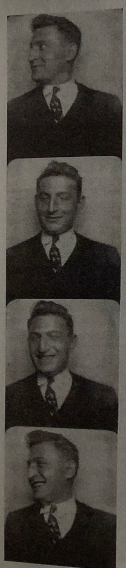


Coble's commitment to his own ideals continued after he returned to Illinois. In 1934, for example, he insisted to a hostile state legislator that he "would hire a foreigner if better than any native prospect."[31] He also remained steadfast in his support of Zariski. As soon as hiring began after the war, he offered Zariski a permanent research professorship at Illinois, and when Harvard invited Zariski the following year Coble wrote, "[Although] we are sorry to lose him … it is my feeling that Professor Zariski would rank among the first five mathematicians of the world."[32]

Zariski's eagerness to succeed in this new mathematical environment was inextricably mixed with his longing for his wife and his need for a position that would make it possible for him to bring her and the baby to the United States. Shortly after his arrival in Baltimore he tried to capture for her the exotic flavor of "an American conversation" with a new colleague at Johns Hopkins:

> After the mathematical talk we spoke of sports, football in particular, and then seeing a dirigible in the sky we spoke of aviation and then of automobilism, and he advises me to buy a car for $100 and to cross America, going to Niagara Falls and to California. Now you have an idea of what an American conversation is like—in one way it is delectable—could one speak to Professor Castelnuovo with such deep interest about football?

He continued, on a more personal note,

> I have had to make a supreme effort to prevent the emptiness of my private life from overwhelming me and destroying my work life. I have had to definitively divide these two worlds—sad temporary necessity, but a question of life and death for me. … I think my work is tied with a very fine but very strong tie to the emotional movement of the soul, and its productivity depends in good part on its strength.

[31] Coble, letter to R. G. D. Richardson, 30 June 1934 (quoted by Nathan Reingold in "Refugee Mathematicians," *Century* I, 182).

[32] Coble, letter to D. V. Widder, 6 May 1947 (Harvard Math Department files).

Eight pictures Zariski took at the Woolworth to send to Yole (courtesy of Yole Zariski)

Although he was impatient to go to Princeton to work with Lefschetz, who had repeatedly invited him, he decided to finish his work on an "interesting but difficult problem" because, as he wrote at the beginning of December, "I connect especially to its publication the whole issue of your coming here":

> Now I am going through a feverish period. … If I will succeed in bringing this research to a good outcome … it will be an affirmation of my worth to the professors who have been vitally interested in this problem. I have an extreme need to make such an American coup. Then I will be able to do as I please and dedicate myself exclusively to the study of the work of Lefschetz. … You see, once I am recognized I will be more secure about the future and I will be psychologically more tranquil for the coming years.

Two weeks later he wrote to say that he had again postponed the trip to Princeton because of difficulties with this paper:

> I have worked intensely and I feel rather tired from the effort. My research has nailed me to the desk, to my papers. … I am no longer going to Princeton after Christmas. I have postponed this trip for two reasons: first, I want to finish my work (paper); then I wish to read more of the papers of Lefschetz in order to be prepared.

At the end of January, he was still at work on the same problem:

> Ah, well, I have had a very bad period. I have gotten stuck on a hard point that I have not been able to overcome for weeks. Now I have had the saving idea and I hope to finish soon. I've been close to despair, even though Prof. Coble consoled me, saying that one must always take these things philosophically and that mathematics is a "slow game." These English and Americans have a sense of humor. I have worked intensely—I hardly left my room.

A month later, he at last wrote of his success, "My most important research is almost finished. I am trying to put some order into the disorder of the first drafts and to organize my work in a methodical fashion. I hope it will be published in the April issue."[33]

[33] This was perhaps "On hyperelliptic 0-functions with rational characteristics" [14].

Intro Geo Alg

15 - Abril - 21

ZARISKI

§1.10 Extensiones de cuerpo

Prop: (Zariski) $K \subset L$ cuerpos tal que L es f.g. como K-álgebra $\Rightarrow$ L es f.g. como K-módulo.

["así no era necesario gen. para mult"]

Dem: Será por inducción en el número de generadores

$$K \subset L = K[v_1, \dots, v_n]$$

(caso 1) $L = K[v]$ y tenemos $K[x] \xrightarrow{\varphi} K[v]$ (sobrey.).

Luego $\ker(\varphi) = (F)$ o (0) con F irreducible.

1. Si $\ker(\varphi) = (0) \Rightarrow K[x] \xrightarrow{\sim} K[v] = L$ cuerpo f.g. como K-alg.

es decir $K[v] = K(x)$ $\rightarrow\leftarrow$ $K(x)$ no es f.g. como K-alg. (o $K[x]$ no es cuerpo)

$K[x]$ anillo pol
$K(x)$ cuerpo fracc.

2. Si $\ker(\varphi) = (F) \Rightarrow L$ es g.g. como K-módulo

ya que $L \simeq k[x]/(F)$ $\quad F = x^n + \dots + a_0$

$K \subset L = K[v_1, \dots, v_n] = K(v_1)[v_2, \dots, v_n]$

$\Rightarrow x^n = -\sum_{i=n}^{1} a_{n-i} x^{n-i} \Rightarrow \bar{1}, \bar{x}, \dots, \bar{x}^{n-1}$ son gen. de L.

(Caso 2) Asumir resultado para toda extensión generada por $n-1$ elementos

- Sea $K_1 = K(v_1)$ (generado como cuerpo)
- Por inducción $L = K(v_1)[v_2, \dots, v_n]$ es g.g. como $K(v_1)$ - módulo.
- Asumir que v_1 no es algebraico sobre K ya que sino

$$\underbrace{\underbrace{K \subset K(v_1)}_{\text{finito}} \subset \underbrace{K(v_1)[v_2, \dots, v_n]}_{\text{finito}}}_{\text{finito}}$$

- Es decir, $K(v_1) \cong$ cuerpo de fracciones de $k[x] \subset L$.

- Tenemos que cada $v_2, \dots, v_n$ es algebraico y así

$$v_i^{n_i} + a_{i1} v_i^{n_i - 1} + \dots + a_{in_i} = 0 \qquad a_{ij} \in K(v_1).$$

- Sea $a \in K[v_1]$ un mult. común a TODOS los denominadores de los a_{ij} $\forall i, j$
- Multiplicando tenemos para cada i:

$$(a v_i)^{n_i} + a a_{i1} (a v_i)^{n_i - 1} + \dots = 0 .$$

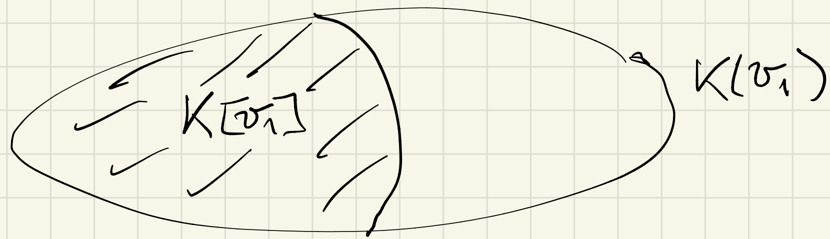
$\Rightarrow$ $a v_i$ es integral sobre $K[v_1]$ $\forall i$.

- Luego, si $z \in K[v_1, \dots, v_n] \Rightarrow \exists N \gg 0$ tal que $a^N \cdot z$ es integral sobre $K[v_1]$, ya que z es "un polinomio" en $v_1, \dots, v_n$ y por a^N, tendríamos una combinación algebraica de elementos integrales, y vimos que eso es <u>integral</u> (Prop3 clase pasada)

$a^N v_2^{\alpha_2} \cdots v_n^{\alpha_n}$

$(a v_2)^{\alpha_2} (a v_3)^{\alpha_3} \cdots (a v_n)^{\alpha_n}$

• Pero $K[v_1, \dots, v_n] = L \supset K(v_1)$ = cuerpo fracciones polin en v_1
y así estaríamos diciendo que todo elemento en $K(v_1)$ es integral sobre $K[v_1]$ $\rightarrow\leftarrow$



Pero $K[v_1]$ es íntegramente cerrado en $K(v_1)$ por el "asunto de las raíces racionales". (Sirve para todo dominio UFD)

Cor: Si $k = \overline{k}$ y $k \subset L$ y L es f.g. k-álgebra
$\Rightarrow L$ es f.g. como k-módulo, $L = k(v_1, \dots, v_n)$

$v_1^n + \dots = 0$ integrales = algebraico

$k = \overline{k}$, implica $v_i \in k \Rightarrow L = k$ ∎

Fin Chapter I.

§2. Variedades afines.

$k = \overline{k}$ (cuerpo base)

<u>Def.</u>- Una variedad afín V es un conj. algebraico irred $V \subseteq \mathbb{A}^n_k$ $\forall n$.

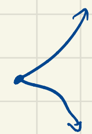
<u>Def.</u>- Sea $V \subseteq \mathbb{A}^n_k$ variedad. Luego $I(V) \subseteq k[x_1, \dots, x_n]$ es ideal primo.

$$\Gamma(V) := k[x_1, \dots, x_n] / I(V)$$

(anillo coord. de V) } Dominio

Este anillo representa a las funciones polinomiales en V.

Ej: $V = \{ y^2 = x^3 \} \subset k[x,y]$, $\Gamma(V) = k[x,y] / (y^2 - x^3)$



Función polinomial: $V \xrightarrow{F} k$, F polinomio $\in k[x_1, \dots, x_n]$ (nuestro ejemplo $k[x,y]$)

$$\begin{array}{ccc} V & \xrightarrow{F} & k \\ \rotatebox{90}{$\in$} & & \\ P & \longmapsto & F(P) \end{array}$$

ahora 2 cosas: (1)

$$\begin{array}{ccc} V & \xrightarrow{F} & \mathbb{A}^1_k \\ \rotatebox{90}{$\in$} & & \\ (a_1, \dots, a_n) & \longmapsto & F(a_1, \dots, a_n) \end{array}$$

(2) F no está únicamente definido, si $G \in I(V)$

$\Rightarrow$ $F + G$ es la misma función.

$$F(x,y) = x^2+y^2 \qquad V \longrightarrow \mathbb{A}^1_k$$

$$\psi \ni (0,0) \longmapsto 0$$

$$\#$$

$$(1,1) \longmapsto 2$$

$$H(x,y) = x^2+y^2-(y^2-x^3) = x^2+x^3$$

$\Rightarrow$ $V \xrightarrow{H} \mathbb{A}^1_k$ es lo mismo que $V \xrightarrow{F} \mathbb{A}^1_k$

$\therefore$ $\Gamma(V)$ = El anillo de las funciones polinomiales en V.

§2.2 mapeos polinomiales.

Def.- Sean $V \subseteq \mathbb{A}^n_k$, $W \subseteq \mathbb{A}^m_k$ variedades. Un <u>mapeo polinomial</u> es $\varphi : V \longrightarrow W$

$$\varphi(a_1,\dots,a_n) = (T_1(a_1,\dots,a_n),\dots,T_m(a_1,\dots,a_n))$$

con $T_i \in k[X_1,\dots,X_n]$.

Prop.-

$$\left\{ \begin{array}{c} \varphi : V \longrightarrow W \\ \text{mapeo pol.} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \tilde{\varphi} : \Gamma(W) \longrightarrow \Gamma(V) \\ \text{homomorfismo de } k\text{-álgebra} \end{array} \right\}$$

$V \underset{\varphi}{\xrightarrow{\sim}} W$ si $\exists\, \psi : W \to V$ tal que $\varphi \circ \psi = 1\!\!1_W$ $\psi \circ \varphi = 1\!\!1_V$

Meta: Identificar isomorfismos entre variedades.