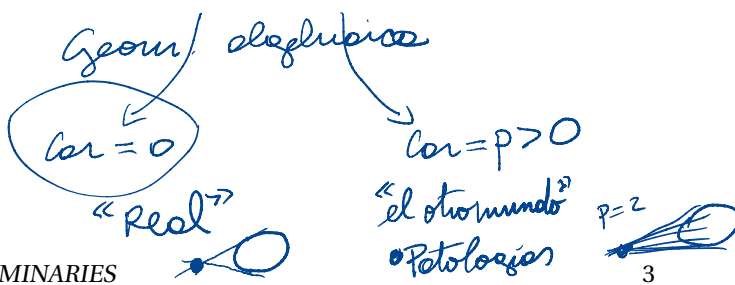
homogeneous:

$$F = X^2 + XY + Y^2$$

$$\lambda \cdot F(x_1, \dots, x_n) = F(\lambda x_1, \dots, \lambda x_n)$$

homogeneous.

$$F = \underbrace{F_0}_{\text{const}} + \underbrace{F_1}_{\text{lineal}} + F_2 + \dots + F_d, \quad \deg(F) = d$$



1.1. ALGEBRAIC PRELIMINARIES

$G \in R[X]$. A field k is algebraically closed if any non-constant $F \in k[X]$ has a root. It follows that $F = \mu \prod (X - \lambda_i)^{e_i}$, $\lambda_i \in k$, where the λ_i are the distinct roots of F , and e_i is the multiplicity of λ_i . A polynomial of degree d has d roots in k , counting multiplicities. The field \mathbb{C} of complex numbers is an algebraically closed field.

Let R be any ring. The derivative of a polynomial $F = \sum a_i X^i \in R[X]$ is defined to be $\sum i a_i X^{i-1}$ and is written either $\frac{\partial F}{\partial X}$ or F_X . If $F \in R[X_1, \dots, X_n]$, $\frac{\partial F}{\partial X_i} = F_{X_i}$ is defined by considering F as a polynomial in X_i with coefficients in $R[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$.

The following rules are easily verified:

- (1) $(aF + bG)_X = aF_X + bG_X$, $a, b \in R$.
- (2) $F_X = 0$ if F is a constant.
- (3) $(FG)_X = F_X G + F G_X$, and $(F^n)_X = nF^{n-1} F_X$.
- (4) If $G_1, \dots, G_n \in R[X]$, and $F \in R[X_1, \dots, X_n]$, then

$$F(G_1, \dots, G_n)_X = \sum_{i=1}^n F_{X_i}(G_1, \dots, G_n)(G_i)_X.$$

- (5) $F_{(X_i X_j)} = F_{(X_j X_i)}$ where we have written $F_{X_i X_j}$ for $(F_{X_i})_{X_j}$.
- (6) (Euler's Theorem) If F is a form of degree m in $R[X_1, \dots, X_n]$, then

$$mF = \sum_{i=1}^n X_i F_{X_i}.$$

derivative found.

\mathbb{R} no lo es
 $f(x) = x^2 + 1$
no tiene raíces.

$\mathbb{R} \subset \mathbb{C} = \overline{\mathbb{R}}$
Teo fund del algebra

$\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

$\overline{\mathbb{F}_p}$ es infinito
Car = p > 0

$k = \overline{k} \Rightarrow \# \{k_i\} = \infty$

Demr Sino
 $k = \{x_1, \dots, x_n\}$
 $\Rightarrow f(x) = (x-x_1) \dots (x-x_n) + 1$
no tiene raíces en $k \rightarrow$

Problems

1.1.* Let R be a domain. (a) If F, G are forms of degree r, s respectively in $R[X_1, \dots, X_n]$, show that FG is a form of degree $r+s$. (b) Show that any factor of a form in $R[X_1, \dots, X_n]$ is also a form.

1.2.* Let R be a UFD, K the quotient field of R . Show that every element z of K may be written $z = a/b$, where $a, b \in R$ have no common factors; this representative is unique up to units of R .

1.3.* Let R be a PID, Let P be a nonzero, proper, prime ideal in R . (a) Show that P is generated by an irreducible element. (b) Show that P is maximal.

1.4.* Let k be an infinite field, $F \in k[X_1, \dots, X_n]$. Suppose $F(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in k$. Show that $F = 0$. (Hint: Write $F = \sum F_i X_n^i$, $F_i \in k[X_1, \dots, X_{n-1}]$. Use induction on n , and the fact that $F(a_1, \dots, a_{n-1}, X_n)$ has only a finite number of roots if any $F_i(a_1, \dots, a_{n-1}) \neq 0$.)

1.5.* Let k be any field. Show that there are an infinite number of irreducible monic polynomials in $k[X]$. (Hint: Suppose F_1, \dots, F_n were all of them, and factor $F_1 \dots F_n + 1$ into irreducible factors.)

1.6.* Show that any algebraically closed field is infinite. (Hint: The irreducible monic polynomials are $X - a$, $a \in k$.)

1.7.* Let k be a field, $F \in k[X_1, \dots, X_n]$, $a_1, \dots, a_n \in k$. (a) Show that

$$F = \sum \lambda_{(i)} (X_1 - a_1)^{i_1} \dots (X_n - a_n)^{i_n}, \quad \lambda_{(i)} \in k.$$

(b) If $F(a_1, \dots, a_n) = 0$, show that $F = \sum_{i=1}^n (X_i - a_i) G_i$ for some (not unique) G_i in $k[X_1, \dots, X_n]$.

